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OLD AND NEW RESULTS ON ALLAN'S GB^*-ALGEBRAS

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This paper is dedicated to the memory of Graham Robert Allan

Abstract. This is an expository paper on the importance and applications of GB^*-algebras in the theory of unbounded operators, which is closely related to quantum field theory and quantum mechanics. After recalling the definition and the main examples of GB^*-algebras we exhibit their most important properties. Then, through concrete examples we are led to a question concerning the structure of the completion of a given C^*-algebra $\mathcal{A}_0[\|\cdot\|_0]$, under a locally convex $*$-algebra topology τ, making the multiplication of $\mathcal{A}_0$ jointly continuous. We conclude that such a completion is a GB^*-algebra over the τ-closure of the unit ball of $\mathcal{A}_0[\|\cdot\|_0]$. Further, we discuss some consequences of this result; we briefly comment the case when τ makes the multiplication of $\mathcal{A}_0$ separately continuous and illustrate the results by examples.

1. Introduction. The motivation of what we shall present comes from mathematical physics, quantum mechanics, and in particular, from the fact that our physical world is mainly represented by unbounded operators. The best known of them is the Hamiltonian operator H representing the observable energy, and the operators P, Q representing the

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observable momentum and observable position respectively (see [9, 14, 31]). The following algebraic equations involving the previous operators

$$[P, Q] = PQ - QP = -i\hbar I, \quad H = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2},$$

correspond to the 1-dimensional harmonic oscillator [9, p. 9, Chapter II], where i is the imaginary unit, $\hbar$ the Planck constant, I the identity operator and m, ω the mass respectively frequency of the oscillator. According to quantum mechanics a physical observable is represented by a linear operator, while certain algebraic relations, as before, correspond to a physical system whose mathematical image is an operator $*$-algebra in an inner product space. In the case of the 1-dimensional harmonic oscillator the respective $*$-algebra is the one generated by the hermitian operators, H, P, Q.

From the above it is clear and quite natural why scientists were led to the study of unbounded operator algebras. Among them are the so-called GB^*-algebras initiated and studied first by G. R. Allan [2], in 1967. In 1970, P. G. Dixon [10] extended the concept of a GB^*-algebra, in order to include also examples of such algebras that are topological $*$-algebras but not locally convex $*$-algebras (see discussion after Examples 2.2). GB^*-algebras generalize the notion of a C^*-algebra. Because of their importance they have been investigated in different directions by various authors (see, for example, [6, 11, 12, 25, 24, 27, 28, 35]). Even more, GB^*-algebras occur among the so-called *unbounded Hilbert algebras* [18, 19, 20, 21, 22, 23, 34], which are very important for the Tomita Takesaki theory for unbounded operator algebras developed in [26], by the second named author.

The structure of this expository paper is as follows: In Section 2, we exhibit some basic definitions and notation and present briefly the most representative and important examples and results on the structure of GB^*-algebras. In Section 3, we discuss a recent result on GB^*-algebras (see [16, Theorem 2.1]), which provides a very handy characterization of this sort of algebras, since using it one does not need to go through all the requirements of the definition of a GB^*-algebra. We reach at the question we deal with through concrete examples. In Section 4, we discuss briefly another aspect of the problem we put in Section 3 (see [4, 5]).

2. Basic examples and results on GB^*-algebras. All algebras we consider are complex and all topological spaces are supposed to be Hausdorff. If an algebra $\mathcal{A}$ has an identity element, this will be denoted by *1*, and an algebra $\mathcal{A}$ with identity *1* will be called *unital*.

Let us begin with a *locally convex $*$-algebra* $\mathcal{A}[\tau]$. At the moment we suppose that the multiplication is separately continuous. We shall always assume that the involution is continuous. An element $x \in \mathcal{A}$ is called (Allan-) *bounded* if there is a non-zero complex number λ, such that the set $\{(\lambda x)^n : n = 1, 2, ...\}$ is bounded in $\mathcal{A}[\tau]$. Denote by $\mathfrak{B}^*$ the family of all subsets B of $\mathcal{A}[\tau]$ such that:

(1) B is absolutely convex, $B^2 \subseteq B$ and $B^* = B$, where $B^* = \{x^* \in B, \forall x \in B\}$;
(2) B is bounded and closed.

For any $B \in \mathfrak{B}^*$, let $\mathcal{A}[B]$ be the $*$-subalgebra of $\mathcal{A}$ generated by B. Then

$$\mathcal{A}[B] = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}.$$

$\mathcal{A}[B]$ becomes a normed $*$-algebra under the gauge function $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$, $x \in \mathcal{A}[B]$. If for every $B \in \mathfrak{B}^*$, $\mathcal{A}[B]$ is a Banach $*$-algebra, then $\mathcal{A}[\tau]$ is called *pseudo-complete* [1, (2.5) Definition]. When $\mathcal{A}[\tau]$ is sequentially complete, then it is pseudo-complete (ibid., (2.6) Proposition).

Definition 2.1. A unital pseudo-complete locally convex $*$-algebra $\mathcal{A}[\tau]$ is called a *GB^*-algebra* (cf. [2, 10]), if $\mathfrak{B}^*$ has a greatest member B_0 and for each $x \in \mathcal{A}$, $(1 + x^*x)^{-1}$ exists in $\mathcal{A}[\tau]$ and it is bounded.

For every GB^*-algebra $\mathcal{A}[\tau]$, the Banach $*$-algebra $\mathcal{A}[B_0]$ is a C^*-algebra and $(1 + x^*x)^{-1} \in \mathcal{A}[B_0]$, for each $x \in \mathcal{A}$ (see [2, (2.6) Lemma]).

A *pro-C^*-algebra* (or locally C^*-algebra) is a complete locally convex algebra $\mathcal{A}[\tau]$ with involution, whose topology τ is defined by a directed family of C^*-seminorms. Each pro-C^*-algebra is topologically $*$-isomorphic to an inverse limit of C^*-algebras [15].

If $\mathcal{A}[\|\cdot\|]$ is a normed algebra we shall use the symbol $\|\cdot\|$ to denote its topology.

Examples 2.2. **1.** Every unital C^*-algebra $\mathcal{A}[\|\cdot\|]$ is a GB^*-algebra over its unit ball.

2. Every unital pro-C^*-algebra $\mathcal{A}[\tau]$ is a GB^*-algebra over the unit ball of its bounded part $\mathcal{D}(p_\Gamma)$, where $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ is a directed family of C^*-seminorms defining τ. More precisely,

$$\mathcal{D}(p_\Gamma) := \{x \in \mathcal{A} : \sup_\lambda p_\lambda(x) < \infty\}$$

is a $*$-subalgebra of $\mathcal{A}$, which is a C^*-algebra under the C^*-norm $p_\Gamma(x) := \sup_\lambda p_\lambda(x), x \in \mathcal{D}(p_\Gamma)$ and moreover it is τ-dense in $\mathcal{A}[\tau]$ [15, Theorem 10.23]. If, e.g., we take the pro-C^*-algebra $\mathcal{C}(\mathbb{R}) = \varprojlim_n \mathcal{C}[-n, n]$ of all $\mathbb{C}$-valued continuous functions on $\mathbb{R}$, with the topology of compact convergence, then $\mathcal{D}(p_\Gamma) = \mathcal{C}_b(\mathbb{R})$, the C^*-algebra of all bounded continuous functions on $\mathbb{R}$, and $\mathcal{C}(\mathbb{R})[B_0] = \mathcal{C}_b(\mathbb{R})$, with B_0 the unit ball of $\mathcal{D}(p_\Gamma)$.

3. Suppose now that $\mathcal{A}[\tau]$ is a locally convex algebra with involution, where τ is defined by a directed family $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms with the following properties: for each $\lambda \in \Lambda$ there is $\lambda' \in \Lambda$ such that $p_\lambda(xy) \le p_{\lambda'}(x)p_{\lambda'}(y)$, $p_\lambda(x^*) \le p_{\lambda'}(x)$ and $p_\lambda(x)^2 \le p_{\lambda'}(x^*x)$, for all $x, y \in \mathcal{A}$. That is, $\mathcal{A}[\tau]$ is a locally convex $*$-algebra with continuous multiplication, whose seminorms determining τ fulfill a C^*-like condition. $\mathcal{A}[\tau]$, as before, is called a *C^*-like locally convex $*$-algebra* if it is moreover complete and the normed $*$-algebra $\mathcal{D}(p_\Gamma)[p_\Gamma]$ is τ-dense in $\mathcal{A}$ [3, 24]. In [24, Theorem 2.1], it is proved that *every C^*-like locally convex $*$-algebra $\mathcal{A}[\tau]$ is a GB^*-algebra over the unit ball B_0 of $\mathcal{D}(p_\Gamma)$*.

4. The Arens algebra $L^\omega[0, 1] = \cap_{1\le p<\infty} L^p[0, 1]$ endowed with the topology induced by the family Γ of the L^p-norms, $1 \le p < \infty$, is a C^*-like locally convex $*$-algebra [24], hence a GB^*-algebra (see also [2, p. 96]) over the unit ball B_0 of $L^\infty[0, 1] = \mathcal{D}(p_\Gamma) = L^\omega[0, 1][B_0]$.

5. We should also note that *every closed $*$-subalgebra of a GB^*-algebra $\mathcal{A}[\tau]$ is also a GB^*-algebra if it contains the identity 1 of $\mathcal{A}$* [2, (2.9) Proposition].

P. G. Dixon, in 1969, extended Allan's definition of a GB^*-algebra to an arbitrary topological $*$-algebra, in order to include also examples of non-locally convex topological GB^*-algebras [10, (2.5) Definition]. Such an example is given by the algebra $\mathcal{M}[0,1]$ of all measurable functions on $[0,1]$ (modulo equality a.e.), endowed with the topology of convergence in measure (ibid., p. 696, (3.4)). More precisely, $\mathcal{M}[0,1]$ is a complete metrizable non-locally-convex GB^*-algebra, with jointly continuous multiplication and $\mathcal{M}[0,1][B_0] = L^\infty[0,1]$, with B_0 the corresponding greatest member of $\mathfrak{B}^*$ in the modified definition of Dixon.

In [11, Sections 6,7], P. G. Dixon constructs another example of a non-locally convex topological GB^*-algebra by considering a W^*-algebra $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$ and $\mathfrak{Q}(\mathcal{A})$ the locally convex $*$-algebra of all operators on $\mathcal{H}$ "quasi-measurable" with respect to $\mathcal{A}$ (ibid., (6.4) Definition). He endows $\mathfrak{Q}(\mathcal{A})$ with a vector space topology resembling to the topology of convergence in measure on a space of measurable functions and he proves that, under this topology, $\mathfrak{Q}(\mathcal{A})$ becomes a topological GB^*-algebra.

Furthermore, the same author points out in [10, p. 696, (3.3)] that if $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ a normed $*$-algebra of bounded linear operators on $\mathcal{H}$, containing the identity operator, then endowed with the weak operator topology, $\mathcal{A}$ becomes a locally convex GB^*-algebra. Moreover, in [12, p. 160, Example 1], Dixon shows that there is a $*$-algebra $\mathcal{A}$ of functions with no GB^*-topology. In fact $\mathcal{A}$ is the algebra of all $\mathbb{C}$-valued Borel functions on $[0,1]$, modulo a.e., where a.e. in this case means "outside a set of first category".

Other interesting examples of GB^*-algebras are given by the L^ω-algebras of operators associated with an unbounded Hilbert algebra; see [19, 21, 22] and in particular [20, Theorem 3.3, Corollary 3.4], [23, p. 32, Theorem1].

Furthermore, S.J.L. van Eijndhoven and P. Kruszyński have considered in [13] a directed family $\mathcal{R}$ of commuting positive bounded operators on a Hilbert space $\mathcal{H}$ and constructed an inductive limit $\mathcal{S}_{\mathcal{R}} \subseteq \mathcal{H}$ of Hilbert spaces, that serves as the maximal common dense domain for the unbounded operator algebras $\mathcal{R}^c, \mathcal{R}^{cc}$, corresponding to the strong commutant respectively strong bicommutant of $\mathcal{R}$. It is proved (ibid., (3.10) Theorem) that both of $\mathcal{R}^c, \mathcal{R}^{cc}$ are GB^*-algebras.

Before we proceed to a selection of representative results on GB^*-algebras, we should stress their contribution to the introduction of the so called EW^*-algebras (extended W^*-algebras) [11, 23, 26], which play a decisive role in the development of the unbounded Tomita-Takesaki theory by the second named author.

We also remark, that using techniques of GB^*-algebras, A. B. Patel [32, Theorem 2.2] proved a "joint spectral theorem" for an n-tuple of doubly commuting unbounded normal operators, and independently A. Danrun-Zh. Dianzhou [17] studied various properties of the joint spectrum of a commuting n-tuple of unbounded normal operators in terms of GB^*-algebras and EC^*-algebras (extended C^*-algebras; see [18]).

• In the rest of the paper we use Allan's definition of a GB^*-algebra.

The first structure results for GB^*-algebras were given by G. R. Allan [2] in the commutative case and concern functional representation and functional calculus [2, Theorems (3,9), (3.12)]. We discuss briefly the functional representation result. If $\mathcal{A}[\tau]$ is a commutative GB^*-algebra, then $\mathcal{A}[B_0]$ coincides with the subalgebra $\mathcal{A}_0$ of all bounded elements of $\mathcal{A}$. So, in this case, for brevity's sake, we shall use the notation $\mathcal{A}_0$ in place of $\mathcal{A}[B_0]$. Denote by $\mathfrak{M}_0$ the Gel'fand space of $\mathcal{A}_0$ and by $\mathbb{C}^*$ the one point compactification of $\mathbb{C}$. Then, the following holds:

THEOREM 2.3 (Allan). *Let $\mathcal{A}[\tau]$ be a commutative GB^*-algebra. Then $\mathcal{A}[\tau]$ is $*$-isomorphic to a $*$-algebra of $\mathbb{C}^*$-valued continuous functions on the compact space $\mathfrak{M}_0$.*

P. G. Dixon studied the structure of non-commutative GB^*-algebras. He showed that they are algebras of unbounded operators, and he investigated their functional calculus and their $*$-representation theory [10].

We briefly refer to the notion of an "algebra of closed operators", as this is used by P. G. Dixon [10, p. 705] in Theorem 2.4, below. Let $\mathcal{H}$ be a Hilbert space and T a linear operator in $\mathcal{H}$, with domain a dense subspace $\mathcal{D}$ of $\mathcal{H}$. T is called *closed* if its graph $\mathcal{G}(T)$ is closed in $\mathcal{H}\oplus\mathcal{H}$. A linear operator T in $\mathcal{H}$ is said to be *closable* if it has a closed extension. Every closable operator has a smallest closed extension, called its *closure*, denoted by $\overline{T}$ [3, pp. 6, 7]. A set $\mathcal{A}$ of closed operators in $\mathcal{H}$ will be called a *$*$-algebra of closed operators* [10, p. 705, (7.1)] if it forms a $*$-algebra under the operations of addition, multiplication and involution defined as follows:

$$(T,S) \mapsto \overline{T+S},\ (T,S) \mapsto \overline{TS} \ \text{ and } \ T \mapsto T^*,$$

where T^* is the adjoint of T. The definition of addition and multiplication implicitly means that $T+S$, TS are closable operators for all $T, S \in \mathcal{A}$.

A *$*$-representation* of a $*$-algebra $\mathcal{A}$ (in the sense of P. G. Dixon) is a $*$-homomorphism π of $\mathcal{A}$ onto a $*$-algebra of closed operators on a Hilbert space $\mathcal{H}$, such that $\pi(1)$ is the identity operator on $\mathcal{H}$; π is called *faithful* if it is injective [10, p. 705, (7.2)].

THEOREM 2.4 (Dixon). *Every GB^*-algebra $\mathcal{A}[\tau]$ over B_0 is *-algebraically realized as an algebra of closed operators on a Hilbert space $\mathcal{H}$. In other words, there exists a faithful $*$-representation π of $\mathcal{A}$ onto a $*$-algebra of closed operators on a Hilbert space $\mathcal{H}$, with common dense domain, such that the elements of B_0 correspond to the operators $T \in \pi(\mathcal{A})\cap\mathfrak{B}(\mathcal{H})$ with $\|T\| \le 1$, where $\mathfrak{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}$ under the operator norm $\|\cdot\|$.*

It is clear that Theorem 2.3 is a Gel'fand-type representation theorem, while Theorem 2.4 is a non-commutative Gel'fand-Naimark-type theorem for GB^*-algebras.

Some further results on GB^*-algebras are listed in the following

PROPOSITION 2.5. (1) [27, p. 10] (i) *If $\mathcal{A}_\lambda[\tau_\lambda]$, $\lambda \in \Lambda$, is a family of GB^*-algebras, then the product $\mathcal{A} = \prod_{\lambda\in\Lambda}\mathcal{A}_\lambda$, endowed with algebraic operations defined coordinatewise and with the product topology, is a GB^*-algebra, with $\mathcal{A}[B_0] = \{(x_\lambda) \in \mathcal{A} : x_\lambda \in \mathcal{A}_\lambda[B_{0\lambda}]$ and $\sup_{\lambda\in\Lambda}\|x_\lambda\|_{B_{0\lambda}} < \infty\}$.*

(ii) *If $\mathcal{A}_\lambda[\tau_\lambda]$, $\lambda \in \Lambda$, is an inverse system of GB^*-algebras, with Λ a directed index set, then the inverse limit algebra $\mathcal{A}[\tau] = \varprojlim_{\lambda}\mathcal{A}_\lambda[\tau_\lambda]$ is a GB^*-algebra, with $\mathcal{A}[B_0] = \{(x_\lambda) \in \mathcal{A} : x_\lambda \in \mathcal{A}_\lambda[B_{0\lambda}]$ and $\sup_{\lambda\in\Lambda}\|x_\lambda\|_{B_{0\lambda}} < \infty\}$. $B_{0\lambda}$ is a greatest member in $\mathfrak{B}^*_{\mathcal{A}_\lambda}$.*

(iii) *Let* $\mathcal{A}[\tau]$ *be a* GB^**-algebra with* $x(1+x^*x)^{-1}$ *bounded, for every* $x \in \mathcal{A}$. *Let* I *be a closed 2-sided ideal in* $\mathcal{A}[\tau]$. *Then* I *is a* $*$*-ideal and the quotient algebra* $\mathcal{A}/I$, *with the quotient topology, is a* GB^**-algebra such that* $(\mathcal{A}/I)[B_0'] = \mathcal{A}[B_0]/I \cap \mathcal{A}[B_0]$, *with respect to an isometric* $*$*-isomorphism,* B_0' *being a greatest member in* $\mathfrak{B}^*_{\mathcal{A}/I}$.

(2) [6, Corollary 3, II] *Every* GB^**-algebra* (not necessarily with identity) *has a bounded approximate identity.*

(3) [6, (2) Theorem, I] *If* $\mathcal{A}[\tau]$ *is a* GB^**-algebra over* B_0, *then the* C^**-algebra* $A[B_0]$ *is sequentially dense in* $\mathcal{A}[\tau]$.

(4) [35, (8.16) Theorem] (Vidav-Palmer theorem) *Let* $\mathcal{A}[\tau]$ *be a Fréchet locally convex algebra. Then,* $\mathcal{A}[\tau]$ *is a* GB^**-algebra if and only if there exists a directed family of seminorms defining the topology* τ *of* $\mathcal{A}$, *with respect to which* $\mathcal{A} = H + iH$, *where* H *stands for the hermitian elements of* $\mathcal{A}$, *i.e., those elements of* $\mathcal{A}$ *having real numerical range.*

3. The completion of a C^*-algebra under a locally convex $*$-algebra topology is a GB^*-algebra. There are examples of C^*-algebras $\mathcal{A}_0[\|\cdot\|_0]$ endowed with a locally convex topology τ, coarser than the C^*-topology $\|\cdot\|_0$ and such that $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra with jointly continuous multiplication, whose completion $\widetilde{\mathcal{A}_0}[\tau]$ contains continuously $\mathcal{A}_0[\|\cdot\|_0]$. Take, for instance, the C^*-algebra $\mathcal{C}_b(\mathbb{R})$ of all bounded continuous functions on $\mathbb{R}$ and the algebra $\mathcal{C}(\mathbb{R})$ of all continuous functions on $\mathbb{R}$, with the topology τ of uniform convergence on compacta. Then,

$$\mathcal{C}_b(\mathbb{R}) \hookrightarrow \mathcal{C}(\mathbb{R}) = \widetilde{\mathcal{C}_b(\mathbb{R})}[\tau].$$

The same happens if we take the GB^*-algebra $L^\omega[0,1]$ (Arens algebra) of the example 2.2(4) and the C^*-algebra $L^\infty[0,1]$. More precisely,

$$L^\infty[0,1] \hookrightarrow L^\omega[0,1] = \widetilde{L^\infty[0,1]}[\tau],$$

with τ the topology on $L^\omega[0,1]$ induced by the L^p-norms $\|\cdot\|_p$, $1 \le p < \infty$. Moreover, if $(f_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $L^\infty[0,1]$ such that $f_\lambda \underset{\tau}{\longrightarrow} f \in L^\infty[0,1]$, then also $f_\lambda \underset{\|\cdot\|_\infty}{\longrightarrow} f$. Topologies with the last property are called *normal* [3, p. 352].

In particular, if $\mathcal{A}_0[\|\cdot\|_0]$ and $\mathcal{A}_0[\tau]$ are like at the beginning of this Section and $\tau \preceq \|\cdot\|_0$, we say that $\tau, \|\cdot\|_0$ are *compatible* if each net in $\mathcal{A}_0$, which is a Cauchy net in both of these topologies and converges in one of them, converges also with respect to the other one. The topology τ of the Arens algebra and the C^*-norm $\|\cdot\|_\infty$ on $L^\infty[0,1]$ are compatible. Such examples lead to the following natural

Question: *Suppose* $\mathcal{A}_0[\|\cdot\|_0]$ *is a unital* C^**-algebra and* τ *a locally convex topology on* $\mathcal{A}_0$ *making the involution continuous and the multiplication jointly continuous. Suppose also that* $\tau \preceq \|\cdot\|_0$ *and* $\tau, \|\cdot\|_0$ *are compatible. What can be said about the completion* $\widetilde{\mathcal{A}_0}[\tau]$ *of* $\mathcal{A}_0[\tau]$*?*

We shall see that $\widetilde{\mathcal{A}_0}[\tau]$ has the structure of a GB^*-algebra. Before we outline the proof of this result, let B_τ denote the τ-closure of the unit ball $U_0 = \{x \in \mathcal{A}_0 : \|x\|_0 \le 1\}$ of $\mathcal{A}_0[\|\cdot\|_0]$ in $\widetilde{\mathcal{A}_0}[\tau]$. By the joint continuity of the multiplication in $\widetilde{\mathcal{A}_0}[\tau]$, it is easily seen that B_τ belongs to the (corresponding to p. 171) family $\mathfrak{B}^*$ of subsets in $\widetilde{\mathcal{A}_0}[\tau]$. Moreover (see [16, p. 157, (2.1)]),

$$\forall\ x \in \widetilde{\mathcal{A}_0}[\tau],\ \text{the element } (1 + x^*x)^{-1} \text{ exists and } (1 + x^*x)^{-1} \in B_\tau.$$

THEOREM 3.1. *$\widetilde{\mathcal{A}_0}[\tau]$ is a GB^*-algebra over B_τ* [16, Theorem 2.1].

Proof. We present the main steps of the proof:

Since $\widetilde{\mathcal{A}_0}[\tau]$ is a complete locally convex $*$-algebra, it suffices to show that B_τ is a greatest member in $\mathfrak{B}^*$ and that $(1 + x^*x)^{-1} \in \mathcal{A}[B_\tau]$, $\forall x \in \widetilde{\mathcal{A}_0}[\tau]$. The latter follows from the preceding discussion.

• A crucial point for showing that B_τ is greatest in $\mathfrak{B}^*$, is to show that $\mathcal{A}[B_\tau]$ is a C^*-algebra.

For this consider the C^*-algebra $\mathfrak{A}$ of all $\|\cdot\|_0$-bounded nets $(x_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{A}_0[\|\cdot\|_0]$, where the index set Λ consists of a 0-neighborhood basis for τ, and the C^*-norm is given by $\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty := \sup_{\lambda\in\Lambda}\|x_\lambda\|_0$, $(x_\lambda)_{\lambda\in\Lambda} \in \mathfrak{A}$. Take now

$$\begin{aligned}\mathfrak{A}_c &:= \{(x_\lambda)_{\lambda\in\Lambda} \in \mathfrak{A} : (x_\lambda)_{\lambda\in\Lambda} \text{ is a } \tau\text{-Cauchy net}\},\\ \mathfrak{A}_0 &:= \{(x_\lambda)_{\lambda\in\Lambda} \in \mathfrak{A}_c : \tau\text{-}\lim_\lambda x_\lambda = 0\}.\end{aligned}$$

It is easily checked that $\mathfrak{A}_c$ is a closed $*$-subalgebra of the C^*-algebra $\mathfrak{A}[\|\cdot\|_\infty]$, hence a C^*-algebra, and that $\mathfrak{A}_0$ is a closed ideal (hence a $*$-ideal) in $\mathfrak{A}_c$. Therefore, the quotient $\mathfrak{A}_c/\mathfrak{A}_0$ is a C^*-algebra. Now, an element $a \in B_\tau$ is of the form τ-$\lim_\lambda x_\lambda$ where $(x_\lambda)_{\lambda\in\Lambda}$ is a net in $\mathcal{A}_0$ with $\|x_\lambda\|_0 \le 1$, for all $\lambda \in \Lambda$. In other words, $a = \tau$-$\lim_\lambda x_\lambda$ with $(x_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{A}_c$. Thus, the following correspondence

$$\Theta : \mathfrak{A}_c \ \rightarrow\ A[B_\tau] : (x_\lambda)_{\lambda\in\Lambda} \mapsto\ \tau\text{-}\lim_\lambda x_\lambda,$$

is a well-defined $*$-homomorphism, with $\ker\Theta = \mathfrak{A}_0$. Θ induces an isometric $*$-isomorphism from $\mathfrak{A}_c/\mathfrak{A}_0$ onto $A[B_\tau]$. Hence, the Banach $*$-algebra $A[B_\tau]$ is a C^*-algebra.

• Let now B be an arbitrary element in $\mathfrak{B}^*$. We shall show that $B \subseteq B_\tau$. Take an element $h = h^*$ in B. Consider the maximal commutative $*$-subalgebra of $\widetilde{\mathcal{A}_0}[\tau]$ containing h, 1. Using C^*-algebra theory and properties of the elements of $\mathfrak{B}^*$ from [1], we conclude that (see [16, p. 158])

$$A[B] \subseteq A[B_\tau].$$

Take now an arbitrary element $x \in B$. Then, by the preceding inclusion, $x \in A[B_\tau]$. Moreover, x^*x as a self-adjoint element in B belongs also to B_τ, which is the unit ball of $A[B_\tau]$. Therefore, $\|x\|^2_{B_\tau} = \|x^*x\|_{B_\tau} \le 1$. Hence, $x \in B_\tau$ and $B \subseteq B_\tau$, which proves that B_τ is a greatest member in $\mathfrak{B}^*$. ∎

It is clear that the C^*-algebra $\mathcal{A}_0[\|\cdot\|_0]$ that determines the locally convex $*$-algebra $\widetilde{\mathcal{A}_0}[\tau]$ is not unique. Thus, if $\mathcal{C}^*(\mathcal{A}_0, \tau)$ denotes the collection of all C^*-algebras $\mathcal{A}[\|\cdot\|]$ such that $\mathcal{A}_0 \subseteq \mathcal{A} \subseteq \widetilde{\mathcal{A}_0}[\tau]$, $\tau \preceq \|\cdot\|$ and $\|x\| = \|x\|_0, \forall\ x \in \mathcal{A}_0$, then $\mathcal{C}^*(\mathcal{A}_0, \tau)$ has an order defined as follows:

$$\mathcal{A}_1[\|\cdot\|_1] \preceq \mathcal{A}_2[\|\cdot\|_2] \iff \mathcal{A}_1 \subseteq \mathcal{A}_2 \text{ and } \|x\|_1 = \|x\|_2,\ \forall\ x \in \mathcal{A}_1.$$

From Theorem 3.1 it follows that the C^*-algebra $A[\mathbf{B}_\tau]$ is the largest member in $\mathcal{C}^*(\mathcal{A}_0, \tau)$. The same theorem implies the following characterization, which is related to some previous results in [6, 27, 28].

Corollary 3.2. *The following statements are equivalent:*

(i) $\widetilde{\mathcal{A}_0}[\tau]$ *is a GB^*-algebra over U_0.*

(ii) U_0 *is τ-closed.*

After Theorem 3.1 one naturally is led to a functional calculus for $\widetilde{\mathcal{A}_0}[\tau]$, in the commutative and non-commutative case, as well as to the investigation of the existence of (unbounded) faithful $*$-representations on this $*$-algebra. Such results are exhibited in [16, Theorems 2.3, 2.4, 2.6].

Corollary 3.2 provides a very useful characterization, which allows us to check more easily whether $\widetilde{\mathcal{A}_0}[\tau]$ is a GB^*-algebra, instead of showing all the properties of the definition given in Section 2.

Other relatively recent results on the existence of either continuous or "well-behaved" (unbounded) $*$-representations on GB^*-algebras can be found in [7, Proposition 4.8, Corollary 5.4] and [8, Corollary 4.5, 5.5 and Example 5.7].

4. A brief discussion on the structure of $\widetilde{\mathcal{A}_0}[\tau]$ when the multiplication in $\mathcal{A}_0[\tau]$ is separately continuous. Let $\mathcal{A}[\tau]$ be a locally convex $*$-algebra with separately continuous multiplication. Taking the completion $\widetilde{\mathcal{A}}[\tau]$ of $\mathcal{A}[\tau]$ we have no more a locally convex $*$-algebra, since multiplication among the elements of $\widetilde{\mathcal{A}}[\tau]$ cannot be defined everywhere. For instance, $x \cdot y$ is well defined if $x, y \in \mathcal{A}[\tau]$ or $x \in \widetilde{\mathcal{A}}[\tau], y \in \mathcal{A}[\tau]$ and vice-versa. Such objects are called partial $*$-algebras. More precisely [3, p. 44], a *partial $*$-algebra* is a pair $(\mathcal{A}, \Gamma)$ consisting of a complex vector space $\mathcal{A}$ endowed with a vector space involution and a subset Γ of $\mathcal{A} \times \mathcal{A}$, with the properties:

(i) $(x, y) \in \Gamma$ yields $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ and $\lambda, \mu \in \mathbb{C}$ yield $(x, \lambda y_1 + \mu y_2) \in \Gamma$;

(iii) for every $(x, y) \in \Gamma$, a product $xy \in \mathcal{A}$ is defined, such that xy depends linearly on x and y and $(xy)^* = y^* x^*$.

Saying that $\mathcal{A}$ is a partial $*$-algebra we shall always mean a pair $(\mathcal{A}, \Gamma)$ as before.

A *locally convex partial $*$-algebra* is a partial $*$-algebra equipped with a locally convex topology such that the involution is continuous and the partial multiplication is separately continuous [3, p. 45].

An important subclass of partial $*$-algebras is the so-called *quasi $*$-algebras*. A partial $*$-algebra $\mathcal{A}$, which contains a $*$-algebra $\mathcal{A}_0$, such that $(x, y) \in \Gamma \Leftrightarrow x$ or y belongs to $\mathcal{A}_0$, is called a *quasi $*$-algebra over* $\mathcal{A}_0$. A locally convex partial $*$-algebra $\mathcal{A}[\tau]$ is said to be a *locally convex quasi $*$-algebra* if $\mathcal{A}$ is a quasi $*$-algebra over $\mathcal{A}_0$ and $\mathcal{A}_0$ is dense in $\mathcal{A}[\tau]$ (ibid., p. 46). For the basic theory and applications of the preceding classes of algebras the reader is referred to [3]. The impetus for the introduction of partial $*$-algebras and quasi $*$-algebras has again its roots in mathematical physics. Partial $*$-algebras were initiated by J.-P. Antoine and W. Karwowski in 1983 and quasi $*$-algebras by G. Lassner in 1981. For more details, motivation and all relevant literature, see [3].

The completion of a locally convex $*$-algebra $\mathcal{A}[\tau]$ with separately continuous multiplication may not be (as we noticed above) a locally convex $*$-algebra, but it is a quasi $*$-algebra.

Suppose, for instance, that Ω is a compact subset of $\mathbb{R}^n$ that coincides with the closure of its interior and let $\mathcal{C}(\Omega)$ be the algebra of all continuous functions on Ω. Then, endowed with the L^p-norm, $1 \leq p < \infty$, $\mathcal{C}(\Omega)$ becomes a locally convex $*$-algebra. Its completion coincides with the space $L^p(\Omega)$, hence $L^p(\Omega)$ is a locally convex quasi $*$-algebra over $\mathcal{C}(\Omega)$.

Furthermore, if $\mathcal{A}[\tau]$ is a GB^*-algebra over B_0, then $A[B_0]$ is a C^*-algebra under the gauge function $\|\cdot\|_{B_0}$ of B_0, as we noticed in Section 2. If τ is determined by a directed family $\{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms, with the property:

$$\forall\ \lambda \in \Lambda\ \exists\ \lambda' \in \Lambda \text{ such that } p_\lambda(xy) \leq \|x\|_{B_0} p_{\lambda'}(y),\ \forall\ x, y \in A[B_0] : xy = yx,$$

then $\widetilde{A[B_0]}[\tau] = \mathcal{A}[\tau]$, and it readily follows that $\mathcal{A}[\tau]$ is a locally convex quasi $*$-algebra over $A[B_0]$.

If $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-algebra (or even a normed $*$-algebra with the C^*-property) and τ a locally convex topology on $\mathcal{A}_0$ making the multiplication of $\mathcal{A}_0$ not jointly continuous, then as we noticed before, the completion $\widetilde{\mathcal{A}_0}[\tau]$ of $\mathcal{A}_0$ with respect to τ, has the structure of a locally convex quasi $*$-algebra. The investigation of the structure and the (unbounded) $*$-representation theory of this sort of algebras started in [16, Section 3] and continued in [4, 5], with a plethora of examples illustrating the results. We also note that Allan's spectral theory for locally convex algebras [1], as well as the related to GB^*-algebras theory of unbounded operator algebras of P. G. Dixon [11], play an important role in the development of the aforementioned study of $\widetilde{\mathcal{A}_0}[\tau]$ as a locally convex quasi $*$-algebra.

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