ON THE PROJECTIVITY AND FLATNESS OF SOME GROUP MODULES

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Abstract. In the sequel of the work of H. G. Dales and M. E. Polyakov we give a few more examples of modules over the Banach algebra $L^1(G)$ whose projectivity resp. flatness implies the compactness resp. amenability of the locally compact group $G$.

Let $L^1(G)$ be the $L^1$-algebra associated with a left invariant Haar measure on the locally compact group $G$. In the sequel of the work of H. G. Dales and M. E. Polyakov, [D-P], we will give a few more examples supporting Helemskii’s philosophy on the relation between the projectivity of $L^1(G)$-modules and the compactness of $G$ on the one hand, and between the flatness of $L^1(G)$-modules and the amenability of $G$ on the other; see for instance [He1, p. 238], or [He1, IV. Theorem 5.13, p. 190] and [He1, VII. Theorem 2.35, p. 260].

If $A$ is an abstract Banach algebra and $A_+ = A \oplus \mathbb{C}$ its unitization, $L_a$ will denote the operator of left multiplication by $a \in A$ on either $A$ or $A_+$. A Banach left $A$-module $X$ will always be contractive such that the action $\pi : A \hat{\otimes} X \to X$, $\pi(a \otimes x) = ax$, is a linear contraction; $\hat{\otimes}$ denotes the projective tensor product of Banach spaces, and $L$ the space of all bounded linear mappings. For any Banach left $A$-module $X$, its dual Banach space, $X^*$, becomes a Banach right $A$-module by defining $\langle x, x^* a \rangle = \langle ax, x^* \rangle$, for $x \in X$, $x^* \in X^*$, $a \in A$. We shall always use the canonical isometrical isomorphism $(A \hat{\otimes} X)^* = L(A, X^*)$.

A Banach left $A$-module $X$ is called essential if the linear span of the products $ax$ ($a \in A, x \in X$) is dense in $X$. In case $A = L^1(G)$, every essential Banach left $L^1(G)$-module is a Banach $G$-module such that for any $x \in X$ the map $s \mapsto sx$ is continuous from $G$ into $X$ and satisfies $\|sx\| = \|x\|$ for all $s \in G$. Conversely, every Banach $G$-module

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is an essential Banach $L^1(G)$-module. Left translation by $s \in G$ will be denoted by $L_s f(t) = f(s^{-1} t)$, for any function $f$ on $G$.

1. Projectivity over $L^1(G)$. Instead of giving the original definition, cf. [D-P, Definition 1.1, p. 392], we shall use the following criterion, [D-P, Proposition 1.2, p. 392].

1.1. Criterion. Let $A$ be a Banach algebra and $X$ be a Banach left $A$-module. $X$ is projective if and only if there is a bounded linear map $\rho$ such that $\pi \circ \rho = 1_X$ and $\rho(ax) = (L_a \otimes 1_X)(\rho x)$, for $x \in X, a \in A$:

$$X \xrightarrow{\rho} A_+ \otimes X \xrightarrow{\pi} X.$$ 

$X$ is called $c$-projective, for some constant $c > 0$, if there is such a $\rho$ of norm $\|\rho\| \leq c$, cf. [W] Proposition 2.8, p. 158. If $X$ is essential, $A_+$ may be replaced by $A$.

1.2. Let $A$ be $L^1(G)$. If $G$ is compact, then every essential Banach left $L^1(G)$-module $X$ is 1-projective. Denoting the continuous contractive action of $s \in G$ on $x \in X$ by $sx$, and identifying $L^1(G) \otimes X$ with $L^1(G, X)$, we see that $(\rho x)(t) = t^{-1}x (x \in X, t \in G)$, defines a linear contraction $\rho$ from $X$ into $L^1(G, X)$ such that for all $s \in G$ and $x \in X$,

$$\rho(sx)(t) = t^{-1}(sx) = (s^{-1} t)^{-1} x = (L_s \otimes 1_X)(\rho x)(t) \quad (t \in G),$$

and

$$\pi(\rho x) = \int t(\rho x)(t) dt = \int t(t^{-1} x) dt = \int x dt = x,$$

provided the Haar measure of $G$ having been chosen equal to one. This implies the 1-projectivity of $X$.

Here we are rather interested in the converse question: given an $L^1(G)$-module $X$, when does the projectivity of $X$ imply the compactness of $G$? The main tool for deciding this is the following lemma of Yu. V. Selivanov, cf. [S1, Lemma 1.4, p. 389] and [S2, Corollary 1. p. 212].

1.3. Lemma (Selivanov). Let $A$ be a Banach algebra and $X$ be an essential Banach left $A$-module such that either $A$ or $X$ satisfy Grothendieck’s approximation condition. If $X$ is projective then there exists for every $x \neq 0$ in $X$ an $A$-module homomorphism $\varphi : X \to A$ with $\varphi(x) \neq 0$.

1.4. Proposition. Let $G$ be a locally compact group. If there exists a projective essential Banach left $L^1(G)$-module $X$ with $X^*$ being either norm separable or weakly sequentially complete, then $G$ is compact.

Proof. Since $A = L^1(G)$ enjoys the approximation property, the projectivity of $X$ implies by (1.3) the existence of a non-zero $L^1(G)$-module homomorphism $\varphi : X \to L^1(G)$ such that $\varphi(sx) = L_s(\varphi x)$, for all $s \in G, x \in X$. Since the dual map of $\varphi, \varphi^* : L^\infty(G) \to X^*$, is weakly compact, in the case of $X^*$ being norm separable by [G] Corollaire 1, p. 168] and in the case of $X^*$ being weakly sequentially complete by [D-S VI.7.6 Theorem, p. 494], $\varphi$ is weakly compact as well. Since for any $x \in X$ its $G$-orbit $\{sx : s \in G\}$ is norm bounded in $X$, it follows from $\varphi(sx) = L_s(\varphi x)$, $s \in G$, that its image is relatively weakly compact in $L^1(G)$. Since $\varphi(x) \neq 0$ in $L^1(G)$ for some $x \in X$, the Dunford-Pettis theorem implies the compactness of $G$, cf. [L] Theorem 4.8, p. 137] or [R] Lemma 1.1.(a), p. 602].
1.5. Example ([D-P, Theorem 5.1, p. 415]). Let $X = L^p(G)$, $1 < p < \infty$, be endowed with any action making it an essential $L^1(G)$-module. Then we have:

$\begin{align*}
L^p(G) \text{ projective } &\iff G \text{ compact }
\end{align*}$

Let us remark that $L^1(G)$ is a projective left $L^1(G)$-module for any $G$, by [He1, IV. Theorem 2.17, p. 175].

1.6. Example. Let $\pi$ be a continuous unitary representation of the locally compact group $G$ on a Hilbert space $h$ and let $X = \mathcal{L}^p(h)$, $1 < p < \infty$, be the space of all operators $T$ on $h$ such that $\text{trace}(T^*T)^{\frac{p}{2}} < \infty$. Then $X$ is a reflexive Banach space that becomes an essential left $L^1(G)$-module under the action $sT = \pi(s)T\pi(s^{-1})$, for $s \in G$, $T \in \mathcal{L}^p(h)$. Endowing the $C^*$-algebra, $K(h)$, of all compact operators on $h$ with the same action and noting that the dual of any $C^*$-algebra is weakly sequentially complete, [T1, III. Corollary 5.2, p. 148], we have

$\begin{align*}
K(h), \mathcal{L}^p(h) \text{ projective } &\iff G \text{ compact }
\end{align*}$

1.7. Example. Let $X$ be either $C^*(G)$, the full $C^*$-algebra of $G$, or $C_r^*(G)$, the reduced $C^*$-algebra of $G$, endowed with left translation. Then $C^*(G)$ and $C_r^*(G)$ are essential left $L^1(G)$-modules whose duals are weakly sequentially complete such that

$\begin{align*}
C^*(G), C_r^*(G) \text{ projective } &\iff G \text{ compact }
\end{align*}$

The same applies to the $C^*$-algebra, $K(L^2(G))$, of compact operators on $L^2(G)$ with $G$ acting as $sT = L_sT L_{s^{-1}}$, $T \in K(L^2(G))$, a special case of 1.6.

1.8. Example. Let $X$ be $K(L^p(G))$ the space of compact operators on $L^p(G)$, $1 < p < \infty$. Then $K(L^p(G))$ is an essential $L^1(G)$-module under $sT = L_sT L_{s^{-1}}$, whose dual Banach space is isometrically isomorphic to $L^p(G) \otimes L^p(G)$, which is norm separable whenever the topology of $G$ has a countable base:

$\begin{align*}
K(L^p(G)) \text{ projective } &\text{ and } G \text{ 2nd-countable } \implies G \text{ compact }
\end{align*}$

1.9. Example. Let $A(G)$ be the Fourier algebra of $G$, and $VN(G)$ its von Neumann algebra such that $A(G)^* = VN(G)$. If $\varphi$ is a function on $G$ satisfying $\varphi u \in A(G)$ for all $u \in A(G)$, then $\varphi$ is continuous and bounded and defines a bounded linear operator, $m_\varphi$, on the Banach space $A(G)$, $m_\varphi(u) = \varphi u$ ($u \in A(G)$). With this in mind we define

$\begin{align*}
MA(G) = \{ \varphi \in C^b(G) : \varphi u \in A(G) \quad \forall u \in A(G) \}, \\
M_0A(G) = \{ \varphi \in MA(G) : (m_\varphi)^* : VN(G) \to VN(G) \text{ completely bounded} \},
\end{align*}$

with norms

$\begin{align*}
\|\varphi\|_{MA(G)} = \|m_\varphi : A(G) \to A(G)\|, \\
\|\varphi\|_{M_0A(G)} = \|(m_\varphi)^* : VN(G) \to VN(G)\|_{cb}.
\end{align*}$

$M_0A(G)$ is called the space of completely bounded multipliers, and $MA(G)$ the space of all multipliers of $A(G)$. Denoting by $Q_0(G)$ and $Q(G)$ the completions of $L^1(G)$ with
2. Flatness over \( L^1(G) \). Rather than giving the original definition, \cite{Hein} VII. Definition 1.1, p. 239, we shall use the following criterion, due to O. Yu. Aristov \cite{A} Lemma 1.2, p. 1558, and its dual.

2.1. Criterion (Aristov). Let \( A \) be a Banach algebra and \( X \) be a Banach left \( A \)-module. \( X \) is flat if and only if there is a bounded linear map \( \rho \) from \( X \) into the bidual \((A_+ \widehat{\otimes} X)^{\ast\ast}\) such that \( \pi^{\ast\ast} \circ \rho = \iota_X \), the canonical embedding of \( X \) into \( X^{\ast\ast} \), and \( \rho(ax) = (L_a \widehat{\otimes} 1_X)^{\ast\ast}(\rho x) \), for \( x \in X \) and \( a \in A \):

\[
X \xrightarrow{\rho} (A_+ \widehat{\otimes} X)^{\ast\ast} \xrightarrow{\pi^{\ast\ast}} X^{\ast\ast}.
\]

\( X \) is called \( c \)-flat, for some constant \( c > 0 \), if there is such a \( \rho \) of norm \( \| \rho \| \leq c \), cf. \cite{W} Definition 4.2, p. 164. If \( X \) is essential, \( A_+ \) may be replaced by \( A \).

2.2. Criterion (dual). Let \( A \) be a Banach algebra, \( X \) be a Banach left \( A \)-module and \( X^\ast \) its dual right \( A \)-module. \( X \) is flat if and only if there is a bounded linear map \( \lambda \) from \( \mathcal{L}(A_+, X^\ast) \) into \( X^\ast \) such that \( \lambda \circ \pi = 1_{X^\ast} \) and \( \lambda(T \circ L_a) = (\lambda T)a \), for all \( T \in \mathcal{L}(A_+, X^\ast) \) and \( a \in X \):

\[
X^\ast \xrightarrow{\pi^\ast} \mathcal{L}(A_+, X^\ast) \xrightarrow{\lambda} X^\ast.
\]

In this case, \( X^\ast \) is called an injective right \( A \)-module, and \( c \)-injective if there is such a \( \lambda \) of norm \( \| \lambda \| \leq c \). If \( X \) is essential, \( A_+ \) may again be replaced by \( A \).

Clearly, a left \( A \)-module \( X \) is \( c \)-flat if and only if its dual right \( A \)-module \( X^\ast \) is \( c \)-injective. For a discussion of injectivity see, for instance, Definition 1.5 and Propositions 1.6 and 1.7 in \cite{DP} p. 394.

2.3. Let \( A = L^1(G) \). If \( G \) is amenable then every essential Banach left \( L^1(G) \)-module \( X \) is 1-flat. Indeed, let \( M \) be a left invariant mean on \( L^\infty(G) \). Using the isometric isomorphism of \( \mathcal{L}(L^1(G), X^\ast) \) with \( L^\infty_{w^\ast}(G, X^\ast) \), the space of all bounded functions \( \Phi : G \to X^\ast \) such that, for any \( x \in X \), \( t \mapsto \langle x, \Phi(t) \rangle \) is measurable on \( G \), there corresponds to every

\[
\| f \|_{Q_0(G)} = \sup \left\{ \left\| \int f(t) \varphi(t) dt \right\| : \varphi \in M_0 A(G), \| \varphi \|_{M_0 A(G)} \leq 1 \right\},
\]

\[
\| f \|_{Q(G)} = \sup \left\{ \left\| \int f(t) \varphi(t) dt \right\| : \varphi \in M A(G), \| \varphi \|_{M A(G)} \leq 1 \right\} \quad (f \in L^1(G)),
\]

we get two translation invariant Banach spaces whose duals are isometrically isomorphic with \( M_0 A(G) \) and \( MA(G) \), respectively:

\[
Q_0(G)^\ast = M_0 A(G), \quad Q(G)^\ast = MA(G),
\]

cf. \cite{C-H} 1.10 Proposition, p. 466. It follows from 1.9 Lemma, p. 465 in \cite{C-H}, that \( M_0 A(G) \) and \( MA(G) \) are weakly sequentially complete. Since left translation is continuous and isometric on \( Q_0(G) \) and \( Q(G) \), these are essential left \( L^1(G) \)-modules such that we have

\[
Q_0(G), Q(G) \text{ projective } \iff \ G \text{ compact}
\]
\( T \in \mathcal{L}(L^1(G), X^\ast) \) a unique function \( \Phi \in L^\infty_w(G, X^\ast) \) via the formula
\[
\langle x, Tf \rangle = \int_G f(t)\langle x, \Phi(t) \rangle dt \quad (x \in X, f \in L^1(G)),
\]
cf. [T1, IV. Proposition 7.16, p. 262]. Considering \( X \) as a (continuous, contractive) Banach \( G \)-module, the function \( t \mapsto \langle t^{-1}x, \Phi(t) \rangle \) is bounded and measurable in \( t \in G \) such that
\[
\langle x, \lambda(\Phi) \rangle = \int (t^{-1}x, \Phi(t))dM(t) \quad (x \in X, \Phi \in L^\infty_w(G, X^\ast))
\]
defines a linear contraction \( \lambda \),
\[
X^\ast \xrightarrow{\pi^*} L^\infty(G, X^\ast) \xrightarrow{\lambda} X^\ast,
\]
satisfying \( \lambda \circ \pi^* = 1_{X^\ast} \) and \( \lambda \circ (L_s \hat{\otimes} 1_X)^*(\Phi) = (\lambda \Phi)s \), for all \( \Phi \in L^\infty_w(G, X^\ast) \) and \( s \in G \). It follows that \( X^\ast \) is 1-injective and \( X \) 1-flat.

2.4. Remark. In spite of the similarity of the diagrams in 1.1 and 2.1 one must not expect that every flat Banach left module \( X \) over a Banach algebra \( A \) admit a non-zero \( A \)-module homomorphism \( \varphi : X \to A^{**} \). Indeed, let \( G \) be an amenable locally compact group and let \( A = L^1(G) \) and \( X = L^p(G), 2 < p < \infty \). Then every non-zero left \( L^1(G) \)-module homomorphism \( \varphi : L^p(G) \to L^1(G)^{**} \) gives rise to a non-zero left invariant operator \( \varphi^t : L^\infty(G) \to L^p(G) \) which forces \( G \) to be compact, cf. [L-vR, Theorem 3, p. 308] or [R] Proposition 1.2, p. 603].

Again, we are interested in the question: given an \( L^1(G) \)-module \( X \), when does flatness of \( X \) imply amenability of \( G \)? Several examples are given in [D-P], and we will add a few more.

2.5. Example. Let \( X = K(L^p(G)), 1 < p < \infty \), be the space of compact operators on \( L^p(G) \) with the action \( sT = L_sT L_{s^{-1}} \) for \( s \in G, T \in K(L^p(G)) \). Then \( K(L^p(G)) \) becomes an essential Banach left \( L^1(G) \)-module whose dual module \( L^p(G) \hat{\otimes} L^p(G) \) is endowed with the right action \( (f \otimes g)s = L_{s^{-1}}f \otimes L_{s^{-1}}g \), for \( s \in G \) and \( f \otimes g \in L^p(G) \hat{\otimes} L^p(G) \). By the left invariance of Haar measure, the duality \( \tau : L^p(G) \hat{\otimes} L^p(G) \to \mathbb{C} \) is \( G \)-invariant such that we infer from Theorem 4.6 in [D-P, p. 414]: if \( L^p(G) \hat{\otimes} L^p(G) \) is injective under the above action then \( G \) is amenable. Dually, if \( K(L^p(G)) \) is flat then \( G \) is amenable, i.e. together with 2.3:

\[
K(L^p(G)) \text{ flat } \iff \ G \text{ amenable}
\]

2.6. Example. Let \( \pi \) be a continuous unitary representation of \( G \) on the Hilbert space \( h, K(h) \) the \( C^* \)-algebra of compact operators on \( h \) with \( sT = \pi(s)T\pi(s^{-1}) \), for \( s \in G, T \in K(h) \), such that its dual module \( h \hat{\otimes} h \) has the action \( (\xi \otimes \overline{\eta})s = \pi(s^{-1})\xi \otimes \overline{\pi(s^{-1})\eta} \), for \( s \in G \) and \( \xi \otimes \overline{\eta} \in h \otimes \overline{h} \) (\( h \) and \( \pi \) denoting the complex-conjugates of \( h \) and \( \pi \), respectively). Therefore the trace \( \tau : h \otimes \overline{h} \to \mathbb{C} \) is \( G \)-invariant, and we conclude as in 2.5:

\[
K(h) \text{ flat } \iff \ G \text{ amenable}
\]

2.7. Example. Let \( C^*(G) \) be the full \( C^* \)-algebra of \( G \), and \( Q_0(G) \) be the Banach space defined in 1.9. Endowing both of them with left translation by \( G \), we have
\[
C^*(G), Q_0(G) \text{ flat } \iff G \text{ amenable}
\]

Proof. One direction follows from 2.3. To prove the other one we shall show that the injectivity of the dual modules, \( C^*(G)^* \) and \( Q_0(G)^* \), implies the amenability of \( G \). Identifying \( C^*(G)^* \) with the space, \( B(G) \), of coefficients of all continuous unitary representations of \( G \), and \( Q_0(G)^* \) with \( M_0A(G) \), we see that \( B(G) \) is contained in \( M_0A(G) \). By a theorem of Bożejko and Fendler, [B-F] or [J], every \( \varphi \in M_0A(G) \) can be written as \( \varphi(t^{-1}s) = (\Phi_1(s)|\Phi_2(t)) \) for \( (s,t) \in G \times G \), where \( \Phi_1, \Phi_2 : G \to h \) are two continuous bounded functions with values in some Hilbert space \( h \). It follows that every such \( \varphi \) is weakly almost periodic: \( M_0A(G) \subset WAP(G) \). Denoting by \( 1_G \) the constant function corresponding to the trivial representation of dimension one, we have \( 1_G \in B(G) \subset M_0A(G) \subset WAP(G) \), and so it suffices to prove the statement for \( M_0A(G) \).

If \( M_0A(G) \) is injective as a right Banach \( G \)-module, we have a map \( \lambda \) as in 2.2,

\[
M_0A(G) \xrightarrow{\pi^*} \mathcal{L}(L^1(G), M_0A(G)) \xrightarrow{\lambda} M_0A(G)
\]
such that \( \lambda(\pi^*\varphi) = \varphi \) for \( \varphi \in M_0A(G) \), and \( \lambda(T \circ L_s) = L_{s^{-1}}(\lambda T) \), for \( T \in \mathcal{L}(L^1, M_0A) \) and \( s \in G \). Associating with every \( \varphi \in L^\infty(G) \) the operator \( T_{\varphi} \), as kindly suggested to us by N. Monod, [M],

\[
T_{\varphi} : L^1(G) \to M_0A(G), \quad T_{\varphi}(f) = \langle f, \varphi \rangle 1_G \quad (f \in L^1(G)),
\]
we get by left invariance of Haar measure

\[
T_{L_{s}\varphi}(f) = \langle f, L_{s}\varphi \rangle 1_G = (L_{s^{-1}}f, \varphi) 1_G = T_{\varphi}(L_{s^{-1}}f) \quad (s \in G, f \in L^1(G)),
\]
and

\[
T_{1_G}(f) = \langle f, 1_G \rangle 1_G = 1_G \otimes 1_G(f) \quad (f \in L^1(G)),
\]
such that \( T_{1_G} = \pi^*(1_G) \). Denoting by \( m \) the left invariant mean on \( WAP(G) \), we see that the composition \( M = m \circ \lambda \circ T \) is a non-zero left invariant functional on \( L^\infty(G) \). Indeed, we have, for any \( \varphi \in L^\infty(G) \) and \( s \in G \),

\[
M(L_{s}\varphi) = m(\lambda(T_{L_{s}\varphi})) = m(\lambda(T_{\varphi} \circ L_{s^{-1}})) = m(L_{s}(\lambda(T_{\varphi}))) = m(\lambda(T_{\varphi})) = M(\varphi),
\]
and

\[
M(1_G) = m(\lambda(T_{1_G})) = m(\lambda(\pi^*(1_G))) = m(1_G) = 1,
\]
from which the amenability of \( G \) follows. ■

2.8. In [S2] Theorem 1, p. 211], Selivanov showed that for any projective module \( X \) over a Banach algebra \( A \) there is a bounded linear projection from \( \mathcal{L}(X) \) onto the subspace, \( \mathcal{L}_A(X) \), of \( A \)-module homomorphisms. In the same vein, there is for any flat \( X \) a bounded linear projection from \( \mathcal{L}(X^*) \) onto \( \mathcal{L}_A(X^*) \), the space of homomorphisms of the dual module \( X^* \), and if \( X \) is \( c \)-flat the projection can be chosen of norm \( \leq c \). Since, in this case, \( X^* \) is injective, this follows from the following lemma which we shall formulate only for left modules.

**Lemma.** Let \( Y \) be a Banach left module over the Banach algebra \( A \). If, for some constant \( c > 0 \), \( Y \) is \( c \)-injective, then there is a bounded linear projection, \( P \), of norm \( \|P\| \leq c \) from \( \mathcal{L}(Y) \) onto the subspace, \( \mathcal{L}_A(Y) \), of \( A \)-module homomorphisms.
Proof. According to the definition, cf. [D-P, Proposition 1.6, p. 394], there is a bounded linear map \( \lambda \) of norm \( \| \lambda \| \leq c \), satisfying \( \lambda(T \circ R_a) = a(\lambda T) \) and \( \lambda(\alpha y) = y \), for \( T \in \mathcal{L}(A_+, Y) \), \( a \in A \) and \( y \in Y \),
\[
Y \xrightarrow{\alpha} \mathcal{L}(A_+, Y) \xrightarrow{\lambda} Y,
\]
\( \alpha \) being given by \( (\alpha y)(a) = ay \), for \( y \in Y \) and \( a \in A_+ \), and \( R_a \) denoting right multiplication by \( a \) on \( A_+ \). Defining \( P \) by \( (PT)(y) = \lambda(T \circ \alpha y) \), for \( T \in \mathcal{L}(Y) \), \( y \in Y \), we see that \( P \) is a bounded linear operator on \( \mathcal{L}(Y) \) of norm \( \| P \| \leq \| \lambda \| \) satisfying, for \( T \in \mathcal{L}(Y) \) and \( a \in A \),
\[
(PT)(ay) = \lambda(T \circ \alpha(ay)) = \lambda(T \circ \alpha y \circ R_a) = a\lambda(T \circ \alpha y) = a(PT)(y),
\]
and, for \( T \in \mathcal{L}_{\lambda}(Y) \), in virtue of \( T \circ \alpha y = \alpha(Ty) \),
\[
(PT)(y) = \lambda(T \circ \alpha y) = \lambda(\alpha(Ty)) = Ty,
\]
such that \( P \) is a linear projection from \( \mathcal{L}(Y) \) onto \( \mathcal{L}_{\lambda}(Y) \) of norm \( \| P \| \leq c \). ■

2.9. A von Neumann algebra \( \mathcal{M} \) on a Hilbert space \( h \) is called injective if there is a linear projection of norm one from \( \mathcal{L}(h) \) onto \( \mathcal{M} \). By a theorem of Helemskii, [He3, Corollary 1, p. 77], the injectivity of \( \mathcal{M} \) implies the injectivity of the Banach left \( \mathcal{M} \)-module \( h \). As a partial converse we have

**Corollary.** Let \( \mathcal{M} \) be a von Neumann algebra on \( h \). If the Banach left \( \mathcal{M} \)-module \( h \) is 1-injective, then \( \mathcal{M} \) is injective.

**Proof.** Let the elements of \( \mathcal{M} \) act on \( h \) as operators. From (2.8), with \( A = \mathcal{M} \) and \( Y = h \), follows the existence of a linear projection of norm \( c = 1 \) from \( \mathcal{L}(h) \) onto \( \mathcal{L}_{\mathcal{M}}(h) = \mathcal{M}' \), the commutant of \( \mathcal{M} \). Hence \( \mathcal{M}' \) is injective, and so is \( \mathcal{M} \), cf. e.g. [T2, XV. Proposition 3.2(iii), p. 174]. ■

**Remark.** The question of how the bound of the projection can be relaxed is discussed by Pisier in [P] and by Christensen and Sinclair in [C-S1] and [C-S2].

2.10. Example. Let \( G \) be a discrete group and let \( l^1(G) \) act on \( l^2(G) \) by left or right convolution. Then the Banach \( l^1(G) \)-module \( l^2(G) \) is 1-flat if and only if \( G \) is amenable:

\[
l^2(G) \text{ 1-flat} \iff G \text{ amenable}
\]

**Proof.** Let us consider \( l^2(G) \) as a right \( l^1(G) \)-module such that \( G \) acts on \( l^2(G) \) by right translation \( (R_s f)(t) = f(ts) \), for \( s \in G \) and \( f \in l^2(G) \). If \( l^2(G) \) is 1-flat, it is 1-injective such that, by 2.8, there is a projection, \( P \), of norm one from \( \mathcal{L}(l^2(G)) \) onto \( \mathcal{L}_{\lambda}(l^2(G)) \), the subspace of all operators commuting with \( R_s \), \( s \in G \), which coincides with the von Neumann algebra, \( VN(G) \), generated by the left translation operators \( L_s \), \( s \in G \). By Tomiyama’s Theorem, [T1, III. Theorem 3.4, p. 131], \( P \) is actually a \( VN(G) \)-bimodule homomorphism such that \( P(L_s T L_s^{-1}) = L_s (PT)L_s^{-1} \), for all \( T \in \mathcal{L}(l^2(G)) \) and \( s \in G \). Denoting the multiplication representation of \( l^\infty(G) \) on \( l^2(G) \) by \( \pi, \pi(\varphi)f = \varphi f \) for \( \varphi \in l^\infty(G), f \in l^2(G) \), and the canonical trace on \( VN(G) \) by \( \tau, \tau(T) = (T\varepsilon | \varepsilon) \) for \( T \in VN(G) \), the composition \( M = \tau \circ P \circ \pi \) will be a left invariant mean on \( l^\infty(G) \), as is well known, cf. [Sch, 7. Lemma, p. 23]. The other direction follows, of course, from 2.3. ■
3. Questions and remarks. $G$ will denote a locally compact group and $p'$ the exponent conjugate to $1 < p < \infty$.

3.1. Question (Dales and Polyakov). Let $G$ act by left translation on $L^p(G)$, $1 < p < \infty$. Does the flatness of $L^p(G)$ as a Banach left module over $L^1(G)$ imply the amenability of $G$? Or, equivalently, does the injectivity of $L^{p'}(G)$ imply the amenability of $G$? H. G. Dales and M. E. Polyakov showed in [D-P], Theorem 5.9 and Theorem 5.12, that for no discrete group $G$ containing the free group on two generators $L^p(G)$ is injective, and they conjecture that this remains true for all non-amenable discrete groups, [D-P], p. 425]. All that is known today is contained in the recent preprint of P. Ramsden [Ra].

3.2. Remark. Let $G$ be a discrete amenable group acting contractively on a Banach space $X$. If $\lambda : \mathcal{L}(L^1(G), X^*) \to X^*$ is the map associated, as in (2.3), with an invariant mean on $G$, then $\lambda(T)$ is contained in the weak *-closed convex hull of the set $\{T(\varepsilon_t)\varepsilon_{t-1} : t \in G\}$, for every $T \in \mathcal{L}(L^1(G), X^*)$.

Proof. Let $T : L^1(G) \to X^*$ be bounded linear and let $\phi : G \to X^*$ be defined by $\phi(t) = T(\varepsilon_t)$, $\varepsilon_t$ being the point measure at $t \in G$. If $\lambda$ is associated with the left invariant mean $M$ on $G$, 2.3, we have

$$\langle x, \lambda T \rangle = \int \langle t^{-1}x, \phi(t) \rangle dM(t) = \int \langle x, \phi(t)t^{-1} \rangle dM(t),$$

for $T \in \mathcal{L}(L^1(G), X^*)$ and $x \in X$. If the assertion were wrong there would exist such $T$ and $x$ and two real numbers $\alpha < \beta$ satisfying

$$\text{Re} \langle x, \lambda T \rangle \leq \alpha < \beta \leq \text{Re} \langle x, \phi(t)t^{-1} \rangle \quad (t \in G),$$

such that averaging with respect to $M$ gives the desired contradiction. (We have written $\phi(t)t^{-1} = T(\varepsilon_t)\varepsilon_{t-1}$.)

3.3. Remark. Let $G$ act by left translation on $L^p(G)$, $1 < p < \infty$. If $G$ is amenable, but non-compact, then any map $\lambda : \mathcal{L}(L^1(G), L^{p'}(G)) \to L^{p'}(G)$ associated with an invariant mean on $G$, 2.3, vanishes on the subspace of compact operators from $L^1(G)$ into $L^{p'}(G)$.

Proof. Since the space of compact operators from $L^1(G)$ into $L^{p'}(G)$ can be identified with $L^\infty(G) \otimes L^{p'}(G)$, the injective tensor product of $L^\infty(G)$ with $L^{p'}(G)$, and $\lambda$ is linear and continuous, it suffices to show that $\lambda(\varphi \otimes g) = 0$ for all $\varphi \in L^\infty(G)$ and $g \in L^{p'}(G)$. But, for any $f \in L^p(G)$, the definition of $\lambda$ associated with the invariant mean $M$, (2.3) with $x = f$ and $\phi = \varphi \otimes g$, implies

$$\langle f, \lambda(\varphi \otimes g) \rangle = \int \langle L_{t-1}f, \varphi(t)g \rangle dM(t) = \int \langle L_{t-1}f, g \rangle \varphi(t) dM(t) = 0,$$

since the convolution $g \ast \tilde{f}$, $\tilde{f}(t) = f(t^{-1})$, vanishes at infinity.

3.4. Denoting by $WAP(G)$ the space of weakly almost periodic functions on $G$ and by $\otimes$ the injective tensor product of Banach spaces, we have for any $1 < p < \infty$ isometric inclusions

$$C^o(G) \otimes L^{p'}(G) \subset WAP(G) \otimes L^{p'}(G) \subset L^\infty(G) \otimes L^{p'}(G) \subset L^\infty(G, L^{p'}(G)),$$
the last space being equal to $\mathcal{L}(L^1(G), L^{p'}(G))$, in this case, and $C^0(G)$ denoting the space of continuous functions on $G$ vanishing at infinity.

**Remark.** Let $G$ be non-compact and $1 < p < \infty$. Then any bounded linear map $\lambda : \mathcal{L}(L^1(G), L^{p'}(G)) \to L^{p'}(G)$ satisfying $\lambda(T \circ L_s) = L_{s-1}(\lambda T)$, for $T \in \mathcal{L}(L^1(G), L^{p'}(G))$ and $s \in G$, vanishes on the subspace $WAP(G) \otimes L^{p'}(G).

**Proof.** It suffices to show that $\lambda(\varphi \otimes g) = 0$ for all $\varphi \in WAP(G)$ and $g \in L^{p'}(G)$. For any fixed $g \in L^{p'}(G)$, we consider the bounded linear operator $\lambda_1$,

$$\lambda_1 : L^\infty(G) \to L^{p'}(G), \quad \lambda_1(\varphi) = \lambda(\varphi \otimes g) \quad (\varphi \in L^\infty(G)),$$

satisfying $\lambda_1(L_s \varphi) = L_s(\lambda_1 \varphi), s \in G$ and $\varphi \in L^\infty(G)$, because of

$$\lambda_1(L_s \varphi) = \lambda(L_s(\varphi \otimes g)) = \lambda(\varphi \otimes g \circ L_{s-1}) = L_s \lambda(\varphi \otimes g) = L_s(\lambda_1 \varphi).$$

Let $\varphi \in WAP(G)$. The set $\{L_s \varphi : s \in G\}$ being relatively weakly compact in $L^\infty(G)$, we obtain in virtue of the Dunford-Pettis property of $L^\infty(G)$, [G, Proposition 1, p. 135, and Théorème 1(a), p. 139], and the weak compactness of $\lambda_1$, that the set $\{\lambda_1(L_s \varphi) : s \in G\} = \{L_s(\lambda_1 \varphi) : s \in G\}$ is relatively norm compact in $L^{p'}(G)$, implying $\lambda_1 \varphi = 0$, by [L] Theorem 4.6, p. 136] or [R Lemma 1.1.(b), p. 602].

**3.5. Remark.** Let $G$ be non-compact and $2 < p < \infty$. Then any bounded linear map $\lambda : \mathcal{L}(L^1(G), L^{p'}(G)) \to L^{p'}(G)$ satisfying $\lambda(T \circ L_s) = L_{s-1}(\lambda T)$, $T \in \mathcal{L}(L^1(G), L^{p'}(G))$ and $s \in G$, vanishes on the subspace of all compact operators from $L^1(G)$ into $L^{p'}(G).

**Proof.** For any fixed $g \in L^{p'}(G)$, let $\lambda_1 : L^\infty(G) \to L^{p'}(G), \lambda_1(\varphi) = \lambda(\varphi \otimes g), \varphi \in L^\infty(G)$, be the left invariant operator considered in the proof of (3.4). Since $1 < p' < 2$, it follows from [L-vR] Theorem 3, p. 308, that $\lambda_1 = 0$ such that $\lambda(\varphi \otimes g) = 0$ for all $\varphi \in L^\infty(G)$ and $g \in L^{p'}(G)$, implying the assertion.

**3.6. Question (Gordin).** Let $G$ act by left translation on $C^*_r(G)$, the reduced $C^*$-algebra of $G$. Does the flatness of $C^*_r(G)$ as a Banach left module over $L^1(G)$ imply the amenability of $G$? This question, related to (2.7), is due to M. Gordin, [Go]. The proof for $C^*_r(G)$ in (2.7) does not apply directly since the constant function $1_G$ is in $(C^*_r(G))^*$ if and only if $G$ is amenable.

**3.7. Question.** Let $G$ act by left translation on $Q(G)$, the predual of $MA(G)$ described in (1.9). Does the flatness of $Q(G)$ as a Banach left module over $L^1(G)$ imply the amenability of $G$? The proof for $Q_0(G)$, as given in (2.7), does not apply since the dual $Q(G)^* = MA(G)$ may contain functions which are not weakly almost perodic.

**3.8. Question.** Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space $h$. By a theorem of A.Ya. Helemskii, [He3 Theorem, p. 77], the injectivity of the von Neumann algebra $\mathcal{M}$ implies the injectivity of any normal dual Banach module over the Banach algebra $\mathcal{M}$. Is any such module already 1-injective in the sense of (2.2)? To be more explicit, let $\mathcal{M}$ be injective, $X$ be a Banach left $\mathcal{M}$-module with dual right module $X^*$ such that, for all $(x, x^*) \in X \times X^*$, the linear form $a \mapsto \langle ax, x^* \rangle$, $a \in \mathcal{M}$, is $\sigma$-weakly continuous on $\mathcal{M}$. Does there exist a linear map $\lambda$ satisfying $\lambda(T \circ L_a) = (\lambda T)a$, for $T \in \mathcal{L}(\mathcal{M}, X^*), a \in \mathcal{M}$,
and being left inverse to $\pi^*$, $(\pi^* x^*)(a) = x^* a$, for $x^* \in X^*$, $a \in \mathcal{M}$,
\[ X^* \xrightarrow{\pi^*} \mathcal{L}(\mathcal{M}, X^*) \xrightarrow{\lambda} X^* \]
such that $\|\lambda\| = 1$?

**Note added in proof.** The answer to 3.8 seems to be yes; cf. the forthcoming paper “On injective von Neumann algebras”, to appear in Proc. Amer. Math. Soc.

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**References**


