BANACH ALGEBRAS 2009
BANACH CENTER PUBLICATIONS, VOLUME 91
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

ON DERIVATIONS AND CROSSED HOMOMORPHISMS

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Abstract. We discuss some results about derivations and crossed homomorphisms arising in the context of locally compact groups and their group algebras, in particular, $L^1(G)$, the von Neumann algebra $VN(G)$ and actions of G on related algebras. We answer a question of Dales, Ghahramani, Grønbæk, showing that $L^1(G)$ is always permanently weakly amenable. Then we show that for some classes of groups (e.g. IN-groups) the homology of $L^1(G)$ with coefficients in $VN(G)$ is trivial. But this is no longer true, in general, if $VN(G)$ is replaced by other von Neumann algebras, like $\mathcal{B}(L^2(G))$. Finally, as an example of a non-discrete, non-amenable group, we investigate the case of $G = SL(2,\mathbb{R})$ where the situation is rather different.

0. Introduction. Let $\mathcal{A}$ be a Banach algebra, X a Banach $\mathcal{A}$-bimodule. A linear mapping $D : \mathcal{A} \to X$ is called a *derivation*, if $D(ab) = aD(b) + D(a)b$ for all $a, b \in \mathcal{A}$ ([D] Def. 1.8.1). For $f \in X$, we define the inner derivation $\mathrm{ad}_f : \mathcal{A} \to X$ by $\mathrm{ad}_f(a) = fa - af$ (as in [GRW]; $\mathrm{ad}_f = -\delta_f$ in the notation of [D] (1.8.2)). $\mathcal{Z}^1(\mathcal{A}, X)$ denotes the space of bounded derivations from $\mathcal{A}$ to X, $\mathcal{B}^1(\mathcal{A}, X)$ the subspace of inner derivations. $\mathcal{H}^1(\mathcal{A}, X) = \mathcal{Z}^1/\mathcal{B}^1$ is called the (first continuous) *cohomology group* of $\mathcal{A}$ with coefficients in X ([D] Def. 2.8.2).

We will concentrate on group algebras. For G a locally compact group with a fixed left Haar measure, we consider $L^1(G)$ (integrable functions), $M(G)$ (complex Radon measures on G) with convolution and $VN(G)$, the von Neumann algebra on $L^2(G)$ generated by the left regular representation. If X is a left Banach G-module (the action of G is denoted by $\circ$), a mapping $\Phi : G \to X$ is called a *crossed homomorphism* if $\Phi(xy) = \Phi(x) + x \circ \Phi(y)$ for all $x, y \in G$ (in the terminology of [D] Def. 5.6.35, this is a G-derivation, if we consider the trivial right action of G on X). Φ is called bounded if $\|\Phi\| = \sup_{x\in G}\|\Phi(x)\| < \infty$. For $f \in X$, the special example $\Phi_f(x) = f - x \circ f$ is called a

2010 *Mathematics Subject Classification*: Primary 43A20; Secondary 22F05, 37B05, 46H99, 46L55, 47B47.

Key words and phrases: cohomology of Banach algebras, group algebras, amenability.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc91-0-12

principal crossed homomorphism (this follows [GRW], the sign is taken opposite to [D]). If $X = Y'$ is the dual of an essential Banach $L^1(G)$-bimodule Y, the actions of $L^1(G)$ on X can be extended to actions of $M(G)$. We consider the action of G on X defined by $x \circ f = \delta_x f \delta_{x^{-1}}$ for $x \in G$, $f \in X$ (δ_x denotes the point measure at x). Then there is a bijective correspondence between bounded derivations $L^1(G) \to X$ and bounded crossed homomorphisms $G \to X$, it associates to the inner derivation ad_f the principal crossed homomorphism Φ_f ([D] Th. 5.6.39). In particular, every bounded derivation $L^1(G) \to X$ is inner if and only if every bounded crossed homomorphism $G \to X$ is principal.

In [Lo] we have considered the case where $X = M(\Omega)$ for a locally compact space Ω and the left action of G on X is induced by an action of G by homeomorphisms on Ω. It was shown ([Lo] Th. 1.1) that in this case every bounded crossed homomorphism $G \to X$ is principal. A special case is $\Omega = G$ with the action $x \circ y = xyx^{-1}$ of G on G. Under the correspondence mentioned above this leads to bounded derivations $L^1(G) \to M(G)$ ($L^1(G)$ acting on $M(G)$ by convolution) and it followed from our theorem that every bounded derivation $L^1(G) \to M(G)$ is inner (Johnson's derivation problem). In Section 1 below we will consider a related question going back to Dales, Ghahramani and Grønbæk ([DGG]) about n-weak amenability of $L^1(G)$. This (Theorem 1.2) will provide another application of [Lo] Th. 1.1, considering actions of G on different compact spaces.

In the remaining sections, we will be dealing with derivations $L^1(G) \to VN(G)$. Some partial results have been obtained in [GRW] (see Remark 3.9). In the case where G is amenable, bounded derivations $L^1(G) \to VN(G)$ are always inner by a general result of Johnson. Then Lau ([La]) used a technique based on weak almost periodicity to show that this remains true when G is a SIN-group, in particular for all discrete groups. This is presented in Theorem 2.2. Then in Section 3 we extend this approach and show that it can also be used to handle the case of IN-groups (Theorem 3.1) and more generally when G is unimodular and has an open amenable normal subgroup (Corollary 3.8). One can consider also actions of G on other von Neumann algebras. For $G = F_2$ (free group with two generators), we give an example (Example 2.6) of a non-principal bounded crossed homomorphism $G \to \mathcal{B}(l^2(G))$ (all bounded linear operators on $l^2(G)$). We do not know if such examples exist for all discrete non-amenable groups. In Section 4, as an example of a non-amenable connected group, we use results from the representation theory of $SL(2,\mathbb{R})$. It turns out (Example 4.6) that the cohomology spaces $\mathcal{H}^1(L^1(G), VN(G))$ and $\mathcal{H}^1(L^1(G), \mathcal{B}(L^2(G)))$ are rather big. For irreducible representations, the cohomology spaces $\mathcal{H}^1(L^1(G), \mathcal{B}(\mathcal{H}))$ are one-dimensional in many cases (Example 4.5).

Further Notations. e will always denote the unit element of a group G. If G is a locally compact group, $L^1(G)$, $L^\infty(G)$ are defined with respect to a fixed left Haar measure on G. Convolution is denoted by $*$. Duality between Banach spaces is denoted by $\langle\ \rangle$, thus for $f \in L^\infty(G)$, $u \in L^1(G)$, we have $\langle f, u\rangle = \int_G f(x)u(x)\,dx$. For $f \in L^1(G)$, $\rho(f)u = f * u$ denotes the corresponding convolution operator on $L^2(G)$. The same for $\rho(\mu)$, when $\mu \in M(G)$ is a measure. $C_r^*(G)$ (the reduced C^*-algebra of G) is defined as the norm-closure of $\rho(L^1(G))$ in $\mathcal{B}(L^2(G))$ (all bounded linear operators on $L^2(G)$).

1. Generalization of weak amenability

DEFINITION 1.1 ([DGG] Def. 1.1). A Banach algebra $\mathcal{A}$ is called *n-weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)})$ is trivial ($n \in \mathbb{N}$). $\mathcal{A}$ is called *permanently weakly amenable* if it is n-weakly amenable for each $n \in \mathbb{N}$.

$\mathcal{A}^{(n)}$ denotes the n-th dual space of $\mathcal{A}$. It is an $\mathcal{A}$-bimodule by dualizing the action of $\mathcal{A}$ on itself ([D] p. 240).

THEOREM 1.2. *Let G be a locally compact group, then $\mathcal{A} = L^1(G)$ (with convolution) is permanently weakly amenable.*

Thus each bounded derivation $D : \mathcal{A} \to \mathcal{A}^{(n)}$ is inner. This question was left open in [DGG] p. 42. I thank V. Runde for pointing out this problem to me.

Proof. If n is odd, it was shown in [DGG] Th. 4.1 that $L^1(G)$ is n-weakly amenable, extending a classical result of Johnson (the case $n = 1$; this is just weak amenability of $L^1(G)$).

Assume that n is even. We will make some standard reductions to be able to apply Th. 1.1 of [Lo]. Put $n = 2(k+1)$ with $k \geq 0$, $Y = \mathcal{A}^{(2k+1)} = L^\infty(G)^{(2k)}$. Then $\mathcal{A}^{(n)} = Y'$. Let $Y_e = \mathcal{A} Y \mathcal{A}$ be the essential part of the $\mathcal{A}$-bimodule Y (again using the dual actions). Y_e is a closed subspace of Y (since $L^1(G)$ has a bounded approximate identity), $\pi : Y' \to Y_e'$ shall denote the canonical projection (dual of the inclusion mapping). Let $D : \mathcal{A} \to Y'$ be a bounded derivation. By a result of Johnson (see [D] Cor. 2.9.27), D is inner iff $\pi \circ D$ is inner. Considering pointwise multiplication in $L^\infty(G)$, it follows from classical results ([D] Cor. 2.9.27) that Y (equipped with the corresponding Arens product) is a commutative unital C^*-algebra. $L^\infty(G)$ being an $M(G)$-bimodule, the same is true for Y. Identifying G with the set of point measures, we get a left and a right action of G on Y, denoted by xf and fx ($x \in G,\ f \in Y$). Using the continuity properties of the action of $M(G)$ on Y and the continuity properties of the Arens product, it is easy to see that $f \mapsto xf$ and $f \mapsto fx$ are C^*-algebra automorphisms for each $x \in G$. Furthermore (using the factorization theorem [D] Cor. 2.9.25), we have $Y_e = \mathcal{A}Y \cap Y\mathcal{A} = \{f \in Y : x \mapsto xf \text{ and } x \mapsto fx \text{ are norm-continuous on } G\}$ (compare [D] Prop. 3.3.11). It follows that Y_e is a C^*-subalgebra of Y containing the identity. Hence $Y_e \cong C(\Omega)$ and the actions of G on Y are induced by actions of G on the (compact) Gelfand space Ω. By [D] Th. 5.6.39, $\pi \circ D$ is induced by a bounded crossed homomorphism $\Phi : G \to Y_e'$ (with respect to the dual action of $f \circ x = x^{-1} f x$). By [Lo] Th. 1.1, Φ is principal, hence $\pi \circ D$ is inner. ■

REMARK 1.3. It may be instructive to consider the case $n = 2$ (i.e. $k = 0$). Then $Y = L^\infty(G)$ and $Y_e = UCB(G)$ consists of the bounded uniformly continuous functions, where "uniformly" means both left and right uniform continuity. This corresponds to the lower uniformity of [RD] 2.5, i.e., the infimum of the left and the right uniformity of G. Then Ω is the Samuel compactification of G with respect to this uniformity. If G is discrete, we have $Y_e = l^\infty(G)$ and Ω coincides with the Stone-Čech compactification of G.

2. Derivations from $L^1(G)$ to $VN(G)$

PROPOSITION 2.1. *Let $D : L^1(G) \to VN(G)$ be a derivation. Then the following conditions are equivalent:*

(i) *D is inner (i.e., there exists $T \in VN(G)$ such that $Df = T * f - f * T$ for all $f \in L^1(G)$).*
(ii) *D extends to a derivation $VN(G) \to VN(G)$.*
(iii) *D extends to a derivation $C_r^*(G) \to VN(G)$.*

The extensions in (ii), (iii) *are unique (if they exist) and always bounded. The extension in* (iii) *is always w^*-continuous.*

Proof. This is just a combination of some classical results: If $\mathcal{A}$ is a C^*-algebra, E a Banach-$\mathcal{A}$-bimodule, then every derivation $D : \mathcal{A} \to E$ is bounded ([D] Cor. 5.3.7). Every derivation $D : \mathcal{B} \to \mathcal{B}$ of a von Neumann algebra $\mathcal{B}$ is inner (Sakai's theorem [Sa] Th. 2.5.3) – in particular, D is w^*-continuous. If $\mathcal{B}_0$ is a w^*-dense C^*-subalgebra of a von Neumann algebra $\mathcal{B}$, then every derivation $D : \mathcal{B}_0 \to \mathcal{B}$ extends to a derivation of $\mathcal{B}$ ([SS] Th. 2.2.2; in fact in [SS] it is assumed that $D : \mathcal{B}_0 \to \mathcal{B}_0$, but the proof works for $D : \mathcal{B}_0 \to \mathcal{B}$; alternatively one can use the biduals similar to the proof of [Sa] Cor. 2.5.4). ■

THEOREM 2.2 (Lau [La]). *Assume that the right action of G on Y is weakly almost periodic. Then every bounded crossed homomorphism $\Phi : G \to Y'$ is principal.*

We can find μ in the w^*-closed convex hull of $\Phi(G)$ such that $\Phi(x) = \mu - x \circ \mu$ for all $x \in G$.

Recall that a right action of a group (or semigroup) G on a Banach space Y is called *weakly almost periodic* if the orbits $\{y \circ g : g \in G\}$ are weakly relatively compact for all $y \in Y$.

Proof. This is a counterpart of a classical result of Johnson, saying that if G is an amenable locally compact group, Y an essential right $L^1(G)$-Banach module, then every bounded w^*-continuous crossed homomorphism $\Phi : G \to Y'$ is principal ([D] Th. 5.6.42; equivalently: "$L^1(G)$ is an amenable Banach algebra"). For $y \in Y$ put $u_y(g) = \langle \Phi(g), y\rangle$. Then $T(y) = u_y$ defines a bounded linear mapping $T : Y \to l^\infty(G)$. Φ being a crossed homomorphism, an easy computation shows that $u_y(gh) = u_y(g) + u_{y\circ g}(h)$ for $g, h \in G$. Since T is weakly continuous, it follows from weak almost periodicity of the action on Y that the set of left translates of u_y is weakly relatively compact in $l^\infty(G)$. Thus u_y is a weakly almost periodic function for all $y \in Y$. Let m be the invariant mean for weakly almost periodic functions ([Gr] § 3.1) and define $\mu \in Y'$ by $\langle \mu, y\rangle = m(u_y)$. Then left invariance of m and the formula above for the left translates of u_y give $\langle \mu, y\rangle = u_y(g) + \langle \mu, y\circ g\rangle$, leading to $\Phi(g) = \mu - g \circ \mu$ for all $g \in G$. ■

COROLLARY 2.3 (Lau [La]). *Let G be a* SIN-*group. Then every bounded crossed homomorphism $\Phi : G \to VN(G)$ is principal. Thus $\mathcal{H}^1(L^1(G), VN(G)) = (0)$.*

Proof. Recall that a locally compact group G has small invariant neighbourhoods ("G is a SIN-group") iff there exists a basis of e-neighbourhoods U with the property that $xUx^{-1} = U$ for all $x \in G$. It follows that $VN(G)$ is of finite type ([Di] Prop. 13.10.5; in fact the converse holds as well, see the arguments in Sec. 3 of [Tay]). Then the conjugate action of G on the predual of $VN(G)$ is weakly almost periodic (see the proof of [Ta] Th. V.2.4; compare Remark 2.4). ■

REMARK 2.4. The Corollary applies in particular when G is discrete or compact. Note that no continuity is required (thus it gives slightly more in the compact case than Johnson's theorem).

A more general version can be obtained as follows: Let G be a group acting on a von Neumann algebra $\mathcal{B}$ by $*$-automorphisms. $\mathcal{B}$ is called *G-finite* if the set of G-invariant normal states is faithful on $\mathcal{B}$ (i.e., if $x \in \mathcal{B}$, $x \geq 0$, $x \neq 0$, there exists a G-invariant normal state ϕ such that $\phi(x) \neq 0$). By [St1], $\mathcal{B}$ is G-finite iff the action of G on the predual is weakly almost periodic. Hence, the same argument as above gives for every G-finite von Neumann algebra $\mathcal{B}$ that every bounded crossed homomorphism $\Phi : G \to \mathcal{B}$ is principal. An example for this extended condition would be $\mathcal{B} = L^\infty(\Omega, m)$, when G acts on the locally compact space Ω by homeomorphisms and m is a G-invariant (Radon) probability measure on Ω. Applications of this generalization will come up in the next section (Proposition 3.6 and Remark 3.9).

In the case of a compact group G, one obtains a G-finite von Neumann algebra from an arbitrary action of G by $*$-automorphisms on $\mathcal{B}$, provided the corresponding action on the predual is strongly continuous (which is known to be equivalent to continuity for the weak operator topology of mappings on the predual – this is the same topology that was used in [St1]). In particular, one can take all actions implemented by unitary representations on Hilbert spaces as described in 4.1. Thus (by the more general version above) the cohomology is trivial in these cases.

On the other hand, in Example 2.6 we will show that this method cannot work for general actions of *discrete* groups on von Neumann algebras. The construction is given for $G = F_2$ the free group of two generators and does not generalize immediately to other non-amenable groups. Hence we do not know if there are similar counter-examples (showing that $\mathcal{H}^1(l^1(G), \mathcal{B}(l^2(G)))$ is non-trivial) for all discrete non-amenable groups G. For connected and almost connected groups the situation is different, see Section 4.

For the example we use the following auxiliary result.

LEMMA 2.5. *There exists a bounded sequence* $(u_n) \subseteq l^2(\mathbb{Z})$ *whose members are pairwise orthogonal and such that* $\liminf \|u_n - w_n\|_2 > 0$ *for every bounded sequence* (w_n) *in* $VN(\mathbb{Z})$.

Proof. Recall that for a discrete group G we can identify $VN(G)$ with those sequences in $l^2(G)$ which define bounded convolution operators on $l^2(G)$. Thus the unit ball of $VN(G)$ is closed under pointwise convergence, in particular it is norm-closed in $l^2(G)$. Take $u \in l^2(\mathbb{Z})$ that does not belong to $VN(\mathbb{Z})$ (i.e., the Fourier coefficients of an unbounded L^2-function). Then $\liminf \|u - w_n'\|_2 > 0$ for every bounded sequence (w_n') in $VN(\mathbb{Z})$. If $\lim \|u - u_n'\|_2 = 0$, we conclude that $\liminf \|u_n' - w_n'\|_2 > 0$. Choose u_n' with finite support and then take $u_n = \delta_{x_n} * u_n'$, where $(x_n) \subseteq \mathbb{Z}$ are selected so that the supports of the shifted elements u_n get pairwise disjoint. If (w_n) is a bounded sequence in $VN(\mathbb{Z})$ then $w_n' = \delta_{x_n^{-1}} * w_n$ is again bounded. We have $\|u_n - w_n\|_2 = \|u_n' - w_n'\|_2$ and our claim follows. ■

EXAMPLE 2.6. Let $G = F_2$ be the free group with 2 generators a, b. Consider $\mathcal{B} = \mathcal{B}(l^2(G))$, the von Neumann algebra of all bounded linear operators on $l^2(G)$. For $x \in G, \delta_x \in l^2(G)$ denotes the corresponding unit vector and $\rho(x)u = \delta_x * u$ the left translation operator on $l^2(G)$. In this way, G and $l^1(G)$ embed into $\mathcal{B}$ and $\mathcal{B}$ becomes an $l^1(G)$-bimodule. We claim that $\mathcal{H}^1(l^1(G), \mathcal{B}(l^2(G)))$ is non-trivial. On the level of crossed homomorphisms, we consider the action $x \circ B = \rho(x)B\rho(x^{-1})$ of G by $*$-automorphisms on $\mathcal{B}$ (be aware that here $\circ$ always refers to this action) and we have to show that there exists a bounded crossed homomorphism $\Phi : G \to \mathcal{B}$ which is not principal.

Let G_a, G_b be the cyclic subgroups generated by a, b. On a free group, the values $\Phi(a), \Phi(b)$ can be prescribed arbitrarily and this will always generate a (not necessarily bounded) crossed homomorphism. If Φ is to be bounded, its restriction to the abelian (hence amenable) subgroups G_a, G_b must be principal. Thus it is no restriction to consider Φ defined by $\Phi(a) = 0,\ \Phi(b) = B - b \circ B$ for a certain $B \in \mathcal{B}$.

B is chosen as follows. Identifying $G_a = \{a^n : n \in \mathbb{Z}\}$ with $\mathbb{Z}$, take $(u_n)_{n\in\mathbb{N}} \subseteq l^2(G_a)$ as in Lemma 2.5. Put $Bv = \sum_{n=1}^{\infty} v(b^n)u_n$ for $v \in l^2(G)$. Orthogonality and boundedness of (u_n) implies that this gives a bounded linear operator. We will show that the resulting crossed homomorphism Φ is bounded and that there cannot exist $B_0 \in \mathcal{B}$ such that $\Phi(x) = B_0 - x \circ B_0$ for all $x \in \mathcal{B}$. To show boundedness of Φ, note that $\Phi(a) = 0$ implies that $\Phi(xa) = \Phi(x),\ \Phi(ax) = a \circ \Phi(x)$ for all $x \in G$. For $x = b^{n_1}a^{m_1} \dots b^{n_j}a^{m_j}$, this gives the explicit formula (assuming $j \geq 2$)

$$\begin{aligned}\Phi(x) &= \Phi(b^{n_1}a^{m_1} \dots b^{n_j}) \\ &= B - b^{n_1} \circ B + (b^{n_1}a^{m_1}) \circ (B - b^{m_2} \circ B) + \dots + (b^{n_1} \dots a^{m_{j-1}}) \circ (B - b^{n_j} \circ B) \\ &= B_1 - b^{n_1} \circ B_2\end{aligned}$$

with

$$\begin{aligned}B_1 &= B + (b^{n_1}a^{m_1}) \circ B + \dots + (b^{n_1} \dots a^{m_{j-1}}) \circ B, \\ B_2 &= B + (a^{m_1}b^{n_2}) \circ B + \dots + (a^{m_1} \dots\ b^{n_j}) \circ B.\end{aligned}$$

Observe that $\operatorname{im} B \subseteq l^2(G_a)$, $\operatorname{im} B^* \subseteq l^2(G_b)$ and $\operatorname{im} x \circ B = \delta_x * (\operatorname{im} B)$. For a reduced word beginning with $b^{\pm 1}$ (i.e., $n_i \neq 0$ for all i, $m_i \neq 0$ for $i < j$), it follows that the operators in the sum defining B_1 have pairwise orthogonal images and the same for B_1^*. This gives $\|B_1\| \leq \|B\|$ and similarly $\|B_2\| \leq \|B\|$. Consequently $\|\Phi(x)\| \leq 2\|B\|$ for all $x \in G$.

Assume that $B_0 \in \mathcal{B}$ and $\Phi(x) = B_0 - x \circ B_0$ holds for all $x \in G$. Put $v_0 = B_0(\delta_e)(\in l^2(G))$. From $\Phi(b^n) = B - b^n \circ B$ and $B(\delta_e) = 0$, we get $\Phi(b^n)(\delta_{b^n}) = B(\delta_{b^n}) = u_n$, hence $\Phi(a^m b^n)(\delta_{a^m b^n}) = (a^m \circ \Phi(b^n))(\delta_{a^m b^n}) = \delta_{a^m} * u_n$ for $n \geq 1$. On the other hand, $\Phi(a^m b^n)(\delta_{a^m b^n}) = B_0(\delta_{a^m b^n}) - \delta_{a^m b^n} * (B_0(\delta_e))$. It follows that $B_0(\delta_{a^m b^n}) = \delta_{a^m} * u_n + \delta_{a^m b^n} * v_0$ for all $n \geq 1,\ m \in \mathbb{Z}$. Let $P_a \in \mathcal{B}$ be the orthogonal projection to $l^2(G_a)$ (restriction operator) and put $v_n = -P_a(\delta_{b^n} * v_0),\ w_n = u_n - v_n$. Then $P_a B_0(\delta_{a^m b^n}) = \delta_{a^m} * w_n$. Thus (considering the convolution operator on $l^2(G_a)$ defined by w_n) $\|w_n\|_{VN(G_a)} \leq \|P_a B_0\|$ is bounded. Since $\delta_{b^{-n}} * v_n = -(b^{-n} \circ P_a)(v_0)$ and the projections $b^n \circ P_a$ (projecting onto $\delta_{b^n} * l^2(G_a)$) are pairwise orthogonal, it follows that $\sum_{n\geq 1}\|v_n\|_2^2 \leq \|v_0\|_2^2$ is finite. But $v_n = u_n - w_n$, contradicting Lemma 2.5.

3. IN-groups and unimodular groups. Recall that a locally compact group G is called an IN-group if there exists a relatively compact e-neighbourhood U such that $xUx^{-1} = U$ for all $x \in G$ (see [Pa] for more details). $SL(2,\mathbb{Z}) \rtimes T^2$ (semidirect product, where $T = \mathbb{R}/\mathbb{Z}$) is a standard example of an IN-group that is not a SIN-group.

THEOREM 3.1. *Let G be an* IN*-group. Then every bounded crossed homomorphism $\Phi : G \to VN(G)$ is principal. Thus $\mathcal{H}^1(L^1(G), VN(G)) = (0)$.*

Before giving the proof, we start with some auxiliary results. If H is a closed normal subgroup of G, put $VN(G)_H = \{T \in VN(G) : \rho(h)T = T\rho(h) \text{ for all } h \in H\}$ (relative commutant of H). As usual $\widehat{H}$ denotes the set of (equivalence classes of) irreducible unitary representations of H (following tradition, we choose a fixed representative for each class and think of $\widehat{H}$ as the set of these representatives). For $\pi \in \widehat{H}$ put $x \circ \pi(h) = \pi(x^{-1}hx)$ $(x \in G,\ h \in H)$ and $G_\pi = \{x \in G : x \circ \pi \text{ is equivalent to } \pi\}$. If $H = K$ is compact and $\pi \in \widehat{K}$, let $P_\pi \in VN(K)$ be the corresponding central projection.

LEMMA 3.2. *If K is a compact normal subgroup of G, $\pi \in \widehat{K}$, $T \in VN(G)_K$, then $P_\pi T = TP_\pi$, $\operatorname{supp} TP_\pi \subseteq G_\pi$ and $TP_\pi \in VN(G_\pi/\ker \pi)$.*

Proof. Let $\chi_\pi(k) = \operatorname{tr} \pi(k)$ be the character of π, $d_\pi = \dim \pi$. We have $P_\pi g = d_\pi \bar{\chi}_\pi * g$ for $g \in L^2(K)$. The same formula with $g \in L^2(G)$ defines an element of $VN(G)$ which will also be denoted by P_π (this is the standard embedding of $L^1(K)$ into $M(G)$ which extends to an embedding of $VN(K)$ into $VN(G)$). Since χ_π is central in $L^1(K)$, we even have $P_\pi \in VN(G)_K$. If $T \in VN(G)_K$, then it commutes with all operators $\rho(x)$ $(x \in K)$ and by standard arguments it commutes with the operators from $L^1(K)$ (even with those of $VN(K)$).

The action $x \mapsto x \circ \pi$ is continuous (using e.g. [Di] Prop. 18.1.5). Since $\widehat{K}$ is discrete ([Di] Cor. 18.4.3), it follows that G_π is open in G. Let $R_\pi \subseteq G$ be a set of representatives for the right G_π-cosets. Then every $T \in VN(G)$ can be written uniquely as a sum $T = \sum_{x \in R_\pi} T_x \, \rho(x)$ (strongly converging on $L^2(K)$ and its translates) with some $T_x \in VN(G_\pi)$ (compare [E] Prop. 3.21). If $T \in VN(G)_K$, then for $k \in K$, $\rho(k)T = T\rho(k)$ implies $\rho(k)T_x = T_x\rho(xkx^{-1})$, in other words, T_x intertwines the restrictions of the representations ρ and $x^{-1} \circ \rho$ to K. We will now show that $\operatorname{supp}(TP_\pi) \subseteq G_\pi$ for $T \in VN(G)_K$ (using the notion of support from [E] Def. 4.5).

Since $P_\pi \in VN(G)_K$, we can assume that $T = TP_\pi$. Decompose T as above, then $T_x P_\pi = T_x$ for all $x \in R_\pi$. The restriction of ρ (as a representation of K) to $P_\pi(L^2(G))$ is a multiple of π ([Di] Th. 15.3.12; the isotypic component of π). Now fix some $x \in R_\pi$ such that $T_x \neq 0$. $\rho(k)$ is a multiple of $\pi(k)$ on $P_\pi(L^2(G))$, hence $\rho(xkx^{-1})$ is a multiple of $x^{-1} \circ \pi(k)$. Existence of a non-zero intertwining operator T_x implies that $x^{-1} \circ \pi$ is equivalent to π ([Di] Prop. 5.2.1), hence $x \in G_\pi$.

For the last statement, we use the standard identifications (see [E]): Since G_π is open in G, $VN(G_\pi)$ can be identified with $\{T \in VN(G) : \operatorname{supp} T \subseteq G_\pi\}$ and if $\lambda_{\ker \pi}$ denotes the normalized Haar measure of $\ker \pi$, $VN(G_\pi/\ker \pi)$ is identified with $\{T \in VN(G_\pi) : T\rho(\lambda_{\ker \pi}) = T\}$ ($x \circ \pi = \pi$ implies that $x(\ker \pi)x^{-1} = \ker \pi$, i.e. $\ker \pi$ is normal in G_π). Then for $T \in VN(G_\pi)$, we get $TP_\pi \in VN(G_\pi)$ and since $\chi_\pi(kk') = \chi_\pi(k)$ for $k \in K$, $k' \in \ker \pi$, it follows that $P_\pi \rho(\lambda_{\ker \pi}) = P_\pi$, hence $TP_\pi \in VN(G_\pi/\ker \pi)$. ∎

Next we consider a special case that can be reduced to SIN-groups.

Lemma 3.3. *Assume that G has an open compact normal subgroup K. Then for $\pi \in \widehat{K}$ we have that $G_\pi/\ker\pi$ is a* SIN*-group.*

Proof. Since $\chi_\pi(xkx^{-1}) = \chi_\pi(k)$ for $x \in G_\pi$, $k \in K$ and K is open, every set $\{k \in K : |\chi_\pi(k) - \chi_\pi(e)| < \epsilon\}$, where $\epsilon > 0$, is a G_π-invariant, $\ker\pi$-periodic e-neighbourhood in G_π. Recall that $\ker\pi = \{k \in K : \chi_\pi(k) = \chi_\pi(e)\}$, thus the intersection of these neighbourhoods is $\ker\pi$ and it follows that $G_\pi/\ker\pi$ is a SIN-group. ∎

Unfortunately, this reduction to SIN-groups does not work for general IN-groups. To apply the technique based on weak almost periodicity, we will use an extension procedure, formulated for general W^*-algebras.

Lemma 3.4. *Let $\mathcal{M}$ be a W^*-algebra, $\mathcal{Z}$ a W^*-subalgebra contained in the centre of $\mathcal{M}$. Let σ be an action of a (discrete) group Γ by $*$-automorphisms on $\mathcal{M}$ such that $\mathcal{Z}$ is Γ-invariant.*

$\Gamma_1 = [\Gamma]$ *shall denote the* "full group" *for this action on $\mathcal{Z}$ (see below). Then we can extend the action of Γ on $\mathcal{M}$ to an action of Γ_1.*

If $\Phi : \Gamma \to \mathcal{M}$ is a bounded crossed homomorphism, there exists a bounded crossed homomorphism $\Phi_1 : \Gamma_1 \to \mathcal{M}$ extending Φ and satisfying $\|\Phi_1\| = \|\Phi\|$.

Proof. We consider the discrete crossed product $\mathcal{Z} \underset{\sigma}{\otimes} \Gamma$ (see [Ta] Sec. V.7, [Str] § 22, who use the notation $\mathcal{R}(\mathcal{Z},\sigma)$). Γ and $\mathcal{Z}$ will be considered as subsets and a general element of $\mathcal{Z} \underset{\sigma}{\otimes} \Gamma$ is written as a countable sum $\sum_{i=1}^\infty T_i\gamma_i$ with $T_i \in \mathcal{Z},\ \gamma_i \in \Gamma$ (the representation being unique if γ_i are pairwise different, $T_i \neq 0$). If $\gamma_i \in \Gamma$ and $P_i \in \mathcal{Z}$ are projections such that $\sum_{i=1}^\infty P_i = 1,\ \sum_{i=1}^\infty \sigma(\gamma_i^{-1})(P_i) = 1$ it is easy to see that $u = \sum_{i=1}^\infty P_i\gamma_i$ (the sum being strongly convergent in this case) defines a unitary element of $\mathcal{Z} \underset{\sigma}{\otimes} \Gamma$. We define the *full group* Γ_1 to be the set of all these unitaries (which is easily seen to form a subgroup). Clearly $\Gamma \subseteq \Gamma_1$. The corresponding inner automorphism (restricted to $\mathcal{M}$) is given by

$$\sigma(u)(T) = uTu^* = \sum_{i=1}^\infty \sigma(\gamma_i)(T)\, P_i$$

(this uses the embedding of $\mathcal{Z} \underset{\sigma}{\otimes} \Gamma$ into $\mathcal{M} \underset{\sigma}{\otimes} \Gamma$, recall that $\mathcal{Z}$ is central in $\mathcal{M}$). Let $\sigma_{\mathcal{Z}}(u)$ be the restriction of $\sigma(u)$ to $\mathcal{Z}$ (which is obviously invariant), then $\sigma_{\mathcal{Z}}(\Gamma_1)$ gives the full group as defined in [N] (who considers only point transformations on σ-finite measure spaces, generalizing the earlier notion of Dye [Dy] for finite measure spaces; see also [Str] 17.3 for general W^*-algebras). If the action of Γ on $\mathcal{Z}$ by σ is properly outer and σ is injective (on Γ), then Γ_1 coincides with the set of unitary elements $u \in \mathcal{Z} \underset{\sigma}{\otimes} \Gamma$ such that $u\mathcal{Z}u^* \subseteq \mathcal{Z}$ ([Str] p. 356).

Let $\Phi : \Gamma \to \mathcal{M}$ be a bounded crossed homomorphism. For $u \in \Gamma_1$ (represented as above) we define $\Phi_1(u) = \sum_{i=1}^\infty \Phi(\gamma_i)\, P_i$. Since P_i are pairwise orthogonal and central this defines an element of $\mathcal{M}$, $\|\Phi_1(u)\| \le \sup_{\gamma\in\Gamma} \|\Phi(\gamma)\|$ and it is a routine matter to verify that $\Phi_1 : \Gamma_1 \to \mathcal{M}$ is a crossed homomorphism. ∎

LEMMA 3.5. *Let σ be an action of a group Γ by $*$-automorphisms on a W^*-algebra $\mathcal{M}$. Let $\Phi : \Gamma \to \mathcal{M}$ be a bounded crossed homomorphism. If $\gamma \in \Gamma$ and $P \in \mathcal{M}$ is a central (orthogonal) projection such that $\sigma(\gamma)$ is the identity on $\mathcal{M}P$, then $\Phi(\gamma)P = 0$.*

Proof. The assumption means that $\sigma(\gamma)(TP) = TP = \sigma(\gamma)(T)\,P$ for all $T \in \mathcal{M}$. Induction gives $\Phi(\gamma^n)\,P = n\,\Phi(\gamma)\,P$ for all $n \in \mathbb{N}$. Thus, boundedness of Φ implies $\Phi(\gamma)P = 0$. ∎

PROPOSITION 3.6. *Let $\mathcal{M}$, $\mathcal{Z}$, Γ, σ be as in Lemma 3.4. Assume that there exists a semifinite, Γ-invariant faithful normal trace on $\mathcal{Z}$ and a Γ-invariant faithful normal conditional expectation $E : \mathcal{M} \to \mathcal{Z}$.*

Then every bounded crossed homomorphism $\Phi : \Gamma \to \mathcal{M}$ is principal. There exists $T \in \mathcal{M}$ with $\|T\| \le \|\Phi\|$ such that $\Phi(\gamma) = T - \sigma(\gamma)(T)$ for all $\gamma \in \Gamma$.

Proof. The assumption on $\mathcal{Z}$ means that $\mathcal{Z} \cong L^\infty(\Omega, m)$ where m is a Radon measure on the locally compact space Ω and, if m is σ-finite, each automorphism $\sigma(\gamma)$ is defined by a bimeasurable transformation preserving m (see [Ta] p. 329 and L. IV. 8.22). If the measure m is finite, it defines a Γ-invariant faithful normal state of $\mathcal{Z}$. Composition with E gives a Γ-invariant faithful normal state of $\mathcal{M}$. Thus $\mathcal{M}$ is Γ-finite and the result follows from Theorem 2.2 and Remark 2.4.

In the general case take a "finite" projection $P \in \mathcal{Z}$ (corresponding to a measurable set $A \subseteq \Omega$ with $0 < m(A) < \infty$). Using Lemma 3.4, we consider the extension Φ_1 to the full group $\Gamma_1 = [\Gamma]$. Let $[\Gamma]_P$ be the subgroup of all u such that $u(1-P) = 1-P$ (i.e., $u = \sum_{i=1}^\infty P_i\gamma_i + 1 - P$ with projections $P_i \in \mathcal{Z}$ satisfying $\sum_{i=1}^\infty P_i = \sum_{i=1}^\infty \sigma(\gamma_i^{-1})(P_i) = P$). Then $\sigma_{\mathcal{Z}}([\Gamma]_P)$ consists of all transformations in $\sigma_{\mathcal{Z}}(\Gamma_1)$ that act as the identity on $\mathcal{Z}(1-P)$ ([N] p. 406). Then $\mathcal{M}P$ is $[\Gamma]_P$-finite, hence there exists $T_P \in \mathcal{M}P$ such that $\Phi_1(u) = T_P - \sigma(u)(T_P)$ for all $u \in [\Gamma]_P$. Since $\|T_P\| \le \|\Phi\|$ is uniformly bounded, there exists a w^*-cluster point T of the net (T_P) arising from the "finite" projections in $\mathcal{Z}$.

In [N] Γ-equivalence $\underset{\Gamma}{\sim}$ of projections in $\mathcal{Z}$ is considered (defined by partial isometries in $\mathcal{Z} \underset{\sigma}{\otimes} \Gamma$ of a similar type as in the proof of Lemma 3.4). In [St2] this is called Hopf equivalence (using the notation $\underset{H}{\sim}$) and it is shown (when $\mathcal{Z}$ is σ-finite, [St2] L. 1 and Th. 5) that this coincides with standard Murray–von Neumann equivalence (denoted by $\sim$) of projections in the W^*-algebra $\mathcal{Z} \underset{\sigma}{\otimes} \Gamma$.

Now fix $\gamma \in \Gamma$. Let $P \in \mathcal{Z}$ be a "finite" projection and put $P' = \sigma(\gamma^{-1})(P)$. Clearly $P \underset{\Gamma}{\sim} P'$. If Q is a "finite" projection such that $P, P' \le Q$, then by [N] L. 3, $P \underset{[\Gamma]_Q}{\sim} P'$, thus (by [St2]) $P \sim P'$ in $\mathcal{Z}Q \underset{\sigma}{\otimes} [\Gamma]_Q$ and by [Ta] Prop. V.1.38, $Q - P \sim Q - P'$ in $\mathcal{Z}Q \underset{\sigma}{\otimes} [\Gamma]_Q$. Going back, this implies $Q - P \underset{[\Gamma]_Q}{\sim} Q - P'$. In combination, it follows that there exists $u \in [\Gamma]_Q$ such that $\sigma(u)(P') = P$. Put $u' = u^{-1}\gamma$, then $\sigma(u')(P') = P'$. If $u' = \sum_{i=1}^\infty P_i\gamma_i + 1 - Q$ (with projections $P_i \in \mathcal{Z}$ satisfying $\sum P_i = \sum \sigma(\gamma_i^{-1})(P_i) = Q$), we get $P' = \sigma(u')(P') = \sum \sigma(\gamma_i)(P')P_i$, hence $P'P_i = \sigma(\gamma_i)(P')P_i$ for all i. Put $u'' = \sum_{i=1}^\infty P_iP'\gamma_i + (1-P')$. Then $u'' \in [\Gamma]_{P'} \subseteq [\Gamma]_Q$ and $\sigma(u'') = \sigma(u')$ on $\mathcal{M}P'$ (we have $\sigma(u'')(T) = \sum \sigma(\gamma_i)(T)\,P_iP' + T(1-P')$ for $T \in \mathcal{M}$). Thus $\gamma = uu'$ implies $\sigma(uu'') = \sigma(\gamma)$ on $\mathcal{M}P'$. By Lemma 3.5, $\Phi_1(\gamma^{-1}uu'')\,P' = 0$ and (recall that $\Phi(\gamma^{-1}) = -\sigma(\gamma^{-1})(\Phi(\gamma))$)

we get $\sigma(\gamma^{-1})(\Phi(\gamma)P) = \sigma(\gamma^{-1})(\Phi_1(uu''))\, P'$. Hence $\Phi(\gamma)P = \Phi_1(uu'')P = (T_Q - \sigma(uu'')(T_Q))P = T_Q P - \sigma(uu'')(T_Q P') = (T_Q - \sigma(\gamma)(T_Q))P$. In the limit, this gives $\Phi(\gamma)P = (T - \sigma(\gamma)(T))P$ for all "finite" projections, finishing the proof. ■

Proposition 3.7. *Assume that G has an open normal subgroup H such that H is unimodular. If $\Phi : G \to VN(G)$ is a bounded crossed homomorphism whose restriction to H is principal, then Φ is principal.*

In particular, when the cohomology group $\mathcal{H}^1(L^1(H), VN(G))$ is trivial, it follows that $\mathcal{H}^1(L^1(G), VN(G))$ is trivial.

Proof. Let $\Phi : G \to VN(G)$ be a bounded crossed homomorphism. By assumption, its restriction to H is principal, hence (subtracting a principal homomorphism) we can assume that $\Phi(h) = 0$ for $h \in H$. Then for $x \in G$, it follows that $\Phi(x) \in VN(G)_H$ and $\Phi(x)$ depends only on the coset xH. Put $\Gamma = G/H$. Thus it suffices to consider a bounded crossed homomorphism $\Phi : \Gamma \to VN(G)_H$. Since H is open and normal, the restriction of $T \in VN(G)$ to H defines a conditional expectation $E : VN(G) \to VN(H)$ which is easily seen to be w^*-continuous ([E] Prop. 3.21; see also his definition of $T|H$; writing $T = \sum_{x\in R} T_x\, \rho(x)$ as in the proof of our Lemma 3.2, where R is a set of representatives for the H-cosets with $e \in R$, one has $E(T) = T|H = T_e$). Furthermore, it is faithful and G-invariant (in particular, $VN(G)_H$ is kept invariant). Put $\mathcal{Z} = \mathcal{Z}(VN(H)) = VN(G)_H \cap VN(H)$, the centre of $VN(H)$.

For f a continuous positive definite function on H with compact support (or more generally, $f \in L^1(H) \cap A(H)$), we have the functional $f \mapsto f(e)$. By a theorem of Godement ([Di] 17.2.5, requiring unimodularity), this extends to a faithful semifinite normal trace on $VN(H)$. If G is unimodular, this is also G-invariant. The restriction of this trace to $\mathcal{Z}$ is again semifinite (e.g. by [Ta] Th. 4.6 and Exerc. 1 p. 332). Hence we can apply Proposition 3.6.

In the general case, let G_Δ be the kernel of the modular function. Openness of H implies $H \subseteq G_\Delta$ and G_Δ is open. Then it follows that G_Δ is unimodular and the argument above shows that Φ is principal on G_Δ. As above, we arrive at a bounded crossed homomorphism defined on G/G_Δ. Since G/G_Δ is abelian, this is principal by Johnson's theorem. ■

Corollary 3.8. *Assume that the locally compact group G has an open normal subgroup H which is amenable and unimodular. Then $\mathcal{H}^1(L^1(G), VN(G)) = (0)$.*

Proof. By Johnson's theorem, $\mathcal{H}^1(L^1(H), VN(G)) = (0)$, thus Proposition 3.7 applies. ■

Proof of Theorem 3.1. By the structure theorem ([GM] Th. 2.13), G has an open normal subgroup H such that H is an extension of a compact group K by a vector group. If $\Phi : G \to VN(G)$ is a bounded crossed homomorphism, we can apply Remark 2.4 to its restriction to K. There remains a bounded crossed homomorphism $G/K \to VN(G)_K$ (K is characteristic in H hence normal in G). H/K being abelian, this part can be handled by Johnson's theorem. The remaining part $G/H \to VN(G)_H$ is covered by Proposition 3.7.

If Φ is known to be continuous, one can apply directly Corollary 3.8, since IN-groups are unimodular ([Pa] p. 718) and H is amenable. ■

Remark 3.9. For groups having an open compact normal subgroup K, Theorem 3.1 can be proved in a more elementary manner using Lemma 3.3 and Corollary 2.3 for SIN-groups. As above, it is enough to consider a bounded crossed homomorphism $\Phi' : G/K \to VN(G)_K$. Put $\Phi_\pi(x \ker \pi) = \Phi'(xK)P_\pi$. Then (using Lemma 3.2) $\Phi_\pi : G_\pi/\ker\pi \to VN(G_\pi/\ker\pi)$ is a bounded crossed homomorphism and, by Lemma 3.3 and Corollary 2.3 it is principal. Since $\Phi_\pi(G_\pi/\ker\pi) \subseteq VN(G)_K P_\pi$ we get (by Theorem 2.2) $A_\pi \in VN(G)_K P_\pi$ such that $\Phi'(xK)P_\pi = A_\pi - x \circ A_\pi$ for $x \in G_\pi$ and $\|A_\pi\| \le \|\Phi'\|$. $L^2(G)$ is the l^2-sum of the subspaces $P_\pi(L^2(G))$ where $\pi \in \widehat{K}$. Then we extend this, defining an operator $\tilde{A}_\pi$ on all $P_{u\circ\pi}(L^2(G))$ $(u \in G)$ by $\tilde{A}_\pi = u \circ (A_\pi - \Phi'(u^{-1}K))$. By some computations one can verify that this depends only on the coset uG_π. Doing the same for all G-orbits in $\widehat{K}$, this combines to an operator $A \in VN(G)_K$ which satisfies (again after some computations) $\Phi'(xK) = A - x \circ A$ for all $x \in G$.

We illustrate in the case of an open compact normal subgroup the more abstract technique applied in Lemma 3.4. Take $\mathcal{M} = VN(G)_K$, $\mathcal{Z} = VN(G)_K \cap VN(K) = \mathcal{Z}(VN(K))$ (centre), $\Gamma = G/K$. Then $\mathcal{Z} \cong l^\infty(\widehat{K})$ with the action of Γ as defined at the beginning of this section. The image of the full group $\sigma_{\mathcal{Z}}(\Gamma_1)$ (given by permutations of $\widehat{K}$) contains all permutations of $\widehat{K}$ that respect the Γ-orbits and act trivially outside some countable set. Observe that σ need not be injective on Γ (just consider the case of a direct product $K \times \Gamma$) and the action of Γ need not be free (which is a frequent assumption in ergodic theory).

An easy example of an IN-group having no open compact normal subgroup is a semidirect product $G = \mathbb{R}^n \rtimes \Gamma$ where Γ is some group of orthogonal $n\times n$-matrices (with discrete topology). This is not amenable in general. For $H = \mathbb{R}^n$, the trace on $VN(H)$ considered in the proof of Proposition 3.7 is given by Haar measure m on the dual group $\widehat{H}$ (recall that $VN(H) \cong L^\infty(\widehat{H})$). When Γ is countable, the full group consists of all bimeasurable transformations on $\widehat{H}$ that are bijective outside a set of m-measure 0, keep m invariant and such that the orbits Γx (dual action) are invariant for almost all $x \in \widehat{H}$ ([N] p. 399). Proposition 3.7 and Corollary 3.8 apply more generally when Γ is some group of matrices of determinant 1. In this case, G is not an IN-group unless the closure of Γ in $SL(n,\mathbb{R})$ is compact.

Some results about derivations $L^1(G) \to VN(G)$ were proved in [GRW] sec. 3 and 4. They concentrated on the case where the image is contained in $L^1(G)$ and found a condition for the extendability to $VN(G)$ ([GRW] Th. 3.6) which was shown to be satisfied in the case of IN-groups and groups having open normal amenable subgroups ([GRW] Th. 4.2). A similar condition could be formulated for general bounded derivations $L^1(G) \to VN(G)$ but the proof in [GRW] Prop. 3.5 used some interpolation technique which does not seem to be available in the general case.

4. Connected groups

4.1. In this section, we consider group actions on von Neumann algebras that are implemented by some unitary representation. G shall be a locally compact group, $\pi : G \to \mathcal{B}(\mathcal{H})$ a strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}$. Let $\mathcal{B}$ be a von Neumann algebra on $\mathcal{H}$ that is invariant under the automorphisms induced by π, i.e. $\pi(x)\,\mathcal{B}\,\pi(x^{-1}) \subseteq \mathcal{B}$ for all $x \in G$. Then $x \circ T = \pi(x)\,T\,\pi(x^{-1})$ defines an action of G on $\mathcal{B}$. Examples are $\mathcal{B} = VN(G)$ or $\mathcal{B} = \mathcal{B}(L^2(G))$ with $\mathcal{H} = L^2(G)$ and the left regular representation of G. We concentrate on the case where G is locally isomorphic to $SL(2,\mathbb{R})$. Proposition 4.3 will provide a method to compute the cohomology groups rather explicitly. This is used in Example 4.5 for irreducible representations and in Example 4.6 for the regular representation. It turns out that most of these cohomology groups are non-trivial.

Now assume that G is a connected Lie group with Lie algebra $\mathfrak{g}$. From the general representation theory of Lie groups, recall that given π (as above), there exists a dense subspace $\mathcal{H}^\infty$ of $\mathcal{H}$ (the C^∞-vectors) and a representation $d\pi$ of $\mathfrak{g}$ by operators on $\mathcal{H}^\infty$ (Gårding's theorem, [Wa] Sec. 4.4.1, who uses the notation U_∞). $d\pi(u)h$ is given as the derivative of $\pi(\exp(tu))h$ at $t = 0$ $(u \in \mathfrak{g},\ h \in \mathcal{H}^\infty)$. $\mathcal{H}^\infty$ is a locally convex Fréchet space and the linear operators $\pi(x)$ $(x \in G)$ and $d\pi(u)$ $(u \in \mathfrak{g})$ are continuous on $\mathcal{H}^\infty$. Since $\mathcal{H}^\infty$ is dense in $\mathcal{H}$, we can embed $\mathcal{H}$ into the dual space $\mathcal{H}^{\infty\prime}$ (strictly speaking, using the Riesz representation theorem, one has to use the complex-conjugate space of the dual) and (using that π is unitary) there are extensions of π and $d\pi$ to $\mathcal{H}^{\infty\prime}$. For $\mathcal{B} = \mathcal{B}(\mathcal{H})$ let $\mathcal{B}^\infty$ be the space of continuous linear mappings $\mathcal{H}^\infty \to \mathcal{H}^{\infty\prime}$ (alternatively, continuous sequilinear forms on $\mathcal{H}^\infty \times \mathcal{H}^\infty$). Then $\mathcal{B} \subseteq \mathcal{B}^\infty$ and we have an action of the Lie algebra $\mathfrak{g}$ on $\mathcal{B}^\infty$ by $u \circ T = [d\pi(u), T] = d\pi(u)\,T - T\,d\pi(u)$ for $u \in \mathfrak{g},\ T \in \mathcal{B}^\infty$ (induced by the action of G on $\mathcal{B}$ in 4.1).

Lemma 4.2. *Let $\Phi : G \to \mathcal{B}(\mathcal{H})$ be a bounded crossed homomorphism. For $u \in \mathfrak{g}$, $h_1, h_2 \in \mathcal{H}^\infty$ let $(d\Phi(u)h_1|\, h_2)$ be the derivative of $(\Phi(\exp(tu))h_1 \,|\, h_2)$ at $t = 0$.*

Then $d\Phi : \mathfrak{g} \to \mathcal{B}^\infty$ is linear and satisfies $d\Phi([u,v]) = u \circ d\Phi(v) - v \circ d\Phi(u)$ for $u, v \in \mathfrak{g}$. If $C \in \mathcal{B}(\mathcal{H})$ and $\Phi(x) = C - x \circ C$ for all $x \in G$, then $d\Phi(u) = -u \circ C$ for all $u \in \mathfrak{g}$.

$(\ |\ )$ denotes the inner product of $\mathcal{H}$.

Proof. First, we consider the one-parameter subgroup $\{\exp(tu) : t \in \mathbb{R}\}$. Applying Johnson's theorem (for a discrete abelian group), there exists $C_u \in \mathcal{B}(\mathcal{H})$ such that $\Phi(\exp(tu)) = C_u - \exp(tu) \circ C_u$ for all $t \in \mathbb{R}$. Then existence of the derivative (defining an element of $\mathcal{B}^\infty$) and $d\Phi(u) = -u \circ C_u$ follow easily.

Furthermore, from $\Phi(xx_1x^{-1}) = \Phi(x) + x \circ \Phi(x_1) - (xx_1x^{-1}) \circ \Phi(x)$, we get (for $x_1 = \exp(tv)$, $v \in \mathfrak{g}$; using [Var] (2.13.7) and the product rule) $d\Phi(\operatorname{ad} x(v)) = x \circ d\Phi(v) - (\operatorname{ad} x\,(v)) \circ \Phi(x)$. Then for $x = \exp(t\,u)$, using that $\Phi(e) = 0$ and [Var] Th. 2.13.2, it follows that $d\Phi(\,[u,v]) = u \circ d\Phi(v) - v \circ d\Phi(u)$. ∎

If G is a (connected) semisimple Lie group, we will use the Iwasawa decomposition $G = KAN$ (see [He] Sec. VI.5), where $Z(G) \subseteq K$, $K/Z(G)$ is compact and $S = AN$ is solvable. We will concentrate on groups G that are locally isomorphic to $SL(2,\mathbb{R})$, the group of real 2×2-matrices of determinant 1. We use the notation of [HT] p. 51. We have $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ (the real 2×2-matrices with trace 0) and write $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e^+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),\ e^- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right),\ k = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),\ n^+ = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right),\ n^- = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$. h, e^+ span the Lie algebra $\mathfrak{s}$ of AN, k that of K.

PROPOSITION 4.3. *Let G be a connected semisimple Lie group locally isomorphic to $SL(2,\mathbb{R})$, let $\pi, \mathcal{H}$ be given as in 4.1. Take $T \in \mathcal{B}(\mathcal{H})$ and put $\Phi(x) = T - x \circ T$ for $x \in K$. Then the following statements are equivalent:*

(i) *Φ extends to a crossed homomorphism on G, satisfying $\Phi(x) = 0$ for $x \in AN$.*
(ii) *$T_1 = k \circ T$ ($\in \mathcal{B}^\infty$) satisfies $e^+ \circ T_1 = 0$, $h \circ T_1 = -2T_1$.*

Proof. Assume that (i) holds (the extension is again denoted by Φ). We use the properties of $d\Phi$ in Lemma 4.2 and $[k, e^+] = -h \in \mathfrak{s}$, $e^+ \in \mathfrak{s}$. Then $e^+ \circ T_1 = e^+ \circ (k \circ T) = -e^+ \circ d\Phi(k) = d\Phi([k, e^+]) - k \circ d\Phi(e^+) = 0$.

Furthermore, $[k, h] = 4e^+ + 2k$, giving $h \circ T_1 = -h \circ d\Phi(k) = d\Phi([k, h]) - k \circ d\Phi(h) = 2d\Phi(k) = -2T_1$.

For the converse, we first extend $d\Phi$ to a linear mapping $\mathfrak{g} \to \mathcal{B}^\infty$ by taking $d\Phi(u) = 0$ for $u \in \mathfrak{s}$ (recall that by Lemma 4.2 $d\Phi(k) = -k \circ T$). Considering the defining relations of the Lie algebra $\mathfrak{g}$, one can do similar computations as above and the assumptions on T in (ii) turn out to be equivalent to the functional equation $d\Phi([u, v]) = u \circ d\Phi(v) - v \circ d\Phi(u)$ ($u, v \in \mathfrak{g}$) of Lemma 4.2. For $u \in \mathfrak{g}$, $h_1, h_2 \in \mathcal{H}^\infty$, the function $x \to (x \circ d\Phi(u)\, h_1 \,|\, h_2)$ is C^∞ on G (see [Wa] 4.4.1). Let $\widetilde{G}$ be the universal covering group of G. It follows from classical results of calculus on manifolds (see [Si] Prop. 4.1 for details) that there exists a unique function $\widetilde{\Phi} : \widetilde{G} \to \mathcal{B}^\infty$ with the properties: (†) $\widetilde{\Phi}(e) = 0$ and (‡) $(\tilde{x} \circ d\Phi(u)\, h_1 | h_2)$ is the derivative of $(\widetilde{\Phi}(\tilde{x} \exp(tu))\, h_1 \,|\, h_2)$ at $t = 0$ for $h_1, h_2 \in \mathcal{H}^\infty$, $u \in \mathfrak{g}$, $\tilde{x} \in \widetilde{G}$ (the action of $\widetilde{G}$ is the canonical lifting of the action of G). (†), (‡) and the functional equation for $d\Phi$ imply that $\widetilde{\Phi}$ is a crossed homomorphism ("exponentiation of $d\Phi$", see again [Si]). The Iwasawa decomposition of $\widetilde{G}$ is given by (or isomorphic to) $\widetilde{K}AN$ where $\widetilde{K}$ is the universal covering group of K (in our case, K is one-dimensional, hence $\widetilde{K} \cong \mathbb{R}$; for $G = SL(2,\mathbb{R})$, K is the subgroup of rotations). Let $q : \widetilde{G} \to G$ be the canonical projection. It follows from uniqueness of the solution (restricted to $\widetilde{K}$) that $\widetilde{\Phi} = \Phi \circ q$ holds on $\widetilde{K}$. This implies that $\widetilde{\Phi}(\tilde{x}) = 0$ for $\tilde{x} \in \ker q$, hence $\widetilde{\Phi}$ induces a crossed homomorphism Φ on $\widetilde{G}/\ker q \cong G$, extending the mapping Φ given on K. Furthermore, the differential equation for $\widetilde{\Phi}$ implies that $\Phi(x) = 0$ for $x \in S$ (since $d\Phi = 0$ on $\mathfrak{s}$). ∎

$\mathfrak{g}$ is a real Lie algebra, but the actions on $\mathcal{H}^\infty$ and $\mathcal{B}^\infty$ extend immediately to the complexification $\mathfrak{g}_{\mathbb{C}}$. Then, using $n^\pm = \frac{1}{2}(h \pm \mathtt{i}k) \pm \mathtt{i}e^+$, we get an equivalent condition to (ii) which will be easier to use in examples.

$$n^+ \circ T_1 = -T_1 + \frac{1}{2}\, k \circ T_1, \quad n^- \circ T_1 = -T_1 - \frac{1}{2}\, k \circ T_1. \tag{*}$$

Now we can use this to describe the cohomology groups. For $\mathcal{B}$ as in 4.1, let $\mathcal{Z}_S(\mathcal{B}, \pi)$ be the space of operators $T \in \mathcal{B}$ satisfying the properties (i),(ii) above. $\mathcal{B}_K$ denotes the relative commutant, i.e. those $T \in \mathcal{B}$ for which $k \circ T = 0$, similarly with $\mathcal{B}_S$. Clearly, $\mathcal{B}_K, \mathcal{B}_S \subseteq \mathcal{Z}_S$. Since K and S are amenable, every bounded crossed homomorphism $G \to \mathcal{B}$ is cohomologous to one as in Prop. 4.3, defined by some $T \in \mathcal{Z}_S(\mathcal{B}, \pi)$. This gives a surjective linear mapping $p : \mathcal{Z}_S(\mathcal{B}, \pi) \to \mathcal{H}^1(L^1(G), \mathcal{B})$. The crossed homomorphism defined by T is zero iff $T \in \mathcal{B}_K$ and principal iff $T \in \mathcal{B}_K + \mathcal{B}_S$. Thus $\mathcal{H}^1(L^1(G), \mathcal{B})$ is isomorphic to $\mathcal{Z}_S/(\mathcal{B}_K + \mathcal{B}_S)$.

Below we will compute explicitly a number of cohomology groups. Before, we prove a technical result.

LEMMA 4.4. *If $\nu = a + \mathtt{i}b \notin \mathbb{R}$ is fixed, $a \geq 0$, then $\Gamma(j+\nu)/\Gamma(j+\overline{\nu}) = j^{2\mathtt{i}b} + O(1/j)$ uniformly for $j > 0$.*

Proof. By [Ol] (5.05), p. 119 (see also the extensions made at the bottom of that page) $\Gamma(z+\alpha)/\Gamma(z+\beta) = z^{\alpha-\beta} + \int_0^\infty e^{-zt}\,\phi_1(t)\,dt$ when $\operatorname{Re}(z), \operatorname{Re}(z+\alpha) > 0, \operatorname{Re}(\beta-\alpha) > -1$. Here $\phi_1(t)$ is the remainder arising in the expansion $q(t) = e^{-\alpha t}(1-e^{-t})^{\beta-\alpha-1} = q_0(\alpha,\beta)\,t^{\beta-\alpha-1} + \phi_1(t)$. Thus

$$\phi_1(t) = t^{\beta-\alpha-1}\biggl(e^{-\alpha t}\Bigl(\frac{1-e^{-t}}{t}\Bigr)^{\beta-\alpha-1} - 1\biggr).$$

It does not depend on z. For fixed α,β, it is continuous on $]0,\infty[$ and satisfies $\phi_1(t) = O(t^{\operatorname{Re}(\beta-\alpha)})$ for $t \to 0$ and $\phi_1(t) = O(e^{-t\,\operatorname{Re}(\alpha)} + t^{\operatorname{Re}(\beta-\alpha-1)})$ for $t \to \infty$. In particular, when $\operatorname{Re}(\alpha) = \operatorname{Re}(\beta) \geq 0$, we get that ϕ_1 is bounded and our claim follows easily. ∎

EXAMPLE 4.5. Take $G = SL(2,\mathbb{R})^{\sim}$, the universal covering group of $SL(2,\mathbb{R})$ and π irreducible, $\mathcal{B} = \mathcal{B}(\mathcal{H})$. For the trivial representation, obviously $\mathcal{Z}_S = \mathcal{B}_S = \mathcal{B}_K$, consequently $\mathcal{H}^1$ is trivial. For the (irreducible) principal series representations (see below) we will show that $\mathcal{Z}_S/\mathcal{B}_K$ is two-dimensional, $(\mathcal{B}_K + \mathcal{B}_S)/\mathcal{B}_K$ is one-dimensional, hence $\mathcal{H}^1(L^1(G),\mathcal{B})$ has dimension one. For the complementary series one can show that $\mathcal{Z}_S = \mathcal{B}_K + \mathcal{B}_S$ and $\mathcal{Z}_S/\mathcal{B}_K$ has dimension one. In particular, $\mathcal{H}^1(L^1(G),\mathcal{B})$ is trivial. For the "discrete series", one gets in the same way $\mathcal{Z}_S = \mathcal{B}_K$, thus $\mathcal{H}^1(L^1(G),\mathcal{B})$ is again trivial. This remains even true for the representations $\pi \oplus \overline{\pi}$, when π is a discrete series representation (of $SL(2,\mathbb{R})$ or some finite covering). As an exceptional case, for the mock discrete series $\pi^{\pm}_{md}$ (of $SL(2,\mathbb{R})$), one still gets $\mathcal{Z}_S = \mathcal{B}_K$, but for $\pi^+_{md} \oplus \pi^-_{md}$ (the reducible principal series representation), $\mathcal{Z}_S/\mathcal{B}_K$ is two-dimensional and $\mathcal{B}_S \subseteq \mathcal{B}_K$, so $\mathcal{H}^1(L^1(G),\mathcal{B})$ has dimension two.

The irreducible unitary representations of G have been described in [Pu]. By the argument of [Pu] p. 99, $d\pi(k)$ ($= 2\,H_0$ in the notation of [Pu]) has a complete system of eigenvectors (mutually orthogonal). By [Wa] Th. 4.4.5.15 there are (non-zero) analytic eigenvectors of $d\pi(k)$ and by [Wa] Th. 4.4.5.5, for any analytic vector v the closure of the smallest $\mathfrak{g}$-invariant subspace containing v is π-invariant, hence (π being irreducible), the closure coincides with $\mathcal{H}$ (for $v \neq 0$). These $\mathfrak{g}$-invariant subspaces generated by an eigenvector of $d\pi(k)$ have been computed in [Pu] Sec. 2. It turns out (see also [HT] II. Prop. 1.1.4) that the eigenspaces of $d\pi(k)$ are all one-dimensional (in particular, π is K-finite). The action by $d\pi$ is given by one of the Harish-Chandra modules, listed in [HT] II. Table 1.2.10, irreducible (infinite-dimensional) ones are only among $U(\nu^+,\nu^-), V_\lambda, \overline{V}_\lambda$ (see also the summary in [Pu] p. 102; it is well known that a finite-dimensional unitary representation of G must be trivial, see also the argument in [Pu] p. 101).

We discuss now the case that π is given by $U(\nu^+,\nu^-)$. By [HT] p. 94, we have a complete orthogonal system $(v_j)_{j\in\mathbb{Z}}$ ($\subseteq \mathcal{H}^\infty$) for $\mathcal{H}$ such that

$$d\pi(k)\,v_j = -\mathtt{i}\,(\nu^+ - \nu^- + 2j)\,v_j, \quad d\pi(n^+)\,v_j = (\nu^+ + j)\,v_{j+1},$$
$$d\pi(n^-)\,v_j = (\nu^- - j)\,v_{j-1} \quad \text{for all } j \in \mathbb{Z}.$$

Let (τ_{ij}) be the matrix representation for T_1 with respect to (v_j) (i.e. $T_1 v_j = \sum_i \tau_{ij} v_i$). Then (by elementary computations) $(*)$ is equivalent to

$$\begin{aligned}
&(\nu^+ + i - 1)\,\tau_{i-1\,j} - (\nu^+ + j)\,\tau_{i\,j+1} = (i-1-j)\,\tau_{ij},\\
&(\nu^- - i - 1)\,\tau_{i+1\,j} - (\nu^- - j)\,\tau_{i\,j-1} = (j-i-1)\,\tau_{ij} \qquad \text{for all } i,j \in \mathbb{Z}.
\end{aligned}$$

Recall that $T_1 = k \circ T$ which gives $(T_1 v_j \,|\, v_i) = ((d\pi(k)T - T d\pi(k))\,v_j \,|\, v_i) = (-2\mathtt{i})(i-j)(Tv_j \mid v_i)$. Hence if (τ'_{ij}) is the matrix for T, we have $\tau_{ij} = (-2\mathtt{i})(i-j)\tau'_{ij}$, in particular $\tau_{ii} = 0$ for all i. Then for $i = j$, the first recursion implies $\tau_{i\,i+1} = c^+/(\nu^+ + i)$ (with $c^+ = \nu^+\tau_{01}$), and the second one $\tau_{i\,i-1} = c^-/(\nu^- + i)$ (with $c^- = \nu^-\tau_{0\,-1}$) for all $i \in \mathbb{Z}$. This allows to compute $\tau_{i\,i+k}$ for all $k > 0$ from the second recursion and $\tau_{i\,i-k}$ from the first recursion (thus they are determined uniquely by $c^+, c^- \in \mathbb{C}$).

For unitary representations, we have ([HT] III.Th. 1.1.3) either case A (principal series): $\nu^+ + \overline{\nu^-} = 1$, $\nu^+ \notin \mathbb{Z}$ or case B (complementary series): $0 < \nu^+, \nu^- < 1$ (in particular, $\nu^+, \nu^- \notin \mathbb{Z}$ which holds for arbitrary irreducible representations).

Assume now that $\nu^+ + \nu^- \neq 1$. Then we obtain (using Pochhammer's notation $(\xi)_l = \xi(\xi+1)\ldots(\xi+l-1)$)

$$\begin{aligned}
\tau_{i\,i+l} &= \frac{c^+}{\nu^+ + \nu^- - 1}\left(1 - \frac{(i+1-\nu^-)_l}{(i+\nu^+)_l}\right),\\
\tau_{i\,i-l} &= \frac{c^-}{\nu^+ + \nu^- - 1}\left(1 - \frac{(i-1+\nu^+)_l}{(i-\nu^-)_l}\right) \qquad \text{for } l > 0.
\end{aligned}$$

By direct verification, one can show that for any $c^+, c^- \in \mathbb{C}$ this gives (algebraically) solutions $T_1 = (\tau_{ij})$ of (*).

For case A $(\nu^+ = a + \mathtt{i}b$ with $0 \le a < 1,\ b \neq 0)$ we will now show that the corresponding mapping $T = (\tau'_{ij})$ is bounded on $\mathcal{H}$.

We assume that $c^+ \neq 0$, $c^- = 0$ (similar estimates apply when $c^+ = 0$, $c^- \neq 0$). Since $b \neq 0$, we have $\nu^+ - \nu^- \neq 1$. and we can take $c^+ = 2\mathtt{i}\,(\nu^+ + \nu^- - 1)$. We write $\nu^+ = \nu$ for short, then $1 - \nu^- = \overline{\nu} = a - \mathtt{i}b$.

First we consider the segment $i, j > 0$ of the matrix. We have for $l > 0$, $(i+\nu)_l = \Gamma(\nu+i+l)/\Gamma(\nu+i)$, hence by Lemma 4.4

$$\tau_{i\,i+l} = 2\mathtt{i}\left(1 - \left(\frac{i}{i+l}\right)^{2\mathtt{i}b} + O\left(\frac{1}{i}\right)\right)$$

uniformly in i, l. Recall that $(-2\mathtt{i})\tau'_{ij} = \dfrac{\tau_{ij}}{i-j}$, hence for $0 < i < j$, we get

$$\tau'_{ij} = \frac{1}{j-i}\left(1 - \left(\frac{i}{j}\right)^{2\mathtt{i}b} + O\left(\frac{1}{i}\right)\right),$$

and $\tau'_{ij} = 0$ for $i > j$ (since $c^- = 0$).

We can assume that $\|v_j\| = 1$ for all j ([HT] p. 95). Thus we have to check boundedness of a matrix operator on l^2. The remainder $O(1/i)$ in τ'_{ij} is square-summable, hence it defines a Hilbert-Schmidt operator, in particular it is bounded. The diagonal elements

τ'_{ii} can be chosen arbitrarily (taking some bounded sequence) and they contribute another bounded operator. So we have just to deal with the principal term $\frac{1}{j-i}\bigl(1-\bigl(\frac{i}{j}\bigr)^{2ib}\bigr)$ where $0<i<j$. We split this into two parts. For $0<i<j\le 2i$, we have the uniform estimate

$$\left|\frac{1}{j-i}\left(1-\left(\frac{i}{j}\right)^{2ib}\right)\right| = O\left(\frac{1}{j}\right).$$

Hence for this part, we have (up to a constant) a majorization by $R=(\rho_{ij})$, with $\rho_{ij}=1/j$ for $0<i<j\le 2i$ and 0 otherwise. This is related to the transpose of the Cesàro operator of [BHS] and its boundedness can be shown by the Schur test as in [BHS] Th. 1 (with $p_j=j^{-\frac{1}{2}}$). For the other part, we have a majorization by $R_1=(\rho'_{ij})$, with $\rho'_{ij}=1/(j-i)$ for $0<2i<j$ and 0 otherwise. Then R^tR is dominated by $R_2=(\rho''_{ij})$, with $\rho''_{ij}=1/\max(i,j)$ for $i,j>0$. As above, boundedness follows from the Schur test.

The same arguments apply when $i,j<0$. For $j\le 0<i$, we get a majorization by $1/(i-j)$ and after a reflection on the initial space $(v_j\mapsto v_{1-j})$, this gives just the classical Hilbert matrix which is known to define a bounded operator (e.g., again by the Schur test).

Now we can use this to describe the cohomology in case A (with $\nu^++\nu^-\neq 1$). $\mathcal{B}_K$ corresponds to the space of diagonal matrices with bounded diagonal (all eigenvalues of $d\pi(k)$ have multiplicity one). Thus we have shown that $\mathcal{Z}_S/\mathcal{B}_K$ has dimension two. Now take $T_1=(\tau_{ij})$ satisfying the recursion formulas above. For $i\neq j$, $\tau'_{ij}=\tau_{ij}/(i-j)$. Specifying some bounded sequence τ'_{ii} gives $T=(\tau'_{ij})\in\mathcal{Z}_S$ such that $T_1=k\circ T$ (the coefficients τ'_{ii} are eliminated under the action of k). Checking now for the conditions $h\circ T=0$ and $e^+\circ T=0$ (or equivalently that T commutes with $\pi(AN)$), it turns out that these equations can be satisfied by choosing an appropriate sequence τ'_{ii} if and only if $c^+=c^-$. Thus $(\mathcal{B}_K+\mathcal{B}_S)/\mathcal{B}_K$ is one-dimensional. (Alternatively, one can use the realization of π on $L^2(\mathbb{R})$ which arises from the action of G by fractional linear transformations on $\mathbb{R}$, combined with some cocycle. Using Fourier transform, one can see immediately that the restriction of π to S splits into two irreducible representations. Hence $\mathcal{B}_S$ is two-dimensional and modulo the centre it has dimension one). It follows that $\mathcal{H}^1(L^1(G),\mathcal{B})$ has dimension one.

In the case $\nu^++\nu^-=1$ (this is the intersection of case A and B). We put again $\nu=\nu^+$, by [HT] III.Th. 1.1.3, we can assume that $0<\nu<1$. Here the solutions of (*) are given by

$$\tau_{i\,i+l}=c^+\sum_{m=0}^{l-1}\frac{1}{\nu+i+m},\qquad \tau_{i\,i-l}=-c^-\sum_{m=1}^{l}\frac{1}{\nu+i-m}\qquad\text{for } l>0.$$

These coefficients can be estimated in a similar way, using an asymptotic expansion of the ψ-function. The results are the same.

In case B (with $\nu^++\nu^-\neq 1$) the situation is different. Observe that by [HT] p. 95 we now have $\|v_i\|^2/\|v_{i+l}\|^2=(i+\nu^+)_l/(i+1-\nu^-)_l$ for $l>0$. Using an estimate as in

Lemma 4.4 one can show that

$$\frac{1}{N}\left\|\sum_{j=1}^{N} T\frac{v_j}{\|v_j\|}\right\|^2 \to \infty \quad \text{as } N \to \infty$$

when $c^+ \neq c^-$. Thus there are non-trivial crossed homomorphisms on the infinitesimal level for which the corresponding mapping $T = (\tau'_{ij})$ is not bounded on $\mathcal{H}$. As above, the restriction of π to S splits into two irreducible representations, hence for $c^+ = c^-$ the operator must be bounded. As a result, we get in this case that $\mathcal{Z}_S = \mathcal{B}_K + \mathcal{B}_S$ and $\mathcal{Z}_S/\mathcal{B}_K$ has dimension one. In particular, $\mathcal{H}^1(L^1(G), \mathcal{B})$ is trivial.

This covers all irreducible representations arising from $U(\nu^+, \nu^-)$. Then there are the modules V_λ (lowest weight modules) and $\overline{V}_{-\lambda}$ (highest weight modules). By [HT] III.Th. 1.1.5, we have $\lambda > 0$ (for $\lambda > 1$, rational, this gives discrete series representations of certain quotients of G). As above, (*) leads to a sequence of recursion formulas for τ_{ij}. But (due to the fact that now $i, j \geq 0$) it turns out that there are no non-zero solutions.

More generally, we consider the case $U(0,1) \cong V_1 \oplus \overline{V}_{-1}$ ([HT] p. 63; $V_1, \overline{V}_{-1}$ define the mock discrete series). Here we get the solutions

$$\tau_{ij} = c^+ \text{ for } i \leq 0 < j, \qquad \tau_{ij} = c^- \text{ for } j \leq 0 < i$$

and $\tau_{ij} = 0$ otherwise. The corresponding operator T is again bounded (essentially two copies of the Hilbert matrix). Thus $\mathcal{Z}_S/\mathcal{B}_K$ is two-dimensional. However, among these, there is no $T \in B(\mathcal{H})$ that satisfies also $h \circ T = 0$. This would give the equations $(j-2)\,\tau'_{j-2\,j} - j\,\tau'_{j-1\,j+1} + (1-j)\,(\tau'_{jj} - \tau'_{j-1\,j-1}) = 0$ for all j. For $j = 1$, this is inconsistent with the formulas for τ_{ij} above when $c^+ \neq 0$. Similarly for $c^- \neq 0$. It follows that $\mathcal{B}_S \subseteq \mathcal{B}_K$, so $\mathcal{H}^1(L^1(G), \mathcal{B})$ has dimension two.

Example 4.6. For $G = SL(2,\mathbb{R})$, we consider $\mathcal{H} = L^2(G)$ with the regular representation and $\mathcal{B} = VN(G)$ or $\mathcal{B}(\mathcal{H})$. Using a disintegration ([Di] Th. 8.4.2, Prop. 18.7.4), the regular representation contains as a subrepresentation a direct integral of principal series representations defined by the modules $U(\nu^+, \nu^-)$ with $\nu^+ = \nu^- = \frac{1}{2} + \mathtt{i}b$ and $\nu^+ = \nu^- - 1 = \mathtt{i}b$, $b \geq 0$. It follows from the formulas given in Example 4.5 that the basic solutions depend continuously on ν^+, hence we can generate decomposable elements of $\mathcal{Z}_S$ by multiplying the basic solutions with appropriate measurable weights. This gives a subspace isomorphic to $L^\infty([0,\infty[\times\{0,\frac{1}{2}\})$ in $\mathcal{H}^1(L^1(G), VN(G))$.

Finally, when $\mathcal{B} = \mathcal{B}(L^2(G))$, one can consider arbitrary decomposable operators (not only those in $VN(G)$). This corresponds to countable sums of modules $U(\nu^+, \nu^-)$ (see the Plancherel theorem, [Di] Th. 18.8.1). One gets the same recurrences as in Example 4.5, for $c^\pm$ one can now choose arbitrary operators from $\mathcal{B}(l^2)$. This gives a subspace isomorphic to $L^\infty([0,\infty[\times\{0,\frac{1}{2}\})\,\overline{\otimes}\,\mathcal{B}(l^2)$ in $\mathcal{H}^1\big(L^1(G), \mathcal{B}(L^2(G))\big)$.

Acknowledgements. I wish to thank the referee for carefully reading the paper.

This paper is based on a lecture delivered at the 19th International Conference on Banach Algebras held at Będlewo, July 17–27, 2009. The support for the meeting by the Polish Academy of Sciences, the European Science Foundation, and the Faculty of

Mathematics and Computer Science of the Adam Mickiewicz University at Poznań is gratefully acknowledged.

Added in proof (November 2010). F. Ghahramani has informed me that the statement of our Theorem 1.2 is also proved as Theorem 5.6 in his paper (joint with Y. Choi and Y. Zhang) "Approximate and pseudo-amenability of various classes of Banach algebras", J. Funct. Anal. 256 (2009), 3158–3191.

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