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ON THE PROJECTIVITY AND FLATNESS OF SOME GROUP MODULES

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Abstract. In the sequel of the work of H. G. Dales and M. E. Polyakov we give a few more examples of modules over the Banach algebra $L^1(G)$ whose projectivity resp. flatness implies the compactness resp. amenability of the locally compact group G.

Let $L^1(G)$ be the L^1-algebra associated with a left invariant Haar measure on the locally compact group G. In the sequel of the work of H. G. Dales and M. E. Polyakov, [D-P], we will give a few more examples supporting Helemskii's philosophy on the relation between the projectivity of $L^1(G)$-modules and the compactness of G on the one hand, and between the flatness of $L^1(G)$-modules and the amenability of G on the other; see for instance [He1, p. 238], or [He1, IV. Theorem 5.13, p. 190] and [He1, VII. Theorem 2.35, p. 260].

If A is an abstract Banach algebra and $A_+ = A \oplus \mathbb{C}$ its unitization, L_a will denote the operator of left multiplication by $a \in A$ on either A or A_+. A Banach left A-module X will always be contractive such that the action $\pi : A \widehat{\otimes} X \to X$, $\pi(a \otimes x) = ax$, is a linear contraction; $\widehat{\otimes}$ denotes the projective tensor product of Banach spaces, and $\mathcal{L}$ the space of all bounded linear mappings. For any Banach left A-module X, its dual Banach space, X^*, becomes a Banach right A-module by defining $\langle x, x^*a\rangle = \langle ax, x^*\rangle$, for $x \in X$, $x^* \in X^*, a \in A$. We shall always use the canonical isometrical isomorphism $(A \widehat{\otimes} X)^* = \mathcal{L}(A, X^*)$.

A Banach left A-module X is called essential if the linear span of the products ax ($a \in A$, $x \in X$) is dense in X. In case $A = L^1(G)$, every essential Banach left $L^1(G)$-module is a Banach G-module such that for any $x \in X$ the map $s \mapsto sx$ is continuous from G into X and satisfies $\|sx\| = \|x\|$ for all $s \in G$. Conversely, every Banach G-module

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is an essential Banach $L^1(G)$-module. Left translation by $s \in G$ will be denoted by $L_s f(t) = f(s^{-1}t)$, for any function f on G.

1. Projectivity over $L^1(G)$. Instead of giving the original definition, cf. [D-P, Definition 1.1, p. 392], we shall use the following criterion, [D-P, Proposition 1.2, p. 392].

1.1. Criterion. Let A be a Banach algebra and X be a Banach left A-module. X is projective if and only if there is a bounded linear map ρ such that $\pi \circ \rho = 1_X$ and $\rho(ax) = (L_a \widehat{\otimes} 1_X)(\rho x)$, for $x \in X, a \in A$:

$$X \xrightarrow{\rho} A_+ \widehat{\otimes} X \xrightarrow{\pi} X.$$

X is called c-projective, for some constant $c > 0$, if there is such a ρ of norm $\|\rho\| \leqslant c$, cf. [W, Proposition 2.8, p. 158]. If X is essential, A_+ may be replaced by A.

1.2. Let A be $L^1(G)$. If G is compact, then every essential Banach left $L^1(G)$-module X is 1-projective. Denoting the continuous contractive action of $s \in G$ on $x \in X$ by sx, and identifying $L^1(G) \widehat{\otimes} X$ with $L^1(G, X)$, we see that $(\rho x)(t) = t^{-1}x$ $(x \in X, t \in G)$, defines a linear contraction ρ from X into $L^1(G, X)$ such that for all $s \in G$ and $x \in X$,

$$\rho(sx)(t) = t^{-1}(sx) = (s^{-1}t)^{-1}x = (L_s \widehat{\otimes} 1_X)(\rho x)(t) \qquad (t \in G),$$

and

$$\pi(\rho x) = \int t(\rho x)(t)dt = \int t(t^{-1}x)dt = \int x\,dt = x,$$

provided the Haar measure of G having been chosen equal to one. This implies the 1-projectivity of X.

Here we are rather interested in the converse question: given an $L^1(G)$-module X, when does the projectivity of X imply the compactness of G? The main tool for deciding this is the following lemma of Yu. V. Selivanov, cf. [S1, Lemma 1.4, p. 389] and [S2, Corollary 1. p. 212].

1.3. Lemma (Selivanov). *Let A be a Banach algebra and X be an essential Banach left A-module such that either A or X satisfy Grothendieck's approximation condition. If X is projective then there exists for every $x \neq 0$ in X an A-module homomorphism $\varphi : X \to A$ with $\varphi(x) \neq 0$.*

1.4. Proposition. *Let G be a locally compact group. If there exists a projective essential Banach left $L^1(G)$-module X with X^* being either norm separable or weakly sequentially complete, then G is compact.*

Proof. Since $A = L^1(G)$ enjoys the approximation property, the projectivity of X implies by (1.3) the existence of a non-zero $L^1(G)$-module homomorphism $\varphi : X \to L^1(G)$ such that $\varphi(sx) = L_s(\varphi x)$, for all $s \in G, x \in X$. Since the dual map of φ, $\varphi^* : L^\infty(G) \to X^*$, is weakly compact, in the case of X^* being norm separable by [G, Corollaire 1, p. 168] and in the case of X^* being weakly sequentially complete by [D-S, VI.7.6 Theorem, p. 494], φ is weakly compact as well. Since for any $x \in X$ its G-orbit $\{sx : s \in G\}$ is norm bounded in X, it follows from $\varphi(sx) = L_s(\varphi x)$, $s \in G$, that its image is relatively weakly compact in $L^1(G)$. Since $\varphi(x) \neq 0$ in $L^1(G)$ for some $x \in X$, the Dunford-Pettis theorem implies the compactness of G, cf. [La, Theorem 4.8, p. 137] or [R, Lemma 1.1.(a), p. 602]. ∎

1.5. Example ([D-P, Theorem 5.1, p. 415]). Let $X = L^p(G)$, $1 < p < \infty$, be endowed with any action making it an essential $L^1(G)$-module. Then we have:

$$\boxed{L^p(G) \text{ projective} \iff G \text{ compact}}$$

Let us remark that $L^1(G)$ is a projective left $L^1(G)$-module for any G, by [He1, IV. Theorem 2.17, p. 175].

1.6. Example. Let π be a continuous unitary representation of the locally compact group G on a Hilbert space h and let $X = \mathcal{L}^p(h)$, $1 < p < \infty$, be the space of all operators T on h such that $\operatorname{trace}(T^*T)^{\frac{p}{2}} < \infty$. Then X is a reflexive Banach space that becomes an essential left $L^1(G)$-module under the action $sT = \pi(s)T\pi(s^{-1})$, for $s \in G$, $T \in \mathcal{L}^p(h)$. Endowing the C^*-algebra, $K(h)$, of all compact operators on h with the same action and noting that the dual of any C^*-algebra is weakly sequentially complete, [T1, III. Corollary 5.2, p. 148], we have

$$\boxed{K(h),\ \mathcal{L}^p(h) \text{ projective} \iff G \text{ compact}}$$

1.7. Example. Let X be either $C^*(G)$, the full C^*-algebra of G, or $C^*_r(G)$, the reduced C^*-algebra of G, endowed with left translation. Then $C^*(G)$ and $C^*_r(G)$ are essential left $L^1(G)$-modules whose duals are weakly sequentially complete such that

$$\boxed{C^*(G),\ C^*_r(G) \text{ projective} \iff G \text{ compact}}$$

The same applies to the C^*-algebra, $K(L^2(G))$, of compact operators on $L^2(G)$ with G acting as $sT = L_sT\,L_{s^{-1}}$, $T \in K(L^2(G))$, a special case of 1.6.

1.8. Example. Let X be $K(L^p(G))$ the space of compact operators on $L^p(G)$, $1 < p < \infty$. Then $K(L^p(G))$ is an essential $L^1(G)$-module under $sT = L_sT\,L_{s^{-1}}$, whose dual Banach space is isometrically isomorphic to $L^p(G) \widehat{\otimes} L^{p'}(G)$, which is norm separable whenever the topology of G has a countable base:

$$\boxed{K(L^p(G)) \text{ projective and } G \text{ 2nd-countable} \implies G \text{ compact}}$$

1.9. Example. Let $A(G)$ be the Fourier algebra of G, and $VN(G)$ its von Neumann algebra such that $A(G)^* = VN(G)$. If φ is a function on G satisfying $\varphi u \in A(G)$ for all $u \in A(G)$, then φ is continuous and bounded and defines a bounded linear operator, m_φ, on the Banach space $A(G)$, $m_\varphi(u) = \varphi u$ $(u \in A(G))$. With this in mind we define

$$MA(G) = \{\varphi \in C^b(G) : \varphi u \in A(G) \quad \forall u \in A(G)\},$$
$$M_0A(G) = \{\varphi \in MA(G) : (m_\varphi)^* : VN(G) \to VN(G) \text{ completely bounded}\},$$

with norms

$$\|\varphi\|_{MA(G)} = \|m_\varphi : A(G) \to A(G)\|,$$
$$\|\varphi\|_{M_0A(G)} = \|(m_\varphi)^* : VN(G) \to VN(G)\|_{cb}.$$

$M_0A(G)$ is called the space of completely bounded multipliers, and $MA(G)$ the space of all multipliers of $A(G)$. Denoting by $Q_0(G)$ and $Q(G)$ the completions of $L^1(G)$ with respect to the norms

$$\|f\|_{Q_0(G)} = \sup\left\{\left|\int f(t)\varphi(t)dt\right| : \varphi \in M_0A(G),\ \|\varphi\|_{M_0A(G)} \leqslant 1\right\},$$

$$\|f\|_{Q(G)} = \sup\left\{\left|\int f(t)\varphi(t)dt\right| : \varphi \in MA(G),\ \|\varphi\|_{MA(G)} \leqslant 1\right\} \quad (f \in L^1(G)),$$

we get two translation invariant Banach spaces whose duals are isometrically isomorphic with $M_0A(G)$ and $MA(G)$, respectively:

$$Q_0(G)^* = M_0A(G), \quad Q(G)^* = MA(G),$$

cf. [dC-H, 1.10 Proposition, p. 466]. It follows from 1.9 Lemma, p. 465 in [dC-H], that $M_0A(G)$ and $MA(G)$ are weakly sequentially complete. Since left translation is continuous and isometric on $Q_0(G)$ and $Q(G)$, these are essential left $L^1(G)$-modules such that we have

$$\boxed{Q_0(G), Q(G) \text{ projective} \iff G \text{ compact}}$$

2. Flatness over $L^1(G)$. Rather than giving the original definition, [He1, VII. Definition 1.1, p. 239], we shall use the following criterion, due to O. Yu. Aristov [A, Lemma 1.2, p. 1558], and its dual.

2.1. Criterion (Aristov). Let A be a Banach algebra and X be a Banach left A-module. X is flat if and only if there is a bounded linear map ρ from X into the bidual $(A_+ \widehat{\otimes} X)^{**}$ such that $\pi^{**} \circ \rho = \iota_X$, the canonical embedding of X into X^{**}, and $\rho(ax) = (L_a \widehat{\otimes} 1_X)^{**}(\rho x)$, for $x \in X$ and $a \in A$:

$$X \xrightarrow{\rho} (A_+ \widehat{\otimes} X)^{**} \xrightarrow{\pi^{**}} X^{**}.$$

X is called c-flat, for some constant $c > 0$, if there is such a ρ of norm $\|\rho\| \leqslant c$, cf. [W, Definition 4.2, p. 164]. If X is essential, A_+ may be replaced by A.

2.2. Criterion (dual). Let A be a Banach algebra, X be a Banach left A-module and X^* its dual right A-module. X is flat if and only if there is a bounded linear map λ from $\mathcal{L}(A_+, X^*)$ into X^* such that $\lambda \circ \pi^* = 1_{X^*}$ and $\lambda(T \circ L_a) = (\lambda T)a$, for all $T \in \mathcal{L}(A_+, X^*)$ and $a \in X$:

$$X^* \xrightarrow{\pi^*} \mathcal{L}(A_+, X^*) \xrightarrow{\lambda} X^*.$$

In this case, X^* is called an injective right A-module, and c-injective if there is such a λ of norm $\|\lambda\| \leqslant c$. If X is essential, A_+ may again be replaced by A.

Clearly, a left A-module X is c-flat if and only if its dual right A-module X^* is c-injective. For a discussion of injectivity see, for instance, Definition 1.5 and Propositions 1.6 and 1.7 in [D-P, p. 394].

2.3. Let $A = L^1(G)$. If G is amenable then every essential Banach left $L^1(G)$-module X is 1-flat. Indeed, let M be a left invariant mean on $L^\infty(G)$. Using the isometric isomorphism of $\mathcal{L}(L^1(G), X^*)$ with $L^\infty_{w^*}(G, X^*)$, the space of all bounded functions $\Phi : G \to X^*$ such that, for any $x \in X$, $t \mapsto \langle x, \Phi(t)\rangle$ is measurable on G, there corresponds to every

$T \in \mathcal{L}(L^1(G), X^*)$ a unique function $\Phi \in L^\infty_{w^*}(G, X^*)$ via the formula

$$\langle x, Tf\rangle = \int_G f(t)\langle x, \Phi(t)\rangle dt \qquad (x \in X, f \in L^1(G)),$$

cf. [T1, IV. Proposition 7.16, p. 262]. Considering X as a (continuous, contractive) Banach G-module, the function $t \mapsto \langle t^{-1}x, \Phi(t)\rangle$ is bounded and measurable in $t \in G$ such that

$$\langle x, \lambda(\Phi)\rangle = \int \langle t^{-1}x, \Phi(t)\rangle dM(t) \qquad (x \in X, \Phi \in L^\infty_{w^*}(G, X^*))$$

defines a linear contraction λ,

$$X^* \xrightarrow{\pi^*} L^\infty(G, X^*) \xrightarrow{\lambda} X^*,$$

satisfying $\lambda \circ \pi^* = 1_{X^*}$ and $\lambda \circ (L_s \widehat{\otimes} 1_X)^*(\Phi) = (\lambda\Phi)s$, for all $\Phi \in L^\infty_{w^*}(G, X^*)$ and $s \in G$. It follows that X^* is 1-injective and X 1-flat.

2.4. Remark. In spite of the similarity of the diagrams in 1.1 and 2.1 one must not expect that every flat Banach left module X over a Banach algebra A admit a non-zero A-module homomorphism $\varphi : X \to A^{**}$. Indeed, let G be an amenable locally compact group and let $A = L^1(G)$ and $X = L^p(G)$, $2 < p < \infty$. Then every non-zero left $L^1(G)$-module homomorphism $\varphi : L^p(G) \to L^1(G)^{**}$ gives rise to a non-zero left invariant operator $\varphi^t : L^\infty(G) \to L^{p'}(G)$ which forces G to be compact, cf. [L-vR, Theorem 3, p. 308] or [R, Proposition 1.2, p. 603].

Again, we are interested in the question: given an $L^1(G)$-module X, when does flatness of X imply amenability of G ? Several examples are given in [D-P], and we will add a few more.

2.5. Example. Let $X = K(L^p(G))$, $1 < p < \infty$, be the space of compact operators on $L^p(G)$ with the action $sT = L_s T L_{s^{-1}}$ for $s \in G$, $T \in K(L^p(G))$. Then $K(L^p(G))$ becomes an essential Banach left $L^1(G)$-module whose dual module $L^p(G) \widehat{\otimes} L^{p'}(G)$ is endowed with the right action $(f \otimes g)s = L_{s^{-1}} f \otimes L_{s-1} g$, for $s \in G$ and $f \otimes g \in L^p(G) \widehat{\otimes} L^{p'}(G)$. By the left invariance of Haar measure, the duality $\tau : L^p(G) \widehat{\otimes} L^{p'}(G) \to \mathbb{C}$ is G-invariant such that we infer from Theorem 4.6 in [D-P, p. 414]: if $L^p(G) \widehat{\otimes} L^{p'}(G)$ is injective under the above action then G is amenable. Dually, if $K(L^p(G))$ is flat then G is amenable, i.e. together with 2.3:

$$\boxed{K(L^p(G)) \text{ flat} \iff G \text{ amenable}}$$

2.6. Example. Let π be a continuous unitary representation of G on the Hilbert space h, $K(h)$ the C^*-algebra of compact operators on h with $sT = \pi(s)T\pi(s^{-1})$, for $s \in G$, $T \in K(h)$, such that its dual module $h \widehat{\otimes} \overline{h}$ has the action $(\xi \otimes \overline{\eta})s = \pi(s^{-1})\xi \otimes \overline{\pi}(s^{-1})\overline{\eta}$, for $s \in G$ and $\xi \otimes \overline{\eta} \in h \otimes \overline{h}$ ($\overline{h}$ and $\overline{\pi}$ denoting the complex-conjugates of h and π, respectively). Therefore the trace $\tau : h \widehat{\otimes} \overline{h} \to \mathbb{C}$ is G-invariant, and we conclude as in 2.5:

$$\boxed{K(h) \text{ flat} \iff G \text{ amenable}}$$

2.7. Example. Let $C^*(G)$ be the full C^*-algebra of G, and $Q_0(G)$ be the Banach space defined in 1.9. Endowing both of them with left translation by G, we have

$$\boxed{C^*(G), Q_0(G) \text{ flat } \iff G \text{ amenable}}$$

Proof. One direction follows from 2.3. To prove the other one we shall show that the injectivity of the dual modules, $C^*(G)^*$ and $Q_0(G)^*$, implies the amenability of G. Identifying $C^*(G)^*$ with the space, $B(G)$, of coefficients of all continuous unitary representations of G, and $Q_0(G)^*$ with $M_0A(G)$, we see that $B(G)$ is contained in $M_0A(G)$. By a theorem of Bożejko and Fendler, [B-F] or [J], every $\varphi \in M_0A(G)$ can be written as $\varphi(t^{-1}s) = (\Phi_1(s)|\Phi_2(t))$ for $(s,t) \in G \times G$, where $\Phi_1, \Phi_2 : G \to h$ are two continuous bounded functions with values in some Hilbert space h. It follows that every such φ is weakly almost periodic: $M_0A(G) \subset WAP(G)$. Denoting by 1_G the constant function corresponding to the trivial representation of dimension one, we have $1_G \in B(G) \subset M_0A(G) \subset WAP(G)$, and so it suffices to prove the statement for $M_0A(G)$.

If $M_0A(G)$ is injective as a right Banach G-module, we have a map λ as in 2.2,

$$M_0A(G) \xrightarrow{\pi^*} \mathcal{L}(L^1(G), M_0A(G)) \xrightarrow{\lambda} M_0A(G))$$

such that $\lambda(\pi^*\varphi) = \varphi$ for $\varphi \in M_0A(G)$, and $\lambda(T \circ L_s) = L_{s^{-1}}(\lambda T)$, for $T \in \mathcal{L}(L^1, M_0A)$ and $s \in G$. Associating with every $\varphi \in L^\infty(G)$ the operator T_φ, as kindly suggested to us by N. Monod, [M],

$$T_\varphi : L^1(G) \to M_0A(G), \quad T_\varphi(f) = \langle f, \varphi \rangle 1_G \qquad (f \in L^1(G)),$$

we get by left invariance of Haar measure

$$T_{L_s\varphi}(f) = \langle f, L_s\varphi \rangle 1_G = \langle L_{s^{-1}} f, \varphi \rangle 1_G = T_\varphi(L_{s^{-1}} f) \quad (s \in G, f \in L^1(G)),$$

and

$$T_{1_G}(f) = \langle f, 1_G \rangle 1_G = 1_G \otimes 1_G(f) \qquad (f \in L^1(G)),$$

such that $T_{1_G} = \pi^*(1_G)$. Denoting by m the left invariant mean on $WAP(G)$, we see that the composition $M = m \circ \lambda \circ T$ is a non-zero left invariant functional on $L^\infty(G)$. Indeed, we have, for any $\varphi \in L^\infty(G)$ and $s \in G$,

$$M(L_s\varphi) = m(\lambda(T_{L_s\varphi})) = m(\lambda(T_\varphi \circ L_{s^{-1}})) = m(L_s(\lambda(T_\varphi))) = m(\lambda(T_\varphi)) = M(\varphi),$$

and

$$M(1_G) = m(\lambda(T_{1_G})) = m(\lambda(\pi^*(1_G))) = m(1_G) = 1,$$

from which the amenability of G follows. ■

2.8. In [S2, Theorem 1, p. 211], Selivanov showed that for any projective module X over a Banach algebra A there is a bounded linear projection from $\mathcal{L}(X)$ onto the subspace, $\mathcal{L}_A(X)$, of A-module homomorphisms. In the same vein, there is for any flat X a bounded linear projection from $\mathcal{L}(X^*)$ onto $\mathcal{L}_A(X^*)$, the space of homomorphisms of the dual module X^*, and if X is c-flat the projection can be chosen of norm $\leqslant c$. Since, in this case, X^* is injective, this follows from the following lemma which we shall formulate only for left modules.

LEMMA. *Let Y be a Banach left module over the Banach algebra A. If, for some constant $c > 0$, Y is c-injective, then there is a bounded linear projection, P, of norm $\|P\| \leqslant c$ from $\mathcal{L}(Y)$ onto the subspace, $\mathcal{L}_A(Y)$, of A-module homomorphisms.*

Proof. According to the definition, cf. [D-P, Proposition 1.6, p. 394], there is a bounded linear map λ of norm $\|\lambda\| \leqslant c$, satisfying $\lambda(T \circ R_a) = a(\lambda T)$ and $\lambda(\alpha y) = y$, for $T \in \mathcal{L}(A_+, Y)$, $a \in A$ and $y \in Y$,

$$Y \xrightarrow{\alpha} \mathcal{L}(A_+, Y) \xrightarrow{\lambda} Y,$$

α being given by $(\alpha y)(a) = ay$, for $y \in Y$ and $a \in A_+$, and R_a denoting right multiplication by a on A_+. Defining P by $(PT)(y) = \lambda(T \circ \alpha y)$, for $T \in \mathcal{L}(Y)$, $y \in Y$, we see that P is a bounded linear operator on $\mathcal{L}(Y)$ of norm $\|P\| \leqslant \|\lambda\|$ satisfying, for $T \in \mathcal{L}(Y)$ and $a \in A$,

$$(PT)(ay) = \lambda(T \circ \alpha(ay)) = \lambda(T \circ \alpha y \circ R_a) = a\lambda(T \circ \alpha y) = a(PT)(y),$$

and, for $T \in \mathcal{L}_A(Y)$, in virtue of $T \circ \alpha y = \alpha(Ty)$,

$$(PT)(y) = \lambda(T \circ \alpha y) = \lambda(\alpha(Ty)) = Ty,$$

such that P is a linear projection from $\mathcal{L}(Y)$ onto $\mathcal{L}_A(Y)$ of norm $\|P\| \leqslant c$. ∎

2.9. A von Neumann algebra $\mathcal{M}$ on a Hilbert space h is called injective if there is a linear projection of norm one from $\mathcal{L}(h)$ onto $\mathcal{M}$. By a theorem of Helemskii, [He3, Corollary 1, p. 77], the injectivity of $\mathcal{M}$ implies the injectivity of the Banach left $\mathcal{M}$-module h. As a partial converse we have

Corollary. *Let $\mathcal{M}$ be a von Neumann algebra on h. If the Banach left $\mathcal{M}$-module h is 1-injective, then $\mathcal{M}$ is injective.*

Proof. Let the elements of $\mathcal{M}$ act on h as operators. From (2.8), with $A = \mathcal{M}$ and $Y = h$, follows the existence of a linear projection of norm $c = 1$ from $\mathcal{L}(h)$ onto $\mathcal{L}_{\mathcal{M}}(h) = \mathcal{M}'$, the commutant of $\mathcal{M}$. Hence $\mathcal{M}'$ is injective, and so is $\mathcal{M}$, cf. e.g. [T2, XV. Proposition 3.2(iii), p. 174]. ∎

Remark. The question of how the bound of the projection can be relaxed is discussed by Pisier in [P] and by Christensen and Sinclair in [C-S1] and [C-S2].

2.10. Example. Let G be a discrete group and let $l^1(G)$ act on $l^2(G)$ by left or right convolution. Then the Banach $l^1(G)$-module $l^2(G)$ is 1-flat if and only if G is amenable:

$$\boxed{l^2(G) \text{ 1-flat} \iff G \text{ amenable}}$$

Proof. Let us consider $l^2(G)$ as a right $l^1(G)$-module such that G acts on $l^2(G)$ by right translation $(R_s f)(t) = f(ts)$, for $s \in G$ and $f \in l^2(G)$. If $l^2(G)$ is 1-flat, it is 1-injective such that, by 2.8, there is a projection, P, of norm one from $\mathcal{L}(l^2(G))$ onto $\mathcal{L}_{l^1(G)}(l^2(G))$, the subspace of all operators commuting with R_s, $s \in G$, which coincides with the von Neumann algebra, $VN(G)$, generated by the left translation operators L_s, $s \in G$. By Tomiyama's Theorem, [T1, III. Theorem 3.4, p. 131], P is actually a $VN(G)$-bimodule homomorphism such that $P(L_s T L_{s^{-1}}) = L_s(PT)L_{s^{-1}}$, for all $T \in \mathcal{L}(l^2(G))$ and $s \in G$. Denoting the multiplication representation of $l^\infty(G)$ on $l^2(G)$ by π, $\pi(\varphi)f = \varphi f$ for $\varphi \in l^\infty(G), f \in l^2(G)$, and the canonical trace on $VN(G)$ by τ, $\tau(T) = (T\varepsilon_e|\varepsilon_e)$ for $T \in VN(G)$, the composition $M = \tau \circ P \circ \pi$ will be a left invariant mean on $L^\infty(G)$, as is well known, cf. [Sch, 7. Lemma, p. 23]. The other direction follows, of course, from 2.3. ∎

3. Questions and remarks. G will denote a locally compact group and p' the exponent conjugate to $1 < p < \infty$.

3.1. Question (Dales and Polyakov). Let G act by left translation on $L^p(G)$, $1 < p < \infty$. Does the flatness of $L^p(G)$ as a Banach left module over $L^1(G)$ imply the amenability of G? Or, equivalently, does the injectivity of $L^{p'}(G)$ imply the amenability of G? H. G. Dales and M. E. Polyakov showed in [D-P], Theorem 5.9 and Theorem 5.12, that for no discrete group G containing the free group on two generators $l^p(G)$ is injective, and they conjecture that this remains true for all non-amenable discrete groups, [D-P, p. 425]. All that is known today is contained in the recent preprint of P. Ramsden [Ra].

3.2. Remark. Let G be a discrete amenable group acting contractively on a Banach space X. If $\lambda : \mathcal{L}(L^1(G), X^*) \to X^*$ is the map associated, as in (2.3), with an invariant mean on G, then $\lambda(T)$ is contained in the weak *- closed convex hull of the set $\{T(\varepsilon_t)\varepsilon_{t^{-1}} : t \in G\}$, for every $T \in \mathcal{L}(L^1(G), X^*)$.

Proof. Let $T : L^1(G) \to X^*$ be bounded linear and let $\phi : G \to X^*$ be defined by $\phi(t) = T(\varepsilon_t)$, ε_t being the point measure at $t \in G$. If λ is associated with the left invariant mean M on G, 2.3, we have

$$\langle x, \lambda T\rangle = \int \langle t^{-1}x, \phi(t)\rangle dM(t) = \int \langle x, \phi(t)t^{-1}\rangle dM(t),$$

for $T \in \mathcal{L}(L^1(G), X^*)$ and $x \in X$. If the assertion were wrong there would exist such T and x and two real numbers $\alpha < \beta$ satisfying

$$\mathrm{Re}\langle x, \lambda T\rangle \leqslant \alpha < \beta \leqslant \mathrm{Re}\langle x, \phi(t)t^{-1}\rangle \qquad (t \in G),$$

such that averaging with respect to M gives the desired contradiction. (We have written $\phi(t)t^{-1} = T(\varepsilon_t)\varepsilon_{t-1}$.) ■

3.3. Remark. Let G act by left translation on $L^p(G)$, $1 < p < \infty$. If G is amenable, but non-compact, then any map $\lambda : \mathcal{L}(L^1(G), L^{p'}(G)) \to L^{p'}(G)$ associated with an invariant mean on G, 2.3, vanishes on the subspace of compact operators from $L^1(G)$ into $L^{p'}(G)$.

Proof. Since the space of compact operators from $L^1(G)$ into $L^{p'}(G)$ can be identified with $L^\infty(G) \check{\otimes} L^{p'}(G)$, the injective tensor product of $L^\infty(G)$ with $L^{p'}(G)$, and λ is linear and continuous, it suffices to show that $\lambda(\varphi \otimes g) = 0$ for all $\varphi \in L^\infty(G)$ and $g \in L^{p'}(G)$. But, for any $f \in L^p(G)$, the definition of λ associated with the invariant mean M, (2.3) with $x = f$ and $\phi = \varphi \otimes g$, implies

$$\begin{aligned}\langle f, \lambda(\varphi \otimes g)\rangle &= \int \langle L_{t^{-1}} f, \varphi(t) g\rangle dM(t) = \int \langle L_{t^{-1}} f, g\rangle \varphi(t) dM(t)\\ &= \int g * \check{f}(t)\varphi(t) dM(t) = 0,\end{aligned}$$

since the convolution $g * \check{f}$, $\check{f}(t) = f(t^{-1})$, vanishes at infinity. ■

3.4. Denoting by $WAP(G)$ the space of weakly almost periodic functions on G and by $\check{\otimes}$ the injective tensor product of Banach spaces, we have for any $1 < p < \infty$ isometric inclusions

$$C^o(G) \check{\otimes} L^{p'}(G) \subset WAP(G) \check{\otimes} L^{p'}(G) \subset L^\infty(G) \check{\otimes} L^{p'}(G) \subset L^\infty(G, L^{p'}(G)),$$

the last space being equal to $\mathcal{L}(L^1(G), L^{p'}(G))$, in this case, and $C^o(G)$ denoting the space of continuous functions on G vanishing at infinity.

Remark. Let G be non-compact and $1 < p < \infty$. Then any bounded linear map $\lambda : \mathcal{L}(L^1(G), L^{p'}(G)) \to L^{p'}(G)$ satisfying $\lambda(T \circ L_s) = L_{s^{-1}}(\lambda T)$, for $T \in \mathcal{L}(L^1(G), L^{p'}(G))$ and $s \in G$, vanishes on the subspace $WAP(G) \check{\otimes} L^{p'}(G)$.

Proof. It suffices to show that $\lambda(\varphi \otimes g) = 0$ for all $\varphi \in WAP(G)$ and $g \in L^{p'}(G)$. For any fixed $g \in L^{p'}(G)$, we consider the bounded linear operator λ_1,

$$\lambda_1 : L^\infty(G) \to L^{p'}(G), \quad \lambda_1(\varphi) = \lambda(\varphi \otimes g) \quad (\varphi \in L^\infty(G)),$$

satisfying $\lambda_1(L_s\varphi) = L_s(\lambda_1\varphi)$, $s \in G$ and $\varphi \in L^\infty(G)$, because of

$$\lambda_1(L_s\varphi) = \lambda(L_s\varphi \otimes g) = \lambda(\varphi \otimes g \circ L_{s^{-1}}) = L_s\lambda(\varphi \otimes g) = L_s(\lambda_1\varphi).$$

Let $\varphi \in WAP(G)$. The set $\{L_s\varphi : s \in G\}$ being relatively weakly compact in $L^\infty(G)$, we obtain in virtue of the Dunford-Pettis property of $L^\infty(G)$, [G, Proposition 1, p. 135, and Théorème 1(a), p. 139], and the weak compactness of λ_1, that the set$\{\lambda_1(L_s\varphi) : s \in G\} = \{L_s(\lambda_1\varphi) : s \in G\}$ is relatively norm compact in $L^{p'}(G)$, implying $\lambda_1\varphi = 0$, by [La, Theorem 4.6, p. 136] or [R, Lemma 1.1.(b), p. 602]. ∎

3.5. Remark. Let G be non-compact and $2 < p < \infty$. Then any bounded linear map $\lambda : \mathcal{L}(L^1(G), L^{p'}(G)) \to L^{p'}(G)$ satisfying $\lambda(T \circ L_s) = L_{s^{-1}}(\lambda T)$, $T \in \mathcal{L}(L^1(G), L^{p'}(G))$ and $s \in G$, vanishes on the subspace of all compact operators from $L^1(G)$ into $L^{p'}(G)$.

Proof. For any fixed $g \in L^{p'}(G)$, let $\lambda_1 : L^\infty(G) \to L^{p'}(G)$, $\lambda_1(\varphi) = \lambda(\varphi \otimes g)$, $\varphi \in L^\infty(G)$, be the left invariant operator considered in the proof of (3.4). Since $1 < p' < 2$, it follows from [L-vR, Theorem 3, p. 308], that $\lambda_1 = 0$ such that $\lambda(\varphi \otimes g) = 0$ for all $\varphi \in L^\infty(G)$ and $g \in L^{p'}(G)$, implying the assertion. ∎

3.6. Question (Gordin). Let G act by left translation on $C_r^*(G)$, the reduced C^*-algebra of G. Does the flatness of $C_r^*(G)$ as a Banach left module over $L^1(G)$ imply the amenability of G? – This question, related to (2.7), is due to M. Gordin, [Go]. The proof for $C^*(G)$ in (2.7) does not apply directly since the constant function 1_G is in $(C_r^*(G))^*$ if and only if G is amenable.

3.7. Question. Let G act by left translation on $Q(G)$, the predual of $MA(G)$ described in (1.9). Does the flatness of $Q(G)$ as a Banach left module over $L^1(G)$ imply the amenability of G? – The proof for $Q_0(G)$, as given in (2.7), does not apply since the dual $Q(G)^* = MA(G)$ may contain functions which are not weakly almost periodic.

3.8. Question. Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space h. By a theorem of A.Ya. Helemskii, [He3, Theorem, p. 77], the injectivity of the von Neumann algebra $\mathcal{M}$ implies the injectivity of any normal dual Banach module over the Banach algebra $\mathcal{M}$. Is any such module already 1-injective in the sense of (2.2)? To be more explicit, let $\mathcal{M}$ be injective, X be a Banach left $\mathcal{M}$-module with dual right module X^* such that, for all $(x, x^*) \in X \times X^*$, the linear form $a \mapsto \langle ax, x^*\rangle$, $a \in \mathcal{M}$, is σ-weakly continuous on $\mathcal{M}$. Does there exist a linear map λ satisfying $\lambda(T \circ L_a) = (\lambda T)a$, for $T \in \mathcal{L}(\mathcal{M}, X^*)$, $a \in \mathcal{M}$, and being left inverse to π^*, $(\pi^* x^*)(a) = x^* a$, for $x^* \in X^*$, $a \in \mathcal{M}$,

$$X^* \xrightarrow{\pi^*} \mathcal{L}(\mathcal{M}, X^*) \xrightarrow{\lambda} X^*$$

such that $\|\lambda\| = 1$?

Note added in proof. The answer to 3.8 seems to be yes; cf. the forthcoming paper "On injective von Neumann algebras", to appear in Proc. Amer. Math. Soc.

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