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RANGE TRIPOTENTS AND ORDER IN JBW^*-TRIPLES

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Abstract. In a JBW^*-triple, i.e., a symmetric complex Banach space possessing a predual, the set of tripotents is naturally endowed with a partial order relation. This work is mainly concerned with this partial order relation when restricted to the subset $\mathcal{R}(A)$ of tripotents in a JBW^*-triple B formed by the range tripotents of the elements of a JB^*-subtriple A of B. The aim is to present recent developments obtained for the poset $\mathcal{R}(A)$ of the range tripotents relative to A, whilst also providing the necessary account of the general theory of the lattice of tripotents. Although the *leitmotiv* might be described as seeking to find conditions under which the supremum of a subset of range tripotents relative to A is itself a range tripotent relative to A, other properties are also investigated. Amongst these is the relation between range tripotents and partial isometries and support projections in W^*-algebras.

1. Introduction. Tripotents have played an important rôle in the research concerning JBW^*-triples. Although they allow for investigation under many perspectives, vastly documented in the literature, an interesting characteristic of the set of tripotents in a JBW^*-triple is that it possesses a natural ordering. In fact, the set of tripotents, together with this partial order relation and with a greatest element adjoined, forms a complete lattice, whose properties have been comprehensively investigated (cf. [4], [9]). As a concrete situation, one has, for example, the set of partial isometries in a W^*-algebra, which coincides exactly with the set of tripotents in the JBW^*-triple formed by the algebra, and thus is automatically endowed with a partial order relation.

Recently, tripotents and their ordering have appeared again, for example, in the relatively new field of operator spaces (cf. [5]). In particular, the suprema of increasing nets of range tripotents lying in the bidual of a ternary ring of operators have been given a special emphasis in the study of positivity in operator spaces (cf. [5]). The definition of the range tripotent of an element in a JBW^*-triple, due to Edwards and Rüttimann,

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appears in [9], although under a different name, and is akin to the concept of range projections in JBW-algebras.

The present work is mainly concerned with the subset $\mathcal{R}(A)$ of tripotents in a JBW^*-triple B formed by the range tripotents of the elements of a JB^*-subtriple A of B, when endowed with the partial ordering inherited from the set of all tripotents. The aim is to give an overview of recent developments obtained for the poset $\mathcal{R}(A)$ whilst also providing the necessary account of the general theory of the lattice of tripotents as to render this work as self-contained as possible.

The content of the remaining two sections is as follows. Section 2 mostly includes classical facts on JB^*-triples and JBW^*-triples appearing in the bibliography, being the notions of order and orthogonality amongst tripotents and properties concerning these notions outlined. Some well-known facts on JB^*-algebras and JBW^*-algebras are also included here to facilitate future reference.

The definition of the range tripotent of an element a in a JBW^*-triple B and the definition of range tripotent relative to a JB^*-subtriple A of B are made in Section 3. A crucial result appearing in this section is Lemma 3.3, which identifies the range tripotent of a with its range projection in a particular JBW-algebra contained in B. This lemma leads to establishing some circumstances in which the supremum of a subset of range tripotents relative to A is itself a range tripotent relative to A. As a consequence, it is shown that the weak* limits of a particular kind of increasing sequences in the closed unit ball of A are necessarily range tripotents relative to A.

It is also investigated in Section 3 how the range tripotents are mapped under isomorphisms and how they relate to the partial isometries and the support projections in W^*-algebras. The last results of this section, namely, Theorem 3.6, Corollary 3.7 and Proposition 3.8, are essentially contained in [19].

2. Preliminaries. This section mainly contains general facts concerning JB^*-triples and JBW^*-triples needed in the sequel. Given the interplay between these spaces and JB^*-algebras and JBW^*-algebras, the section begins with selected remarks about these algebras.

Recall that a Jordan *-algebra A, with a multiplication $\circ$, is said to be a JB^**-algebra* if it is a complex Banach space whose norm satisfies

(i) $\|a \circ b\| \leq \|a\|\|b\|$,
(ii) $\|a^*\| = \|a\|$,
(iii) $\|\{a\ a\ a\}\| = \|a\|^3$;

for all elements a and b in A. Here $\{\dots\}$ denotes the Jordan triple product on A defined, for all a, b and c, by

$$\{a\ b\ c\} = a \circ (b^* \circ c) - b^* \circ (c \circ a) + c \circ (a \circ b^*).$$

A JB^*-algebra which is the dual of a Banach space is said to be a JBW^**-algebra*. The self-adjoint parts of JB^*-algebras and JBW^*-algebras, i.e., the subset of the self-adjoint elements, are called JB-algebras and JBW-algebras, respectively. JB-algebras and JBW-algebras can be defined independently but it has been shown that there exists a one-to-one correspondence between these algebras and the self-adjoint parts of the JB^*-algebras and JBW^*-algebras, respectively (cf. [8, 13, 23]).

Denote by A_{sa} the self-adjoint part of A and by A_+ the cone formed by the *positive* elements, i.e., the squares of the elements lying in A_{sa}. The positive cone allows for a partial order $\leq$ to be defined in A_{sa} in the usual manner. In the case of a JBW-algebra, the positive cone is weak* closed, the algebra is monotone complete and a bounded increasing net converges to its supremum in the weak* topology (cf. [8, 13]).

The *projections* in a JBW^*-algebra are the self-adjoint elements which are also idempotent. The set of projections is a complete lattice for the partial order mentioned above. Furthermore, two projections p and q are said to be *orthogonal* if

$$p \circ q = 0.$$

The lattice of projections together with this orthogonality relation becomes a complete orthocomplemented lattice.

The *range projection* p_a of an element a in the JBW^*-algebra A is defined to be the least projection p for which

$$p \circ a = a.$$

The range projection of a is the unit of the JBW^*-subalgebra generated by a. (For the general theory of JB^*-algebras and JBW^*-algebras, and of their self-adjoint parts, JB-algebras and JBW-algebras, respectively, the reader is referred to [1, 8, 13, 22, 23].)

A complex vector space A endowed with a triple product, i.e. a mapping $(a, b, c) \mapsto \{a\ b\ c\}$, from $A \times A \times A$ to A, which is symmetric and linear in the outer variables and conjugate linear in the middle variable, is said to be a *Jordan *-triple* if the triple product satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\ b\ c\}, d) - D(c, \{b\ a\ d\}), \tag{1}$$

where $[\,,]$ denotes the commutator and D is the mapping, from $A \times A$ to the space of linear operators on A, defined by

$$D(a,b)c = \{a\ b\ c\}.$$

A Jordan *-triple A is said to be a *JB^*-triple* if A is a Banach space, the triple product is continuous and, for each element a in A, the operator $D(a,a)$ is hermitian with non-negative spectrum and

$$\|D(a,a)\| = \|a\|^2.$$

A JB^*-triple is said to be a JBW^*-triple if A is the dual of a Banach space A_*, called the *predual* of A. The predual is unique up to isometry and the triple product is separately weak* continuous (cf. [3, 11]) and jointly strong* continuous on bounded sets ([20]). The bidual of a JB^*-triple is a JBW^*-triple.

A *JB^*-subtriple* B of the JB^*-triple A is a norm closed subspace of A such that $\{B\ B\ B\}$ is contained in B. Examples of JB^*-triples and JBW^*-triples are C^*-algebras and W^*-algebras, respectively, for the triple product defined by

$$\{a\ b\ c\} = \frac{1}{2}(ab^*c + cb^*a), \tag{2}$$

for all elements a, b and c. An element u of a JB^*-triple A is said to be a *tripotent* if

$$\{u\ \ u\ \ u\} = u.$$

The space A splits into the direct sum

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)\ ,$$

called the *Peirce decomposition of A relative to u*, where the *Peirce spaces* $A_j(u)$ are defined, for $j = 0, 1, 2$, by

$$A_j(u) = \left\{ a \in A : D(u,u)a = \frac{1}{2} ja \right\}.$$

These spaces satisfy the *Peirce arithmetic*, namely,

$$\{A_j(u)\ \ A_k(u)\ \ A_l(u)\} \subseteq A_{j-k+l}(u),$$

if $j - k + l = 0, 1$ or 2, and

$$\{A_j(u)\ \ A_k(u)\ \ A_l(u)\} = \{0\},$$

otherwise. Furthermore, the Peirce spaces are JB^*-subtriples, which are weak* closed if A is a JBW^*-triple, and such that

$$\{A_2(u)\ \ A_0(u)\ \ A\} = \{A_0(u)\ \ A_2(u)\ \ A\} = \{0\}. \tag{3}$$

The space $A_2(u)$ when endowed with the product

$$a \circ_u b = \{a\ \ u\ \ b\}$$

and the involution

$$a^* = \{u\ \ a\ \ u\}$$

is a JB^*-algebra and, if A is a JBW^*-triple, $A_2(u)$ becomes a JBW^*-algebra.

PROPOSITION 2.1. *Let A be a JB^*-triple and let u and v be tripotents in A. The following assertions are equivalent:*

(i) *u lies in $A_0(v)$,*
(ii) *v lies in $A_0(u)$,*
(iii) $D(v,u) = 0$,
(iv) $D(u,v) = 0$.

Proof. If u lies in $A_0(v)$, then the equalities (3) and the fact that v lies in $A_2(v)$, yield

$$D(u,u)v = 0$$

and, thus, v lies in $A_0(u)$. Consequently, the assertion (i) implies the assertion (ii).

To see that (ii) implies (iii), observe that, since u lies in $A_2(u)$, it follows immediately from (3) that

$$D(v,u) = 0.$$

To show that (iii) implies (iv), notice that by the symmetry of the triple product in the outer variables,

$$D(v,u)u = D(u,u)v = 0$$

and, hence, v lies in $A_0(u)$. Therefore, by (3), the operator $D(u,v)$ coincides with 0.

Finally, (iv) implies (i) because, since $D(u,v)$ coincides with zero, it follows that

$$D(u,v)v = D(v,v)u = 0$$

and clearly u lies in $A_0(v)$. ∎

Two tripotents u and v are said to be *orthogonal* if v lies in $A_0(u)$. A finite or infinite family of tripotents is called an *orthogonal family of tripotents* if it consists of tripotents which are pairwise orthogonal. This notion of orthogonality allows for establishing a partial order relation in the set $\mathcal{U}(A)$ of all tripotents lying in A in the following manner. A tripotent u is said to be *less than or equal to* a tripotent v, written $u \leq v$, if $v - u$ lies in $A_0(u)$. If $u \leq v$, then the tripotent u lies in the JBW^*-algebra $A_2(v)$ and is a projection in this algebra. Moreover, the converse is also true, since it has been shown that a tripotent u which is a projection in $A_2(v)$ is necessarily a tripotent less than or equal to v. Furthermore, if u and w are tripotents less than or equal to v, then $u \leq w$ if, and only if, the same inequality holds also when considering u and w as projections in the JBW^*-algebra $A_2(v)$ (cf. [9], Lemma 2.4).

It is not difficult to realise that in a C^*-algebra the tripotents are exactly the partial isometries, and an easy application of Proposition 2.1 yields that two partial isometries u and v are orthogonal tripotents if, and only if, the initial projections u^*u and v^*v are orthogonal and the final projections uu^* and vv^* are orthogonal.

As proved in [4], it is possible to make the assertions contained in the next lemma.

LEMMA 2.2. *Let A be a JBW^*-triple. Then the following assertions hold.*

(i) *If $\{u_i\}_{i\in I}$ is a family of tripotents in A having an upper bound, then it has a supremum.*

(ii) *If (u_j) is an increasing net of tripotents in A, then (u_j) converges to its supremum in the weak* topology.*

Proof. (i) This assertion is an immediate consequence of [9], Theorem 4.4.

(ii) The net (u_j) has a supremum u in $\mathcal{U}(A)$ as a direct consequence of [9], Theorem 4.6. Hence (u_j) is an increasing net of projections lying in the self-adjoint part $A_2(u)_{sa}$ of the JBW^*-algebra $A_2(u)$. By the general theory of JBW^*-algebras, it is known that the self-adjoint part of a JBW^*-algebra is a JBW-algebra and, consequently, a bounded increasing net in $A_2(u)_{sa}$ converges to its supremum in the weak* topology of this JBW-algebra (cf. [8, 13]). Since this topology is the restriction of the weak* topology of A, the result follows. ∎

A comprehensive account of the general theory of the infima and the suprema of subsets of tripotents in $\mathcal{U}(A)$ is beyond the scope of this work. The interested reader is referred to [4, 9].

A notion required in the sequel is the concept of isomorphism between JBW^*-triples. A linear mapping $T : B \to C$ between JBW^*-triples is said to be a *homomorphism* if $T\{a\ b\ c\}$ coincides with $\{Ta\ Tb\ Tc\}$, for all elements a, b and c lying in B, and a bijective homomorphism is said to be an *isomorphism*. A homomorphism is necessarily continuous, maps tripotents to tripotents and preserves the order amongst tripotents, i.e., if u and v lie in $\mathcal{U}(B)$ and $u \leq v$, then, in $\mathcal{U}(C)$, $Tu \leq Tv$.

3. Range tripotents and order. The JB^*-subtriple generated by an element a of norm one in a JBW^*-triple A coincides with the norm closure $\overline{A(a)}$ of the linear span $A(a)$ of the powers a^{2n+1}, being the powers of a defined recursively by

$$a^1 = a, \qquad a^{2n+1} = \{a \ a^{2n-1} \ a\}.$$

As a JB^*-triple, $\overline{A(a)}$ is isomorphic, and therefore isometrically isomorphic, to the commutative C^*-algebra $C_0(\Omega)$ of continuous functions vanishing at zero, defined on a locally compact subset of $]0,1]$, (cf. [9, 12, 14, 16]). The image of a under this isomorphism is the function $f(t) = t$.

Let $\overline{A(a)}^{w^*}$ be the weak* closure of the space $A(a)$. Clearly, the JBW^*-subtriple $\overline{A(a)}^{w^*}$ generated by a is isomorphic as a JBW^*-triple to a commutative W^*-algebra B_a containing the C^*-algebra $C_0(\Omega)$. In the following, by a slight abuse of notation, no distinction will be made between the spaces $\overline{A(a)}^{w^*}$ and B_a or between their elements.

The proposition below relates the partial ordering existing in the complete lattice of projections in the W^*-algebra B_a with the one existing for tripotents in $\overline{A(a)}^{w^*}$.

Proposition 3.1. *Let A be a JBW^*-triple, let a be an element of A of unit norm, let $\overline{A(a)}^{w^*}$ be the JBW^*-subtriple generated by a and let B_a be the commutative W^*-algebra isomorphic to $\overline{A(a)}^{w^*}$. Then the following assertions hold.*

(i) *For all projections p and q in B_a, $p \leq q$ if, and only if, p is a tripotent less than or equal to the the tripotent q.*
(ii) *Let $\{p_j\}_{j\in\Lambda}$ be an increasing net of projections in the W^*-algebra B_a. Then the supremum of the net $\{p_j\}_{j\in\Lambda}$ taken in the lattice of projections coincides with the supremum existing in the set of tripotents in $\overline{A(a)}^{w^*}$.*

Proof. To prove assertion (i), observe that, if $p \leq q$ are projections lying in B_a, then $q-p$ is a projection orthogonal to p. It follows that $\{p \ p \ q-p\}$ coincides with zero and $q-p$ is a tripotent orthogonal to p, or equivalently the tripotent p is less than or equal to q.

Conversely, if p and q are tripotents such that $p \leq q$, then $q-p$ is a tripotent orthogonal to p, i.e.,

$$0 = \{p \ p \ (p-q)\} = \{p \ p \ q\} - p = \frac{1}{2}(ppq + qpp) - p.$$

Since the algebra B_a is commutative, the previous equality yields that qp coincides with p. Therefore, p is a projection less than or equal to q.

As to the remaining assertion, let p denote the supremum $\sup\{p_j\}_{j\in\Lambda}$ in the complete lattice of projections in B_a, and recall that the net converges in the weak* topology to p. The net $\{p_j\}_{j\in\Lambda}$ is also an increasing net of tripotents for which p is an upper bound. By Lemma 2.2, the net $\{p_j\}_{j\in\Lambda}$ has a supremum u in the set of tripotents and, moreover, converges to this supremum in the weak* topology. Therefore p coincides with u. ∎

Proposition 3.2. *Let A be a JBW^*-triple and let a be an element in A with unit norm. Then, there exists a smallest tripotent $r(a)$ in $\mathcal{U}(A)$ such that a is a positive element in the JBW^*-algebra $A_2(r(a))$ and a is less than or equal to $r(a)$ in this algebra.*

The proof below is based on the proof of [10], Lemma 3.3.

Proof. For all non-negative integers n, the image of a^{2n+1} under the isomorphism mentioned above is the function $f(t) = t^{2n+1}$ (cf. [9, 12, 14, 16]). Since there exists a sequence of real odd polynomials converging pointwisely to the characteristic function of $\Omega\backslash\{0\}$, it follows that there exists a sequence of real odd polynomials in a converging to a tripotent $r(a)$ in the weak* topology.

Clearly a is a self-adjoint element in the JBW^*-algebra $A_2(r(a))$, since

$$\{r(a)\ r(a)\ a\} = a \quad \text{and} \quad \{u\ a\ u\} = a.$$

Moreover, if b is the image of the function $g(t) = \sqrt{t}$ under the isomorphism, then it can be shown analogously that b is a self-adjoint element in $A_2(r(a))$ whose square coincides with a. It follows that a lies in the positive cone of the JBW^*-algebra $A_2(r(a))$. Similarly, it can be shown that $r(a) - a$ is a positive element in the JBW^*-algebra $A_2(r(a))$, which ends the existence part of the proof.

Suppose now that u is a tripotent for which a is a positive element in the JBW^*-algebra $A_2(u)$. The tripotent $r(a)$ lies in $A_2(u)$ because it is the weak* limit of a sequence in the subspace $A(a)$ of $A_2(u)$.

The weak* limit of the sequence (a^{2n+1}) is a tripotent v, whose image coincides with the characteristic function of $\{1\}$. By the Jordan triple identity (1), it follows that

$$a^{2n+1} = \{u\ \ a^{2n+1}\ \ u\}, \qquad b^{2n+1} = \{u\ \ b^{2n+1}\ \ u\},$$

and

$$a^{2n+1} = \{b^{2n+1}\ \ u\ \ b^{2n+1}\}.$$

Taking weak* limits yields that v is positive and $r(a)$ is self-adjoint in $A_2(u)$ (cf. [3]). Clearly, $v \leq u$ and, by [9], Lemma 2.4, v is a projection in the JBW^*-algebra $A_2(u)$. Similarly, since $u - a$ is a positive element in $A_2(u)$, it follows that the weak* limit of the sequence $\big((u-a)^{2n+1}\big)$ is also a projection in this algebra. In the associative JBW^*-algebra generated by a and u, the sequence $\big(u - (u-a)^{2n+1}\big)$ converges to $r(a)$. Consequently, $u - r(a)$ coincides with v and therefore $u - r(a)$ is a positive element of $A_2(u)$. Hence, by [9], Lemma 2.4, the tripotent $r(a)$ is less than or equal to u. ∎

Proposition 3.2 allows for the possibility of assigning a tripotent to each element of a JBW^*-triple A in the following manner. The *range tripotent* $r(a)$ of an element a is defined to be the least tripotent $r(a)$ such that a is a positive element in the JBW^*-algebra $A_2(r(a))$.

The next lemma establishes a relation between range tripotents and range projections. Although the lemma may be found in [19], its proof is included here for the reader's convenience.

LEMMA 3.3. *Let A be a JBW^*-triple and let a be an element in A. Then, if u is a tripotent for which a is positive in the JBW^*-algebra $A_2(u)$, the range projection of a in $A_2(u)$ coincides with the range tripotent $r(a)$.*

Proof. Suppose, without loss of generality, that a has unit norm. Since a is positive in the JBW^*-algebra $A_2(u)$, the range tripotent of a is less than or equal to u. Hence the powers of a in the algebras $A_2(u)_{sa}$ and $A_2(r(a))_{sa}$ coincide and lie in $A_2(r(a))_{sa}$. By spectral theory, the JBW-subalgebra of $A_2(u)_{sa}$ generated by a and u is isometrically isomorphic to a monotone complete algebra $C(X)$ of continuous real functions on a compact Hausdorff space X. Denote by $f(a)$ the inverse image of a function f through this isomorphism.

Let (f_n) be a sequence of continuous real functions defined by

$$f_n(t) = \begin{cases} 1, & |t| \geq 1/n, \\ nt, & 0 \leq t < 1/n, \\ -nt, & -1/n < t \leq 0. \end{cases}$$

By [13], Lemma 4.2.6, the sequence $f_n(a)$ converges in the weak* topology to the range projection p_a of a in $A_2(u)_{sa}$. The Stone–Weierstrass Theorem ensures that, for each n, there exists a sequence of polynomials $(p_{n,k})$ vanishing at zero and converging in norm to f_n, on X. For each n, choose k_n such that

$$\|f_n - p_{n,k_n}\| < \frac{1}{n}$$

and let (q_n) be the sequence defined by $q_n = p_{n,k_n}$. Let φ be any element in the predual of $A_2(u)_{sa}$. Given a positive ϵ, it is possible to choose n such that

$$\begin{aligned} |\varphi(q_n(a) - p_a)| &= |\varphi(q_n(a) - f_n(a) + f_n(a) - p_a)| \\ &\leq |\varphi(q_n(a) - f_n(a))| + |\varphi(f_n(a) - p_a)| \leq \epsilon. \end{aligned}$$

It follows that $(q_n(a))$ converges to the range projection p_a of the element a in the weak* topology. As a consequence p_a lies in the weak* closure of the span of the powers of a and, thus, this projection lies in the smallest JBW-subalgebra of $A_2(u)_{sa}$ containing a. Observe that p_a is the unit in this JBW-subalgebra.

Let $W(a)$ denote the smallest JBW-subalgebra of $A_2(r(a))_{sa}$ containing a. This algebra coincides with the weak* closure of the span of the powers of a. Observing that the weak* topologies in $A_2(r(a))_{sa}$ and in $A_2(u)_{sa}$ coincide with the restriction, to those algebras, of the weak* topology of the JBW^*-triple A, it follows that the smallest JBW-subalgebra of $A_2(u)_{sa}$ containing a is also $W(a)$. Since, by [9], Lemma 3.1, the range tripotent $r(a)$ is the unit of $W(a)$, the lemma is proved. ∎

A direct consequence of the Lemma 3.3 is the corollary below, which asserts that the range tripotent of an element a in a W^*-algebra is precisely the partial isometry appearing in the polar decomposition of a.

Corollary 3.4. *Let A be a W^*-algebra, let a be an element in A and let $a = v|a|$ be the polar decomposition of a. Then, the range tripotent $r(a)$ coincides with the partial isometry v.*

Proof. The W^*-algebra A, when endowed with the triple product defined by the equality (2), is a JBW^*-triple, and $A_2(v)$ together with the product $\circ_v$, defined for all b and c in $A_2(v)$, by

$$b \circ_v c = \{b \ v \ c\},$$

is a JBW^*-algebra. Straightforward computations show that a is positive in $A_2(v)$.

Let $W(a)$ be the JBW-subalgebra, of the JBW-algebra A_{sa}, generated by a. The algebra $W(a)$ is the weak* closure of the span of the powers $a^{(n)}$ of a in the JBW-algebra A_{sa}. The polar decomposition of a in the W^*-algebra A yields that

$$a^{(n)} = v|a|^n,$$

where $.^n$ denotes the n-th power taken in the W^*-algebra A. It follows that

$$W(a) = \overline{vC(|a|)}^{w^*},$$

where $C(|a|)$ is the span of the powers $|a|^n$ and the superscript w^* denotes the weak* closure. Hence $v(\overline{C(|a|)}^{w^*})$ is a subalgebra contained in $W(a)$, and, since $\overline{C(|a|)}^{w^*}$ is unital, the partial isometry v lies in $W(a)$ and, thus, is the unit of this algebra. Hence v is the range projection of a in $A_2(v)$. Now Lemma 3.3 yields that the partial isometry v coincides with the range tripotent $r(a)$. ∎

Recall that the support projection of a self-adjoint element a in a W^*-algebra is the least projection p for which $a = ap$. The next proposition relates the support projection of a positive element b of the W^*-algebra B_a, defined at the beginning of this section, with its range tripotent $r(b)$ when alternatively the element b is seen as lying in the JBW^*-triple $\overline{A(a)}^{w^*}$.

Proposition 3.5. *Let A be a JBW^*-triple, let a be an element of A of unit norm, let $\overline{A(a)}^{w^*}$ be the JBW^*-subtriple generated by a and let B_a be the commutative W^*-algebra isomorphic to $\overline{A(a)}^{w^*}$. Then, the suport projection of any positive element b in the W^*-algebra B_a coincides with its range tripotent in $\overline{A(a)}^{w^*}$.*

Proof. Let b be an element lying in $\overline{A(a)}^{w^*}$ such that b is positive in the commutative W^*-algebra B_a which is isomorphic as a JBW^*-triple to $\overline{A(a)}^{w^*}$. The partial isometry appearing in the polar decomposition of b is, in this case, its support projection $s(b)$ and, by Corollary 3.4, the range tripotent $r(b)$ coincides with $s(b)$. ∎

Let B be a JBW^*-triple, let A be a JB^*-subtriple of B and consider the set $\mathcal{U}(B)$ of tripotents in B equipped with the partial ordering defined in Section 2. A tripotent u lying in B is said to be a *range tripotent relative to A* if there exists a lying in A such that u coincides with $r(a)$. The set of range tripotents relative to A will be denoted by $\mathcal{R}(A)$. To avoid unnecessarily heavy notation, sometimes the range tripotents relative to A will be simply called range tripotents.

A natural question is to ask in what circumstances the supremum of a set of range tripotents is itself a range tripotent. The theorem below is a first step to answer this question.

Theorem 3.6. *Let B be a JBW^*-triple, let A be a JB^*-subtriple of B and let $\{u_i\}_{i\in\Lambda}$ be either a countable orthogonal family of range tripotents relative to A or a finite family of range tripotents relative to A having an upper bound. Then, the supremum $\vee_{i\in\Lambda} u_i$ lies in $\mathcal{R}(A)$ and, moreover, there exists a family $\{a_i\}_{i\in\Lambda}$ in A such that, for all i, the tripotent $r(a_i)$ coincides with u_i and*

$$\bigvee_{i\in\Lambda} u_i = r\Big(\sum_{i\in\Lambda} a_i\Big),$$

where the series converges in norm.

Proof. It will be supposed that not all tripotents are equal to zero. If they were, the assertion would trivially hold.

In the case of the family of tripotents being finite, let $u_1, u_2, \ldots, u_n$ be tripotents in $\mathcal{R}(A)$ having an upper bound in $\mathcal{U}(B)$ and denote by w the supremum of the family, which exists by Lemma 2.2. Let a_i, for all $i = 1, 2, \ldots, n$, be such that $u_i = r(a_i)$. Since, for all $i = 1, 2, \ldots, n$, the element a_i is a positive element in $B_2(u_i)$ and $u_i \leq w$, it follows that a_i is positive in $B_2(w)$ and hence also $\sum_{i=1}^n a_i$ is positive in $B_2(w)$.

Since $r(\sum_{i=1}^n a_i)$ is the least tripotent u' for which $\sum_{i=1}^n a_i$ is positive in $B_2(u')$, it follows that

$$r\Big(\sum_{i=1}^n a_i\Big) \leq w.$$

By [10], Lemma 3.3,

$$0 \leq \frac{\sum_{i=1}^n a_i}{\| \sum_{i=1}^n a_i\|} \leq r\bigg(\frac{\sum_{i=1}^n a_i}{\| \sum_{i=1}^n a_i\|}\bigg) = r\Big(\sum_{i=1}^n a_i\Big)$$

in the JBW^*-algebra $B_2(r(\sum_{i=1}^n a_i))$.

Since the tripotent $r(\sum_{i=1}^n a_i)$ is less than or equal to the tripotent w, the positive cone of the JBW^*-algebra $B_2(r(\sum_{i=1}^n a_i))$ is contained in $B_2(w)_+$. Thus it follows that

$$0 \leq \frac{a_i}{\| \sum_{i=1}^n a_i\|} \leq \frac{\sum_{i=1}^n a_i}{\| \sum_{i=1}^n a_i\|} \leq r\Big(\sum_{i=1}^n a_i\Big),$$

also in the JBW^*-algebra $B_2(w)$.

By [13], 4.1.13, for all $i = 1, 2, \ldots, n$, the range projection of $a_i/\| \sum_{i=1}^n a_i\|$ is less than or equal to $r(\sum_{i=1}^n a_i)$ whence, by Lemma 3.3,

$$u_i \leq r\Big(\sum_{i=1}^n a_i\Big)$$

in $B_2(w)$. It follows that

$$r\Big(\sum_{i=1}^n a_i\Big) \circ_w \sum_{i=1}^n u_i = \sum_{i=1}^n \Big(r\Big(\sum_{i=1}^n a_i\Big) \circ_w u_i\Big) = \sum_{i=1}^n u_i$$

in the JBW^*-algebra $B_2(w)$. Therefore the range projection of $\sum_{i=1}^n u_i$ is less than or equal to $r(\sum_{i=1}^n a_i)$. By [4], Proposition 3.9, the range projection of $\sum_{i=1}^n u_i$ coincides with the supremum w and, therefore,

$$w \leq r\Big(\sum_{i=1}^n a_i\Big).$$

This finally yields that w coincides with $r\,(\sum_{i=1}^n a_i)$.

Suppose now that $\{u_i\}_{i\in\Lambda}$ is a countable orthogonal family of range tripotents. A direct application of the proof above shows that the assertion holds for finite families. Suppose then that Λ coincides with the set of positive integers. Since, for all n, the tripotent u_n is a range tripotent, there exists a sequence (a_n) contained in A satisfying

$$u_n = r(a_n), \qquad \|a_n\| \leq \frac{1}{2n}.$$

Let (b_n) be the increasing Cauchy sequence in the JBW^*-algebra $B_2(u)$ defined, for all positive integers n, by

$$b_n = \sum_{i=1}^{n} a_i.$$

Observing that, by [9], Lemma 2.4, the sequence (u_n) is an orthogonal family of projections in the JBW-algebra $B_2(u)_{sa}$ and applying Lemma 3.3 and [13], Lemma 4.2.2, it follows that, in the closed unit ball of the JBW^*-algebra $B_2(u)_{sa}$,

$$0 \le b_n \le \sum_{i=1}^{n} u_i \le u.$$

Observe that (b_n) is a norm convergent sequence whose limit b lies in the closed unit ball of A and is the supremum of the sequence (b_n). By [8], Lemma 3.1, the positive cone in the JBW^*-algebra $B_2(u)$ is weak* closed and it follows that b is a positive element of this algebra. Hence, by the definition of range tripotent, $r(b)$ is less than or equal to u.

To prove the converse assertion, observe that, in the JBW^*-algebra $B_2(u)$, for all positive integers n, by [10], Lemma 3.3,

$$0 \le b_n \le b \le r(b),$$

and, using the result previously proved for finite families of range tripotents, it follows that

$$\bigvee_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i = r(b_n) \le r(b)$$

in the same algebra. Therefore, in the JBW^*-algebra $B_2(u)$, $r(b)$ is an upper bound for the family $\{u_i\}_{i\in\mathbb{N}}$, and, hence, u is less than or equal to $r(b)$. Consequentially, $r(b)$ coincides with u, as required. ∎

In spite of the good behaviour of the range tripotents displayed in the theorem above, it should be noted that this is not a general feature, namely in what concerns continuity properties, for example. The norm limit of the sequence (a_n) of the 2×2 matrices, defined by

$$a_n = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix},$$

has a range tripotent different from the limit of the corresponding sequence of range tripotents. Nevertheless, one can go as further as to saying:

COROLLARY 3.7. *Let B be a JBW^*-triple and let A be a JB^*-subtriple of B. Then the following assertions hold.*

(i) *If (u_n) is an increasing sequence of range tripotents relative to A such that, for all n in $\mathbb{N}$, the tripotent $u_{n+1} - u_n$ lies in $\mathcal{R}(A)$, then the supremum of the set of tripotents $\{u_n : n \in \mathbb{N}\}$ is a range tripotent relative to A.*

(ii) *If u is a tripotent which is the weak* limit of an increasing sequence (a_n) lying in $B_2(u)_1^+ \cap A$ and such that, for all n in $\mathbb{N}$, $r(a_{n+1}) - r(a_n)$ lies in $\mathcal{R}(A)$, then u is a range tripotent relative to A.*

In the above corollary, $B_2(u)_1^+$ denotes the positive elements of the closed unit ball $B_2(u)_1$ of the JBW^*-algebra $B_2(u)$.

Proof. Denote by u the supremum of the set of tripotents $\{u_n : n \in \mathbb{N}\}$ which, by Lemma 2.2, exists and satisfies

$$u = w^*\text{-}\lim u_i.$$

Since (u_n) is an increasing sequence of range tripotents, for all positive integers n,

$$u_{n+1} = u_n + w_{n+1},$$

where w_{n+1} is a tripotent orthogonal to u_n. Hence, letting w_1 coincide with 0, the sequence (w_n) forms an orthogonal family of range tripotents relative to A, whose supremum w, by Theorem 3.6, is also a range tripotent relative to A.

The supremum u satisfies the equality

$$u = w^*\text{-}\lim u_n = w^*\text{-}\lim\Big(u_1 + \sum_{i=2}^{n} w_i\Big) = u_1 + w^*\text{-}\lim \sum_{i=2}^{n} w_i = u_1 + w.$$

Thus u is the sum of two range tripotents and, as a consequence of Theorem 3.6, is a range tripotent.

To prove assertion (ii), observe that if there exists an increasing sequence (a_n) in $B_2(u)_1^+ \cap A$ such that $u = w^*\text{-}\lim a_n$, then, by Lemma 3.3 and the theory of JBW-algebras, the corresponding sequence of range tripotents $(r(a_n))$ is also an increasing sequence in the JBW^*-algebra $B_2(u)$.

Since, for all n,

$$0 \leq a_n \leq r(a_n) \leq u,$$

it follows that $u = w^*\text{-}\lim r(a_n)$ and, by (i) in this proposition, the tripotent u is a range tripotent relative to A. ∎

An interesting question motivated by Corollary 3.7 is whether the identification of some range tripotents relative to the JB^*-subtriple A as weak* limits of certain nets of positive elements lying in the open unit ball of the JBW^*-triple B might be pursued further.

A description of how range tripotents are mapped under homomorphisms seems not to be available yet, but for isomorphisms the following holds.

Proposition 3.8. *Let B and C be JBW^*-triples, let A be a JB^*-subtriple of B and let $T : B \to C$ be an isomorphism. Then, T maps range tripotents relative to A to range tripotents relative to $T(A)$ and, for all elements a in A, $Tr(a)$ equals $r(Ta)$. Moreover, $T(\mathcal{R}(A))$ coincides with $\mathcal{R}(T(A))$.*

Proof. Let a be an element of A. Then, by the definition of range tripotent, there exists an element b in $B_2(r(a))$ such that

$$a = \{b\ r(a)\ b\} \quad \text{and} \quad b = \{r(a)\ b\ r(a)\}.$$

It follows that the images of a and b under the isomorphism T satisfy

$$Ta = \{Tb\ Tr(a)\ Tb\} \quad \text{and} \quad Tb = \{Tr(a)\ Tb\ Tr(a)\},$$

which implies that the range tripotent $r(Ta)$ is less than or equal to $Tr(a)$. Therefore, by [9], Lemma 2.4, the tripotent $r(Ta)$ is a projection in $C_2(Tr(a))$ such that

$$r(Ta) \circ_{Tr(a)} Tr(a) = r(Ta).$$

Hence, applying the inverse mapping T^{-1} to the equality above, $T^{-1}r(Ta)$ is a projection in the JBW*-algebra $B_2(r(a))$ for which

$$T^{-1}r(Ta) \circ_{r(a)} a = a.$$

Since $r(a)$ is the least projection p in $A_2(r(a))$ for which $p \circ_{r(a)} a$ coincides with a, it follows that $r(Ta)$ and $Tr(a)$ coincide. ■

Finally, it should be noted that many questions regarding the subjects under scrutiny in this work remain unanswered. For example, one may ask in what way Theorem 3.6 may be extended. Indeed, in a JBW^*-triple B, what requirements must a subset of range tripotents relative to a JB^*-subtriple A satisfy so that its supremum remains a range tripotent relative to A? Or, still, how are these range tripotents mapped under homomorphisms?

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