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CONTRACTIBLE QUANTUM ARENS–MICHAEL ALGEBRAS

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Dedicated to Professor Yu. V. Selivanov on the occasion of his 60th birthday

Abstract. We consider quantum analogues of locally convex spaces in terms of the non-coordinate approach. We introduce the notions of a quantum Arens–Michael algebra and a quantum polynormed module, and also quantum versions of projectivity and contractibility. We prove that a quantum Arens–Michael algebra is contractible if and only if it is completely isomorphic to a Cartesian product of full matrix C^*-algebras. Similar results in the framework of traditional (non-quantum) approach are established, at the moment, only under some additional assumptions.

1. Introduction. In this paper, we consider quantum Arens–Michael ($\overset{h}{\otimes}$-) algebras and describe (in Theorem 1) those of them whose completely continuous derivations are all inner, i.e., the so-called contractible algebras. We prove that a quantum Arens–Michael algebra is contractible if and only if it is completely isomorphic to a Cartesian product of full matrix algebras. Similar results in the homology of "classical" Arens–Michael algebras are established, at the moment, only under some additional assumptions (see, e.g., [4, Theorem IV.5.27], [11, Theorem 5] and [3, Theorem 3.3]). Our theorem can be regarded as a generalization of the corresponding result (of V. I. Paulsen and R. R. Smith [8]) concerning operator Banach algebras.

2. Quantum polynormed spaces. We shall follow the non-coordinate approach to the theory of quantum polynormed spaces; its "matricial" variant is presented in [2].

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In formulating definitions and proving essential facts we shall often take after [7], where a quantization of normed spaces is considered.

Let L be an infinite-dimensional separable Hilbert space, $\mathcal{B} = \mathcal{B}(L)$ the space (algebra) of bounded linear operators on L, and $\mathcal{F} = \mathcal{F}(L)$ the subspace (two-sided ideal) in $\mathcal{B}$ consisting of the finite-rank operators on L.

For $\xi, \eta \in L$, let $\xi \bigcirc \eta$ denote the rank 1 operator $\zeta \mapsto \langle \zeta, \eta \rangle \xi$. Recall that each bounded rank 1 operator on L has such a form.

Let now E be a linear space. We denote by $\mathcal{F}E$ the algebraic tensor product $\mathcal{F} \otimes E$, and we write ax for an elementary tensor $a \otimes x$ ($a \in \mathcal{F}$, $x \in E$).

The space $\mathcal{F}E$ is a $\mathcal{B}$-bimodule with respect to the outer multiplications

$$a \cdot bx = (ab)x, \quad bx \cdot a = (ba)x \qquad (a \in \mathcal{B},\ b \in \mathcal{F},\ x \in E).$$

Let $u \in \mathcal{F}E$. A *support* of the element u is, by definition, every projection (i.e., self-adjoint idempotent) $P \in \mathcal{B}$ such that $P \cdot u \cdot P = u$. The supports P and Q of two elements u and v of $\mathcal{F}E$ are called *orthogonal* if $PQ = 0$.

We shall call the space E a *quantum polynormed space* if $\mathcal{F}E$ is equipped with a family of semi-norms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$) distinguishing elements of $\mathcal{F}E$ and satisfying the following two conditions ("Ruan's axioms"):

(R_1) for each $a \in \mathcal{B}$ and $u \in \mathcal{F}E$, we have $\|a \cdot u\|_\lambda, \|u \cdot a\|_\lambda \le \|a\| \|u\|_\lambda$,
(R_2) for each $u, v \in \mathcal{F}E$ with orthogonal supports, we have $\|u + v\|_\lambda = \max\{\|u\|_\lambda, \|v\|_\lambda\}$.

Such semi-norms are called *quantum semi-norms.*

Note that the second condition can be replaced by a weaker one. Namely, if a semi-norm on $\mathcal{F}E$ satisfies the first axiom of Ruan and the following condition:

(R_2') for each $u, v \in \mathcal{F}E$ with orthogonal supports, we have $\|u + v\|_\lambda \le \max\{\|u\|_\lambda, \|v\|_\lambda\}$,

then it satisfies the second axiom of Ruan as well (see [7, Proposition 1.1.4]).

It is easily seen that the maximum of a finite number of quantum semi-norms is also a quantum semi-norm. So we can consider a family of semi-norms on $\mathcal{F}E$ being saturated (i.e., including the maximum of any two semi-norms contained in it) whenever this is convenient.

For each $\lambda \in \Lambda$, we consider the semi-norm on E (denoted also by $\|\cdot\|_\lambda$) defined, for $x \in E$, by $\|x\|_\lambda = \|px\|_\lambda$, where $p \in \mathcal{F}$ is an arbitrary rank 1 projection. As is known [7, Section 1.2], this semi-norm does not depend on a choice of p. Moreover, we have $\|ax\|_\lambda = \|a\| \|x\|_\lambda$ for all $a \in \mathcal{F}$ and $x \in E$ (cf. [7, Proposition 1.2.4]). Thus E becomes a polynormed space. Since the family of semi-norms $\|\cdot\|_\lambda$ distinguishes elements of $\mathcal{F}E$, the space E is Hausdorff as well.

PROPOSITION 1 (cf. [7, Proposition 1.2.2]). *Let a family of quantum semi-norms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$) on $\mathcal{F}E$ be such that the family of the corresponding semi-norms on E distinguishes elements of E. Then the initial family of semi-norms distinguishes elements of $\mathcal{F}E$.*

Proof. Let $u \in \mathcal{F}E$, $u \neq 0$. As is well known (see, e.g., [6, Proposition 2.7.1]), this element can be represented in the form $u = \sum_{k=1}^n a_k x_k$, where $a_1, \dots, a_n \in \mathcal{F}$ are linearly independent, and $x_1 \in E$ is not zero. Then there exist elements $\xi_\ell, \eta_\ell \in L$ ($\ell = 1, \dots, m$) such that $\sum_{\ell=1}^{m} \langle a_k \xi_\ell, \eta_\ell \rangle$ is 1 if $k = 1$, and 0 otherwise (cf. [6, Proposition 4.2.3]).

Fix an arbitrary $e \in L$ with $\|e\| = 1$. Then $p = e \bigcirc e \in \mathcal{F}$ is a rank 1 projection. Let us consider the following element of $\mathcal{F}E$:

$$v = \sum_{\ell=1}^{m} (e \bigcirc \eta_\ell) \cdot u \cdot (\xi_\ell \bigcirc e) = \sum_{\ell=1}^{m} (e \bigcirc \eta_\ell) \cdot \Big(\sum_{k=1}^{n} a_k x_k \Big) \cdot (\xi_\ell \bigcirc e) =$$

$$= \sum_{k,\ell} [(e \bigcirc \eta_\ell) a_k (\xi_\ell \bigcirc e)]\, x_k = \sum_{k,\ell} [\langle a_k \xi_\ell, \eta_\ell \rangle p]\, x_k = \Big(\sum_{\ell=1}^{m} \langle a_1 \xi_\ell, \eta_\ell \rangle p \Big) x_1 = p x_1.$$

By our assumption, there exists $\lambda \in \Lambda$ for which

$$\|x_1\|_\lambda = \|p x_1\|_\lambda = \|v\|_\lambda \neq 0.$$

However, it follows from the triangle inequality and from the first axiom of Ruan that

$$\|v\|_\lambda \le \sum_{\ell=1}^{m} \|(e \bigcirc \eta_\ell) \cdot u \cdot (\xi_\ell \bigcirc e)\|_\lambda \le \sum_{\ell=1}^{m} \|e \bigcirc \eta_\ell\| \|u\|_\lambda \|\xi_\ell \bigcirc e\|,$$

and hence $\|u\|_\lambda \neq 0$. ■

In particular, if $\| \cdot \|_\lambda$ is a norm on E, then the initial semi-norm $\| \cdot \|_\lambda$ is a norm on $\mathcal{F}E$.

If E is a polynormed space, then the space $\mathcal{F}E$ equipped with a family of quantum semi-norms which induce the initial topology on E is called a *quantization* of E.

Note that for a finite-dimensional space there exists a unique, up to a continuous isomorphism, quantization (this is not true in the general case). In order to prove this, we shall first show that we can use an equivalent family of quantum norms instead of a family of quantum semi-norms.

The maximum of several quantum semi-norms is also a quantum semi-norm. Take a semi-norm $\| \cdot \|_{\lambda_1}$ from our family and consider the subspace

$$E_1 = \{x \in E : \ \|x\|_{\lambda_1} = 0\}.$$

If $E_1 = \{0\}$, then $\| \cdot \|_{\lambda_1}$ is a norm. Otherwise choose a non-zero $x \in E_1$ and a semi-norm $\| \cdot \|_{\lambda_2}$ such that $\|x\|_{\lambda_2} \neq 0$. Obviously,

$$E_2 = \{x \in E : \ \max\{\|x\|_{\lambda_1}, \|x\|_{\lambda_2}\} = 0\}$$

is a subspace in E_1 of dimension less then the dimension of E_1. If $E_2 \neq \{0\}$, again choose a non-zero element in it, and so on. After repeating this procedure a finite number of times we get no more than $n = \dim E$ semi-norms $\| \cdot \|_{\lambda_1}, \dots, \| \cdot \|_{\lambda_k}$ such that

$$\max\{\| \cdot \|_{\lambda_1}, \dots, \| \cdot \|_{\lambda_n}\}$$

is a norm on E, and hence on $\mathcal{F}E$. We replace each quantum semi-norm by the quantum norm so obtained, and thus we get a family of quantum norms which is equivalent to the initial family.

However, it is known (cf. [7, Proposition 2.2.2(iii)]) that all quantum norms on a finite-dimensional space are equivalent, which leads to the assertion we prove. In particular, on $\mathcal{F}\mathbb{C} = \mathcal{F}$ the operator norm is the only (non-zero) quantum semi-norm.

Let E be a quantum polynormed space, and let F be a subspace of E. Then, obviously, there exists a quantum polynormed space structure on F determined by the restriction of semi-norms on $\mathcal{F}E$ to $\mathcal{F}F$.

Suppose now that E is a quantum polynormed space with a saturated family of quantum semi-norms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$), and F is a closed subspace of E. Consider the semi-norms $\|\cdot\|_\lambda^\wedge$ on $\mathcal{F}(E/F)$ defined by

$$\|\tilde{u}\|_\lambda^\wedge = \inf\{\|u\|_\lambda : \ u \in \mathcal{F}E, \ (\mathbf{1}_\mathcal{F} \otimes \pi)u = \tilde{u}\},$$

where $\tilde{u} \in \mathcal{F}(E/F)$, and $\pi \colon E \to E/F$ is the quotient map.

Proposition 2. *The family of semi-norms $\|\cdot\|_\lambda^\wedge$ ($\lambda \in \Lambda$) determines a quantization of the space E/F.*

Proof. First we show that each semi-norm $\|\cdot\|_\lambda^\wedge$ on E/F is the quotient semi-norm corresponding to the semi-norm $\|\cdot\|_\lambda$ on E. This follows from the fact that for each $\tilde{x} \in E/F$, a rank 1 projection $p \in \mathcal{B}$ and $u \in \mathcal{F}E$ such that $p\tilde{x} = (\mathbf{1}_\mathcal{F} \otimes \pi)u$ we have

$$(\mathbf{1}_\mathcal{F} \otimes \pi)(p \cdot u \cdot p) = p \cdot p\tilde{x} \cdot p = p\tilde{x},$$

and moreover $\|p \cdot u \cdot p\|_\lambda \le \|u\|_\lambda$ and $p \cdot u \cdot p = py$ for some $y \in E$. So

$$\|\tilde{x}\|_\lambda^\wedge = \|p\tilde{x}\|_\lambda^\wedge = \inf\{\|py\|_\lambda : \ y \in E, \ \pi(y) = \tilde{x}\} = \inf\{\|y\|_\lambda : \ y \in E, \ \pi(y) = \tilde{x}\},$$

which had to be proved. The subspace F is closed, so the family of quotient semi-norms distinguishes elements of E/F, and therefore also elements of $\mathcal{F}(E/F)$.

It remains to prove that the semi-norms $\|\cdot\|_\lambda^\wedge$ satisfy the axioms of Ruan:

1. Let $\tilde{u} \in \mathcal{F}(E/F)$, $a \in \mathcal{B}$. For any $\varepsilon > 0$ there exists $u \in \mathcal{F}E$ such that $(\mathbf{1}_\mathcal{F} \otimes \pi)u = \tilde{u}$ and $\|u\|_\lambda < \|\tilde{u}\|_\lambda^\wedge + \varepsilon$. Then $(\mathbf{1}_\mathcal{F} \otimes \pi)(a \cdot u) = a \cdot \tilde{u}$, and hence

$$\|a \cdot \tilde{u}\|_\lambda^\wedge \le \|a \cdot u\|_\lambda \le \|a\|\|u\|_\lambda < \|a\|\|\tilde{u}\|_\lambda^\wedge + \varepsilon\|a\|.$$

Therefore, $\|a \cdot \tilde{u}\|_\lambda^\wedge \le \|a\|\|\tilde{u}\|_\lambda^\wedge$. Similarly, we obtain the inequality $\|\tilde{u} \cdot a\|_\lambda^\wedge \le \|a\|\|\tilde{u}\|_\lambda^\wedge$.

2. Let $\tilde{u}, \tilde{v} \in \mathcal{F}(E/F)$, and let $P, Q \in \mathcal{B}$ be orthogonal supports of $\tilde{u}$ and $\tilde{v}$, respectively. For every $\varepsilon > 0$ there exist $u, v \in \mathcal{F}E$ such that $(\mathbf{1}_\mathcal{F} \otimes \pi)u = \tilde{u}$, $(\mathbf{1}_\mathcal{F} \otimes \pi)v = \tilde{v}$, $\|u\|_\lambda < \|\tilde{u}\|_\lambda^\wedge + \varepsilon$ and $\|v\|_\lambda < \|\tilde{v}\|_\lambda^\wedge + \varepsilon$. Then

$$\begin{aligned}(\mathbf{1}_\mathcal{F} \otimes \pi)(P \cdot u \cdot P + Q \cdot v \cdot Q) &= P \cdot (\mathbf{1}_\mathcal{F} \otimes \pi)u \cdot P + Q \cdot (\mathbf{1}_\mathcal{F} \otimes \pi)v \cdot Q \\ &= P \cdot \tilde{u} \cdot P + Q \cdot \tilde{v} \cdot Q = \tilde{u} + \tilde{v}.\end{aligned}$$

Therefore $\|\tilde{u} + \tilde{v}\|_\lambda^\wedge \le \|P \cdot u \cdot P + Q \cdot v \cdot Q\|_\lambda$, and, P and Q being supports of $P \cdot u \cdot P$ and $Q \cdot v \cdot Q$ respectively, this implies

$$\begin{aligned}\|\tilde{u} + \tilde{v}\|_\lambda^\wedge &\le \max\{\|P \cdot u \cdot P\|_\lambda, \|Q \cdot v \cdot Q\|_\lambda\} \\ &\le \max\{\|u\|_\lambda, \|v\|_\lambda\} \le \max\{\|\tilde{u}\|_\lambda^\wedge, \|\tilde{v}\|_\lambda^\wedge\} + \varepsilon.\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we have

$$\|\tilde{u}+\tilde{v}\|_\lambda^\wedge \le \max\{\|\tilde{u}\|_\lambda^\wedge, \|\tilde{v}\|_\lambda^\wedge\},$$

and hence the semi-norm $\|\cdot\|_\lambda^\wedge$ satisfies the second axiom of Ruan. ■

Now suppose that E and F are quantum polynormed spaces, and $\varphi\colon E \to F$ is a linear operator. The operator

$$\varphi_\infty = \mathbf{1}_{\mathcal{F}} \otimes \varphi\colon \mathcal{F}E \to \mathcal{F}F, \quad ax \mapsto a\varphi(x),$$

is called *the amplification* of φ. We shall call $\varphi\colon E \to F$ *completely continuous* if the operator $\varphi_\infty\colon \mathcal{F}E \to \mathcal{F}F$ is continuous. The following assertion is a "polynormed" analogue of [7, Theorem 2.2.1].

Proposition 3. *Let E be a quantum polynormed space with respect to a family of quantum semi-norms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$), and let $f\colon E \to \mathbb{C}$ be a continuous linear functional such that for some $\lambda \in \Lambda$ and $C > 0$ we have $|f(x)| \le C\|x\|_\lambda$ for all $x \in E$. Then f is completely continuous. Moreover, $\|f_\infty(u)\| \le C\|u\|_\lambda$ for all $u \in \mathcal{F}E$.*

Proof. Let $u \in \mathcal{F}E$. Since $f_\infty(u) \in \mathcal{F}\mathbb{C} = \mathcal{F}$ is an operator in the Hilbert space L, we have

$$\|f_\infty(u)\| = \sup\{|\langle f_\infty(u)\xi, \eta\rangle| : \xi, \eta \in L,\ \|\xi\|, \|\eta\| \le 1\}.$$

Fix an arbitrary $e \in L$ with $\|e\| = 1$. Then $p = e \bigcirc e \in \mathcal{F}$ is a rank 1 projection. We have

$$\begin{aligned}\langle f_\infty(u)\xi, \eta\rangle p = \langle f_\infty(u)\xi, \eta\rangle(e \bigcirc e) &= (e \bigcirc \eta)(f_\infty(u)\xi \bigcirc e) \\ &= (e \bigcirc \eta) f_\infty(u)(\xi \bigcirc e) = f_\infty[(e \bigcirc \eta)\cdot u \cdot (\xi \bigcirc e)],\end{aligned}$$

and hence

$$|\langle f_\infty(u)\xi, \eta\rangle| = \|f_\infty[(e \bigcirc \eta)\cdot u \cdot (\xi \bigcirc e)]\|.$$

Note that $(e \bigcirc \eta)\cdot u \cdot (\xi \bigcirc e) = px_{\xi,\eta}$ for some $x_{\xi,\eta} \in E$. By virtue of the first axiom of Ruan,

$$\|x_{\xi,\eta}\|_\lambda = \|px_{\xi,\eta}\|_\lambda \le \|e \bigcirc \eta\|\|u\|_\lambda\|\xi \bigcirc e\| \le \|u\|_\lambda$$

whenever $\|\xi\|, \|\eta\| \le 1$. Hence for the same $\xi, \eta \in L$ we have

$$|\langle f_\infty(u)\xi, \eta\rangle| = \|f_\infty(px_{\xi,\eta})\| = \|f(x_{\xi,\eta})p\| = |f(x_{\xi,\eta})| \le C\|x_{\xi,\eta}\|_\lambda \le C\|u\|_\lambda.$$

Taking the supremum over all $\xi, \eta \in L$ with $\|\xi\|, \|\eta\| \le 1$, we see that $\|f_\infty(u)\| \le C\|u\|_\lambda$. ■

A quantum polynormed space E is said to be *complete* if its underlying polynormed space is complete. There exists a way of completing a quantum polynormed space, i.e., of defining such quantization of $\overline{E}$ which provides the initial one when restricted to E, with the obtained quantum polynormed space possessing the following universal property: for every complete quantum polynormed space F and a completely continuous linear operator $\varphi\colon E \to F$ there exists a unique completely continuous linear operator $\overline{\varphi}\colon \overline{E} \to F$ which makes the diagram

$$\begin{tikzcd} E \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \overline{E} \arrow[r, "\overline{\varphi}"] & F \end{tikzcd}$$

(i standing for the continuous embedding of E into its completion) commutative.

This construction repeats the construction of completing of a quantum normed space described in [7, Section 3] with replacement of converging sequences by nets and of bounded and completely bounded linear operators by continuous and completely continuous linear operators, respectively.

If E_σ $(\sigma \in \mathcal{S})$ is a set of quantum polynormed spaces with defining families of semi-norms $\|\cdot\|_\mu^\sigma$ $(\sigma \in \mathcal{S}, \mu \in \Lambda^\sigma)$, then the Cartesian product $\times_\sigma E_\sigma$ is a quantum polynormed space with respect to the family of quantum semi-norms $\|\cdot\|_\mu^\sigma$ $(\sigma \in \mathcal{S}, \mu \in \Lambda^\sigma)$ on $\mathcal{F}(\times_\sigma E_\sigma)$ determined by

$$\|u\|_\mu^\sigma = \|i_\infty^\sigma(u)\|_\mu^\sigma,$$

where $i^\sigma\colon \times_\sigma E_\sigma \to E_\sigma$ is the natural projection onto E_σ.

Now suppose that E, F and G are quantum polynormed spaces with defining families of quantum semi-norms $\|\cdot\|_\lambda$ $(\lambda \in \Lambda)$, $\|\cdot\|_\mu$ $(\mu \in M)$ and $\|\cdot\|_\nu$ $(\nu \in N)$, respectively. Let $R\colon E \times F \to G$ be a bilinear operator. The bilinear operator

$$\mathcal{R}\colon \mathcal{F}E \times \mathcal{F}F \to \mathcal{F}G, \quad (ax, by) \mapsto (ab)R(x,y),$$

is called the *strong amplification* of R. We shall call $R\colon E \times F \to G$ *strongly completely continuous* if the bilinear operator $\mathcal{R}\colon \mathcal{F}E \times \mathcal{F}F \to \mathcal{F}G$ is continuous. This means, assuming that the families of quantum semi-norms on E and F are saturated, that for any $\nu \in N$ there exist $\lambda \in \Lambda$ and $\mu \in M$ such that

$$\|\mathcal{R}(u,v)\|_\nu \le C\|u\|_\lambda\|v\|_\mu$$

for some $C > 0$ and for all $u \in \mathcal{F}E$, $v \in \mathcal{F}F$.

The following assertion is a "polynormed" analogue of [7, Proposition 4.2.2].

Proposition 4. *Let $f\colon E \to \mathbb{C}$ and $g\colon F \to \mathbb{C}$ be continuous linear functionals such that $|f(x)| \le C_1\|x\|_\lambda$ for all $x \in E$ and $|g(y)| \le C_2\|y\|_\mu$ for all $y \in F$. Then the bilinear functional*

$$f \times g\colon E \times F \to \mathbb{C}, \quad (x,y) \mapsto f(x)g(y),$$

is strongly completely continuous. Moreover, for its strong amplification $\mathcal{R}$ we have

$$\|\mathcal{R}(u,v)\| \le C_1C_2\|u\|_\lambda\|v\|_\mu$$

for all $u \in \mathcal{F}E$, $v \in \mathcal{F}F$.

Proof. For elementary tensors, we have

$$\mathcal{R}(ax, by) = (ab)f(x)g(y) = f_\infty(ax)g_\infty(by),$$

which implies, together with the bilinearity of $\mathcal{R}$, that $\mathcal{R}(u,v) = f_\infty(u)g_\infty(v)$ for all $u \in \mathcal{F}E$, $v \in \mathcal{F}F$. By Proposition 3, we have $\|f_\infty(u)\| \le C_1\|u\|_\lambda$ and $\|g_\infty(v)\| \le C_2\|v\|_\mu$, and the desired assertion follows. ∎

3. The Haagerup tensor product. Let E and F be complete quantum polynormed spaces with corresponding saturated families of semi-norms $\|\cdot\|_\lambda$ $(\lambda \in \Lambda)$ on $\mathcal{F}E$ and $\|\cdot\|_\mu$ $(\mu \in M)$ on $\mathcal{F}F$. Consider the bilinear operator $\theta\colon E \times F \to E \otimes F$, $(x,y) \mapsto x \otimes y$, and its strong amplification $\Theta\colon \mathcal{F}E \times \mathcal{F}F \to \mathcal{F}(E \otimes F)$. Denote by

$$\odot\colon \mathcal{F}E \otimes \mathcal{F}F \to \mathcal{F}(E \otimes F)$$

the linear operator associated with Θ, and, for $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$, use the notation $u \odot v$ instead of $\odot(u \otimes v) = \Theta(u, v)$.

Since every element of $\mathcal{F}$ is a product of other elements, every elementary tensor and hence an arbitrary element in $\mathcal{F}(E \otimes F)$ belongs to the image of $\odot$; in other words, the operator $\odot$ is surjective. Therefore we can consider $\mathcal{F}(E \otimes F)$ as a quotient space of $\mathcal{F}E \otimes \mathcal{F}F$, and consequently, for the projective semi-norms on $\mathcal{F}E \otimes \mathcal{F}F$, we obtain the corresponding quotient semi-norms on $\mathcal{F}(E \otimes F)$. Namely, for $U \in \mathcal{F}(E \otimes F)$, $\lambda \in \Lambda$ and $\mu \in M$, we define

$$\|U\|_{\lambda,\mu} = \inf \Big\{ \sum_{k=1}^{n} \|u_k\|_\lambda \|v_k\|_\mu \Big\},$$

where the infimum is taken over all representations of U in the form

$$\sum_{k=1}^{n} u_k \odot v_k \quad (u_k \in \mathcal{F}E,\ v_k \in \mathcal{F}F).$$

We shall show that the family of semi-norms $\|\cdot\|_{\lambda,\mu}$ distinguishes elements of $\mathcal{F}(E \otimes F)$ and that all these semi-norms satisfy the axioms of Ruan, and thus $E \otimes F$ becomes a quantum polynormed space.

PROPOSITION 5. *The semi-norms $\|\cdot\|_{\lambda,\mu}$ satisfy the first axiom of Ruan.*

The proof is easy.

PROPOSITION 6. *Let $\{G, \|\cdot\|_\nu\ (\nu \in N)\}$ be a quantum polynormed space, $R\colon E \times F \to G$ a bilinear operator such that, for its strong amplification $\mathcal{R}$,*

$$\|\mathcal{R}(u, v)\|_\nu \le \|u\|_\lambda \|v\|_\mu$$

for all $u \in \mathcal{F}E$ and $v \in \mathcal{F}F$. Let $\rho\colon E \otimes F \to G$ be the linear operator associated with R. Then $\|\rho_\infty(w)\|_\nu \le \|w\|_{\lambda,\mu}$ for all $w \in \mathcal{F}(E \otimes F)$.

Proof. The proof is analogous to that of [7, Proposition 6.1.4] with replacement of norms by corresponding semi-norms. ∎

PROPOSITION 7 (cf. [7, Proposition 6.1.5]). *The family of semi-norms $\|\cdot\|_{\lambda,\mu}$ distinguishes elements of $\mathcal{F}(E \otimes F)$.*

Proof. By virtue of Proposition 1, it is sufficient to prove that, for each non-zero elementary tensor $aw \in \mathcal{F}(E \otimes F)$, there exist $\lambda \in \Lambda$ and $\mu \in M$ such that $\|aw\|_{\lambda,\mu} \neq 0$. It is not hard to show (cf. [6, Proposition 2.7.6]) that there exist continuous linear functionals $f\colon E \to \mathbb{C}$ and $g\colon F \to \mathbb{C}$ such that $(f \otimes g)(w) \neq 0$. Let $|f(x)| \le C_1 \|x\|_\lambda$ for all $x \in E$ and $|g(y)| \le C_2 \|y\|_\mu$ for all $y \in F$. Then, by Proposition 4, for the strong amplification $\mathcal{R}$ of the bilinear functional $f \times g$ and for all $u \in \mathcal{F}E$, $v \in \mathcal{F}F$ we have $\|\mathcal{R}(u, v)\| \le \|u\|_\lambda \|v\|_\mu$. Therefore, for the functional $f \otimes g$ associated with $f \times g$, we have (see Proposition 6)

$$|(f \otimes g)(w)| \|a\| = \|((f \otimes g)(w))a\| = \|(f \otimes g)_\infty(aw)\| \le \|aw\|_{\lambda,\mu}.$$

Since $(f \otimes g)(w) \neq 0$ and $a \neq 0$, we see that $\|aw\|_{\lambda,\mu} > 0$. ∎

Proposition 8. *Each element $U \in \mathcal{F}(E \otimes F)$ can be represented in the form $u \odot v$ ($u \in \mathcal{F}E$, $v \in \mathcal{F}F$). Moreover, for all $\lambda \in \Lambda$, $\mu \in M$*

$$\|U\|_{\lambda,\mu} = \inf\{\|u\|_\lambda \|v\|_\mu\},$$

where the infimum is taken over all possible representations of U in the indicated form.

Proof. First we shall show that in the definition of the semi-norm $\|U\|_{\lambda,\mu}$ we can take the infimum over the representations $U = \sum_{k=1}^n u_k \odot v_k$ for which, for all k, either $\|u_k\|_\lambda \neq 0$ and $\|v_k\|_\mu \neq 0$, or $\|u_k\|_\lambda = \|v_k\|_\mu = 0$. Indeed, let $\varepsilon > 0$ and let a representation $U = \sum_{k=1}^n u_k \odot v_k$ be such that

$$\sum_{k=1}^n \|u_k\|_\lambda \|v_k\|_\mu < \|U\|_{\lambda,\mu} + \varepsilon/2. \tag{1}$$

Take $u_0 \in \mathcal{F}E$ with $0 < \|u_0\|_\lambda < \varepsilon/4n$ and $v_0 \in \mathcal{F}F$ with $0 < \|v_0\|_\mu < \varepsilon/4n$. In the representation $U = \sum_{k=1}^n u_k \odot v_k$ we replace each summand $u_k \odot v_k$ with $\|u_k\|_\lambda = 0$ and $\|v_k\|_\mu \neq 0$ by the sum

$$(u_k - u_0/\|v_k\|_\mu) \odot v_k + u_0/\|v_k\|_\mu \odot v_k,$$

and similarly we do with those summands for which $\|u_k\|_\lambda \neq 0$ and $\|v_k\|_\mu = 0$. We come to a representation possessing the desired property. Moreover, the difference between the sum of the form (1) for this representation and $\sum_{k=1}^n \|u_k\|_\lambda \|v_k\|_\mu$ is not greater than $\varepsilon/2$, and hence the first sum mentioned does not exceed $\|U\|_{\lambda,\mu} + \varepsilon$. Further argument repeats the proof of [7, Proposition 6.1.6]. ∎

Proposition 9. *The semi-norms $\|\cdot\|_{\lambda,\mu}$ satisfy the second axiom of Ruan.*

Proof. Let elements $U, V \in \mathcal{F}(E \otimes F)$ possess orthogonal supports. If $\|U\|_{\lambda,\mu} = 0$, then

$$\|U + V\|_{\lambda,\mu} = \|V\|_{\lambda,\mu} = \max\{\|U\|_{\lambda,\mu}, \|V\|_{\lambda,\mu}\}.$$

The similar holds in the case $\|V\|_{\lambda,\mu} = 0$. If the semi-norms of both these elements do not equal zero, then for them the same argument holds which is used for the Haagerup norm on the tensor product of quantum normed spaces (see [7, Proposition 6.1.7]). ∎

Thus $E \otimes F$ with the family of semi-norms $\|\cdot\|_{\lambda,\mu}$ is a quantum polynormed space. We denote its (quantum) completion by $E \overset{h}{\otimes} F$. This space is called *the Haagerup tensor product* of E and F. It follows from Proposition 6 and the universal property of quantum completion that $E \overset{h}{\otimes} F$ possesses the universal property with respect to the class of strongly completely continuous bilinear operators. Namely, for any complete quantum polynormed space G and strongly completely continuous bilinear operator $R\colon E \times F \to G$ there exists a unique completely continuous linear operator $\rho\colon E \overset{h}{\otimes} F \to G$ such that the diagram

$$\begin{tikzcd} & E \overset{h}{\otimes} F \arrow[d, "\rho"] \\ E \times F \arrow[ur, "\theta"] \arrow[r, "R"'] & G \end{tikzcd}$$

is commutative.

In the following proposition E_k, F_k $(k = 1, 2)$ are quantum polynormed spaces, and $\varphi\colon E_1 \to E_2$, $\psi\colon F_1 \to F_2$ are two completely continuous linear operators. The proof of the proposition repeats, in essence, the proof of [7, Theorem 6.5.1].

Proposition 10. *There exists a completely continuous linear operator*

$$\varphi \overset{h}{\otimes} \psi\colon E_1 \overset{h}{\otimes} F_1 \to E_2 \overset{h}{\otimes} F_2$$

uniquely determined by

$$(\varphi \overset{h}{\otimes} \psi)(x \otimes y) = \varphi(x) \otimes \psi(y) \qquad (x \in E_1,\ y \in F_1).$$

Now suppose that E_σ $(\sigma \in \mathcal{S})$ and F are complete quantum polynormed spaces with (saturated) defining families of semi-norms $\|\cdot\|_\mu^\sigma$ $(\sigma \in \mathcal{S},\ \mu \in \Lambda^\sigma)$ and $\|\cdot\|_\nu$ $(\nu \in \Lambda)$, respectively. We shall prove the following property of the Haagerup tensor product (cf. [4, Theorem II.5.19]):

Proposition 11. *There exists a complete topological isomorphism between quantum polynormed spaces*

$$i\colon \big(\mathop{\times}_{\sigma\in\mathcal{S}} E_\sigma\big) \overset{h}{\otimes} F \xrightarrow{\ \sim\ } \mathop{\times}_{\sigma\in\mathcal{S}} (E_\sigma \overset{h}{\otimes} F)$$

uniquely determined by $i(\{x_\sigma\} \otimes y) = \{x_\sigma \otimes y\}$, *where* $\{x_\sigma\} \in \times_\sigma E_\sigma$, $y \in F$.

Proof. Consider the bilinear operator

$$R\colon \big(\mathop{\times}_{\sigma\in\mathcal{S}} E_\sigma\big) \times F \to \mathop{\times}_{\sigma\in\mathcal{S}} (E_\sigma \overset{h}{\otimes} F), \quad (\{x_\sigma\}, y) \mapsto \{x_\sigma \otimes y\}.$$

Denote by $\mathcal{R}$ its strong amplification, and by

$$i^\sigma\colon \mathop{\times}_{\sigma\in\mathcal{S}} E_\sigma \to E_\sigma \quad \text{and} \quad j^\sigma\colon \mathop{\times}_{\sigma\in\mathcal{S}} (E_\sigma \overset{h}{\otimes} F) \to E_\sigma \overset{h}{\otimes} F$$

the natural projections onto respective factors. Then we have

$$\|\mathcal{R}(u, v)\|_{\mu,\nu}^\sigma = \|j_\infty^\sigma \mathcal{R}(u, v)\|_{\mu,\nu} = \|i_\infty^\sigma(u) \odot v\|_{\mu,\nu} \le \|u\|_\mu^\sigma \|v\|_\nu.$$

Therefore R is strongly completely continuous. So we define i as the completely continuous linear operator associated with R.

For each $\sigma \in \mathcal{S}$, let L_σ denote the linear span of elements of the form $\{x_\sigma\} \otimes y$, where $y \in F$ and all the coordinates of $\{x_\sigma\} \in \times_\sigma E_\sigma$ apart from the σ-th are zero. Let L denote the linear span of all L_σ. Clearly, L is dense in $(\times_{\sigma\in\mathcal{S}} E_\sigma) \overset{h}{\otimes} F$, and $i(L)$ is dense in $\times_{\sigma\in\mathcal{S}} (E_\sigma \overset{h}{\otimes} F)$.

We fix $\sigma_1, \dots, \sigma_n \in \mathcal{S}$, $\alpha = (\mu_1 \in \Lambda^{\sigma_1}, \dots, \mu_n \in \Lambda^{\sigma_n})$ and $\nu \in \Lambda$, and we denote by $\|\cdot\|_{\alpha,\nu}$ the quantum semi-norm on $\mathcal{F}((\times_\sigma E_\sigma) \overset{h}{\otimes} F)$ corresponding to the semi-norms $\max\{\|\cdot\|_{\mu_1}^{\sigma_1}, \dots, \|\cdot\|_{\mu_n}^{\sigma_n}\}$ and $\|\cdot\|_\nu$. Each element $u \in \mathcal{F}L$ has the form

$$\sum_{k=1}^n u_k + \sum_{\sigma'} v_{\sigma'},$$

where $u_k \in \mathcal{F}L_{\sigma_k}$, $v_{\sigma'} \in \mathcal{F}L_{\sigma'}$ and $\sigma' \in \mathcal{S}$ runs over a finite set of indices other than $\sigma_1, \dots, \sigma_k$. Note that

$$\|v_{\sigma'}\|_{\alpha,\nu} = \|i_\infty(v_{\sigma'})\|^{\sigma_k}_{\mu_k,\nu} = 0 \quad (k = 1, \dots, n).$$

Also, it is not hard to verify that, for all k,

$$\|i_\infty(u_k)\|^{\sigma_k}_{\mu_k,\nu} = \|u_k\|_{\alpha,\nu}.$$

So we have

$$\|u\|_{\alpha,\nu} \le \sum_{k=1}^{n} \|u_k\|_{\alpha,\nu} + \sum_{\sigma'} \|v_{\sigma'}\|_{\alpha,\nu} = \sum_{k=1}^{n} \|i_\infty(u_k)\|^{\sigma_k}_{\mu_k,\nu} \le n \max_k \{\|i_\infty(u_k)\|^{\sigma_k}_{\mu_k,\nu}\} = n \max_k \{\|i_\infty(u)\|^{\sigma_k}_{\mu_k,\nu}\}.$$

However, $\max\{\|\cdot\|^{\sigma_1}_{\mu_1,\nu}, \dots, \|\cdot\|^{\sigma_n}_{\mu_n,\nu}\}$ is a semi-norm from the defining family of semi-norms for $\times_{\sigma\in\mathcal{S}}(E_\sigma \overset{h}{\otimes} F)$. Since $\alpha = (\mu_1 \in \Lambda^{\sigma_1}, \dots, \mu_n \in \Lambda^{\sigma_n})$ and $\nu \in \Lambda$ are arbitrary, it follows from the above inequality that i provides a complete topological isomorphism between the subspaces L and $i(L)$ in $(\times_{\sigma\in\mathcal{S}} E_\sigma) \overset{h}{\otimes} F$ and $\times_{\sigma\in\mathcal{S}}(E_\sigma \overset{h}{\otimes} F)$ respectively, and hence the latter are isomorphic. ■

4. Quantum Arens–Michael algebras. Let A be a quantum polynormed space which is at the same time an algebra with multiplication determined by a bilinear operator $m\colon A \times A \to A$. A quantum semi-norm $\|\cdot\|_\lambda$ is called *strongly completely submultiplicative* if $\|\mathcal{M}(u,v)\|_\lambda \le \|u\|_\lambda \|v\|_\lambda$ for all $u, v \in \mathcal{F}A$, where $\mathcal{M}\colon \mathcal{F}A \times \mathcal{F}A \to \mathcal{F}A$ is the strong amplification of m.

If the structure of a quantum polynormed space in A can be defined by a family of strongly completely submultiplicative quantum semi-norms and A is complete, then A is called an *Arens–Michael* $\overset{h}{\otimes}$*-algebra*. We can consider the category of Arens–Michael $\overset{h}{\otimes}$-algebras with Arens–Michael $\overset{h}{\otimes}$-algebras as objects and completely continuous algebra homomorphisms as morphisms. By a *complete isomorphism* between two Arens–Michael $\overset{h}{\otimes}$-algebras A and B we understand a completely continuous algebraic isomorphism from A onto B with completely continuous inverse.

If the structure of a quantum polynormed space in A is determined by a unique strongly completely submultiplicative quantum norm, then A is called a *Banach* $\overset{h}{\otimes}$*-algebra*. Certainly we can consider the category of Banach $\overset{h}{\otimes}$-algebras as a full subcategory in the category of Arens–Michael $\overset{h}{\otimes}$-algebras.

It is not hard to show that the Cartesian product of a family of Arens–Michael $\overset{h}{\otimes}$-algebras is an Arens–Michael $\overset{h}{\otimes}$-algebra with respect to coordinatewise multiplication.

Suppose now that A is an Arens–Michael $\overset{h}{\otimes}$-algebra, $\|\cdot\|_\lambda$ $(\lambda \in \Lambda)$ is a family of strongly completely submultiplicative semi-norms on $\mathcal{F}A$ that defines the structure of a quantum polynormed space on A, $m\colon A \times A \to A$ is the bilinear operator of multiplication, and $\mathcal{M}\colon \mathcal{F}A \times \mathcal{F}A \to \mathcal{F}A$ is the strong amplification of m.

Since the maximum of strongly completely submultiplicative semi-norms is also a strongly completely submultiplicative semi-norm (this fact is easily checked), we may assume that the family $\|\cdot\|_\lambda$ $(\lambda\in\Lambda)$ is saturated.

For each λ, we put

$$I_\lambda=\{a\in A:\ \|a\|_\lambda=0\}.$$

Obviously, I_λ is a closed two-sided ideal of A. Consider the quotient algebra A/I_λ. Denote the bilinear operator of multiplication in this algebra by $\tilde m$, and its strong amplification by $\widetilde{\mathcal{M}}$. By Proposition 2, the quotient semi-norm $\|\cdot\|_\lambda^\wedge$ on $\mathcal{F}(A/I_\lambda)$ (defined by

$$\|\tilde u\|_\lambda^\wedge=\inf\{\|u\|_\lambda:\ u\in\mathcal{F}A,\ \tau_\infty^\lambda(u)=\tilde u\},$$

where $\tau^\lambda\colon A\to A/I_\lambda$ is the quotient map) satisfies the axioms of Ruan. It follows from the definition of I_λ that $\|\cdot\|_\lambda^\wedge$ is a norm on A/I_λ, and hence on $\mathcal{F}(A/I_\lambda)$. Moreover, if

$$\tau_\infty^\lambda(u)=\tau_\infty^\lambda(v),$$

then $\tau_\infty^\lambda(u-v)=0$, and therefore $u-v\in\mathcal{F}I_\lambda$, which implies that $\|u-v\|_\lambda=0$ and $\|u\|_\lambda=\|v\|_\lambda$. Consequently, $\|u\|_\lambda=\|\tau_\infty^\lambda(u)\|_\lambda^\wedge$ for any $u\in\mathcal{F}A$.

It is not hard to check that this norm is strongly completely submultiplicative. Indeed, let $\tilde u,\tilde v\in\mathcal{F}(A/I_\lambda)$. Then, for $u,v\in\mathcal{F}A$ such that $\tau_\infty^\lambda(u)=\tilde u$ and $\tau_\infty^\lambda(v)=\tilde v$, we have $\tau_\infty^\lambda\mathcal{M}(u,v)=\widetilde{\mathcal{M}}(\tilde u,\tilde v)$, and therefore

$$\|\widetilde{\mathcal{M}}(\tilde u,\tilde v)\|_\lambda^\wedge=\|\mathcal{M}(u,v)\|_\lambda\le\|u\|_\lambda\|v\|_\lambda=\|\tilde u\|_\lambda^\wedge\|\tilde v\|_\lambda^\wedge.$$

We let A_λ denote the completion of A/I_λ with respect to $\|\cdot\|_\lambda^\wedge$, and we still use the notation τ^λ for the map $A\to A_\lambda$, $a\mapsto a+I_\lambda\in A/I_\lambda\subseteq A_\lambda$. Clearly, A_λ is a Banach $\overset{h}{\otimes}$-algebra.

We introduce on Λ an order relation "$\mu\prec\lambda$ if $\|\cdot\|_\mu\le\|\cdot\|_\lambda$". In this way, since our family of semi-norms is saturated, we get a directed set.

If $\mu\prec\lambda$, then $I_\lambda\subseteq I_\mu$. Therefore the map

$$a+I_\lambda\mapsto a+I_\mu$$

of A/I_λ onto A/I_μ is well-defined and is a completely continuous homomorphism. If we extend this by continuity, we get a completely continuous homomorphism $\tau_\lambda^\mu\colon A_\lambda\to A_\mu$. Moreover, $\tau^\mu=\tau_\lambda^\mu\tau^\lambda$ and, if $\nu\prec\mu\prec\lambda$, then $\tau_\lambda^\nu=\tau_\mu^\nu\tau_\lambda^\mu$. Thus

$$\mathcal{A}=(\Lambda,\{A_\lambda\},\{\tau_\lambda^\mu\})$$

is an inverse system in the category of Banach $\overset{h}{\otimes}$-algebras (as well as in the category of Arens–Michael $\overset{h}{\otimes}$-algebras). We shall call an element $\bar a=\{a_\lambda\in A_\lambda\}_{\lambda\in\Lambda}$ of the Cartesian product $\times_\lambda A_\lambda$ a *compatible family* of the system $\mathcal{A}$ if $a_\mu=\tau_\lambda^\mu(a_\lambda)$ whenever $\mu\prec\lambda$.

We let A_0 denote the subalgebra of $\times_\lambda A_\lambda$ consisting of all those "rows" $\bar a=\{a_\lambda\}_{\lambda\in\Lambda}$ that are compatible families; since all the τ_λ^μ are continuous, this subalgebra is closed. Further, we put

$$\tau_0^\lambda\colon A_0\to A_\lambda,\quad \bar a\mapsto a_\lambda\qquad(\lambda\in\Lambda).$$

Just as in the case of classical Arens–Michael algebras [5, Chapter V, §2], it can be proved that both $(A,\{\tau^\lambda\})$ and $(A_0,\{\tau_0^\lambda\})$ are the inverse limit of the system $\mathcal{A}$ in the category of Arens–Michael $\overset{h}{\otimes}$-algebras. Consequently, A is isomorphic in this category (i.e, completely isomorphic) to the closed subalgebra A_0 of the Cartesian product $\times_\lambda A_\lambda$.

5. Quantum polynormed modules and derivations. Suppose that A is an Arens–Michael $\overset{h}{\otimes}$-algebra and X is a complete quantum polynormed space which is also a left A-module. Thus there is a bilinear operator

$$\dot{m}\colon A \times X \to X, \quad (a, x) \mapsto a \cdot x,$$

such that $(ab) \cdot x = a \cdot (b \cdot x)$ for $a, b \in A$, $x \in X$. Then X is called a *left* $\overset{h}{\otimes}$*-module over* A (a *left* A-$\overset{h}{\otimes}$*-module* for short) if the above module operator is strongly completely continuous. Given two left A-$\overset{h}{\otimes}$-modules X and Y, a *morphism* of left A-$\overset{h}{\otimes}$-modules is a completely continuous linear operator $\varphi\colon X \to Y$ such that $\varphi(a \cdot x) = a \cdot \varphi(x)$ for all $a \in A$, $x \in X$. Right A-$\overset{h}{\otimes}$-modules, A-$\overset{h}{\otimes}$-bimodules, and their morphisms are defined similarly.

For example, the algebra A is itself an A-$\overset{h}{\otimes}$-bimodule with respect to the outer multiplications given by the product in A. It is not hard to show that, if X is a left A-$\overset{h}{\otimes}$-module and Y is a right A-$\overset{h}{\otimes}$-module, then $X \overset{h}{\otimes} Y$ is an A-$\overset{h}{\otimes}$-bimodule for the products defined by

$$a \cdot (x \otimes y) = a \cdot x \otimes y, \quad (x \otimes y) \cdot a = x \otimes y \cdot a \qquad (a \in A,\ x \in X,\ y \in Y).$$

In particular, $A \overset{h}{\otimes} A$ is an A-$\overset{h}{\otimes}$-bimodule in this way.

We recall now that a linear operator $\delta\colon A \to X$, where A is an algebra and X is an A-bimodule, is called a *derivation of* A *with values in* X if it satisfies

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \qquad (a, b \in A).$$

A derivation is called an *inner derivation* if there exists $x \in X$ such that, for any $a \in A$, $\delta(a) = a \cdot x - x \cdot a$.

Let A be an Arens–Michael $\overset{h}{\otimes}$-algebra, and let X be an A-$\overset{h}{\otimes}$-bimodule. Obviously, each inner derivation is automatically completely continuous. An Arens–Michael $\overset{h}{\otimes}$-algebra is said to be *contractible* if each completely continuous derivation of A with values in any A-$\overset{h}{\otimes}$-bimodule is inner. The main result of this paper is a description of contractible Arens–Michael $\overset{h}{\otimes}$-algebras (see Theorem 1 below).

For some results concerning contractible Banach and Arens–Michael algebras in the framework of traditional (non-quantum) approach, see [13, 14, 10, 4, 11, 9, 3, 8, 12].

6. Main results. We shall now characterize contractible Arens–Michael $\overset{h}{\otimes}$-algebras; to do this we first introduce the notion of a diagonal.

Let A be an Arens–Michael $\overset{h}{\otimes}$-algebra. Recall that $A \overset{h}{\otimes} A$ is an A-$\overset{h}{\otimes}$-bimodule for products determined by the conditions $a \cdot (b \otimes c) = ab \otimes c$ and $(b \otimes c) \cdot a = b \otimes ca$ for

$a, b, c \in A$. Further, we put

$$\pi_A \colon A \overset{h}{\otimes} A \to A, \quad a \otimes b \mapsto ab,$$

i.e., we take the completely continuous linear operator associated with the bilinear operator of multiplication. Note that $\pi_A \colon A \overset{h}{\otimes} A \to A$ is a *morphism* of A-$\overset{h}{\otimes}$-bimodules, and so the kernel of π_A is a submodule of $A \overset{h}{\otimes} A$.

Suppose now that A is a unital Arens–Michael $\overset{h}{\otimes}$-algebra. An element $d \in A \overset{h}{\otimes} A$ is said to be a *diagonal* for A if $a \cdot d = d \cdot a$ for all $a \in A$ and if $\pi_A(d)$ is an identity element of A.

Proposition 12. *An Arens–Michael $\overset{h}{\otimes}$-algebra A is contractible if and only if it is unital and has a diagonal.*

Proof. $\Rightarrow$ Suppose that A is contractible. Let X be the A-$\overset{h}{\otimes}$-bimodule whose underlying space is A, but on which A acts via

$$a \cdot x = ax \quad \text{and} \quad x \cdot a = 0 \qquad (a \in A,\ x \in X).$$

The map

$$\delta \colon A \to X, \quad a \mapsto a,$$

(i.e, the identity map on A) is a completely continuous derivation, and so there exists $r \in X$ with $\delta(a) = a \cdot r - r \cdot a$ $(a \in A)$. But then $ar = a$ $(a \in A)$, i.e, r is a right identity for A.

A similar argument applies to prove that A has a left identity s. Hence $r = s$, and so A has an identity, say e.

We next consider the A-$\overset{h}{\otimes}$-bimodule $\operatorname{Ker} \pi_A \subseteq A \overset{h}{\otimes} A$. The map

$$\delta \colon A \to \operatorname{Ker} \pi_A, \quad a \mapsto a \otimes e - e \otimes a,$$

is a completely continuous derivation, and so there exists $u \in \operatorname{Ker} \pi_A$ with $\delta(a) = a \cdot u - u \cdot a$ $(a \in A)$. Set $d = e \otimes e - u$. Then d is the required diagonal.

$\Leftarrow$ Suppose that A has an identity $e \in A$ and a diagonal $d \in A \overset{h}{\otimes} A$. Let X be an A-$\overset{h}{\otimes}$-bimodule, and let $\delta \colon A \to X$ be a completely continuous derivation. We put

$$y = \delta(e) - \delta(e) \cdot e - \pi_X(\delta \overset{h}{\otimes} \mathbf{1}_A) d \in X,$$

where

$$\pi_X \colon X \overset{h}{\otimes} A \to X, \quad x \otimes a \mapsto x \cdot a.$$

For each $a \in A$, we have

$$\begin{aligned}
&a \cdot y - y \cdot a \\
&\quad = a \cdot \delta(e) - a \cdot \delta(e) \cdot e - a \cdot \pi_X(\delta \overset{h}{\otimes} \mathbf{1}_A) d - \delta(e) \cdot a + \delta(e) \cdot a + \pi_X(\delta \overset{h}{\otimes} \mathbf{1}_A) d \cdot a \\
&\quad = a \cdot \delta(e) - (\delta(a) - \delta(a) \cdot e) \cdot e - \pi_X(a \cdot (\delta \overset{h}{\otimes} \mathbf{1}_A) d) + \pi_X(\delta \overset{h}{\otimes} \mathbf{1}_A)(d \cdot a) \\
&\quad = a \cdot \delta(e) - \pi_X(a \cdot (\delta \overset{h}{\otimes} \mathbf{1}_A) d) + \pi_X(\delta \overset{h}{\otimes} \mathbf{1}_A)(a \cdot d).
\end{aligned}$$

Let $u \mapsto \delta(a) \cdot u$ be the operator from $A \overset{h}{\otimes} A$ to $X \overset{h}{\otimes} A$ defined on elementary tensors by $\delta(a) \cdot (b \otimes c) = (\delta(a) \cdot b) \otimes c$. (It is not hard to check that this operator is completely continuous.) Then, for any $u \in A \overset{h}{\otimes} A$, we have

$$(\delta \overset{h}{\otimes} \mathbf{1}_A)(a \cdot u) = a \cdot (\delta \overset{h}{\otimes} \mathbf{1}_A)u + \delta(a) \cdot u. \tag{2}$$

This equality can be easily checked on elementary tensors.

Using (2), we continue the calculations:

$$a \cdot y - y \cdot a \;=\; a \cdot \delta(e) - \pi_X(a \cdot (\delta \overset{h}{\otimes} \mathbf{1}_A)d) + \pi_X(a \cdot (\delta \overset{h}{\otimes} \mathbf{1}_A)d) + \pi_X(\delta(a) \cdot d).$$

Since

$$\pi_X(\delta(a) \cdot d) = \delta(a) \cdot \pi_A(d) \tag{3}$$

(again it suffices to check this equality for elementary tensors), we get

$$a \cdot y - y \cdot a = a \cdot \delta(e) + \delta(a) \cdot \pi_A(d) = a \cdot \delta(e) + \delta(a) \cdot e = \delta(a),$$

which means that δ is inner. So A is contractible. ∎

Let A_σ $(\sigma \in \mathcal{S})$ be a family of Arens–Michael $\overset{h}{\otimes}$-algebras. As in the homology of classical polynormed algebras (see [11, Lemma 11] or [12, Lemma 1.4.7]), by using Proposition 11 it is easy to prove the following.

Proposition 13. *If each of the algebras A_σ has a diagonal $d_\sigma \in A_\sigma \overset{h}{\otimes} A_\sigma$, then $\times_\sigma A_\sigma$ has a diagonal $d \in (\times_\sigma A_\sigma) \overset{h}{\otimes} (\times_\sigma A_\sigma)$.*

We now proceed to getting our main result.

Theorem 1. *An Arens–Michael $\overset{h}{\otimes}$-algebra A is contractible if and only if it is completely isomorphic to the Cartesian product of a family of full matrix C^*-algebras.*

Proof. $\Leftarrow$ Suppose that A is completely isomorphic to the Cartesian product of a family of full matrix C^*-algebras. By [5, Proposition VII.1.73], each full matrix algebra has a diagonal, and so, by virtue of Proposition 13, A has a diagonal, too. By Proposition 12, A is contractible.

$\Rightarrow$ Suppose that A is contractible. By Proposition 12, A has an identity $e \in A$ and a diagonal $d \in A \overset{h}{\otimes} A$. Let $\|\cdot\|_\lambda$ $(\lambda \in \Lambda)$ be a saturated family of strongly completely submultiplicative quantum semi-norms that defines the structure of a quantum polynormed space on A. For each λ, we put

$$I_\lambda = \{a \in A : \; \|a\|_\lambda = 0\},$$

and we define, as in Section 4, the Banach $\overset{h}{\otimes}$-algebra A_λ to be the completion of A/I_λ with respect to the quotient norm $\|\cdot\|_\lambda^\wedge$, and also the operators

$$\tau^\lambda \colon A \to A_\lambda, \quad a \mapsto a + I_\lambda \in A/I_\lambda \subseteq A_\lambda.$$

It is easy to check that $e_\lambda = \tau^\lambda(e)$ is the identity in A_λ. For the elements $\tilde{a} \in A_\lambda$ that belong to A/I_λ the equality

$$\tilde{a}e_\lambda = e_\lambda\tilde{a} = \tilde{a}$$

is obvious. The rest follows from the density of A/I_λ in A_λ.

Let us check that $d_\lambda = (\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(d) \in A_\lambda \overset{h}{\otimes} A_\lambda$ is a diagonal for A_λ. For the operators $\pi_A\colon A \overset{h}{\otimes} A \to A$ and $\pi^\lambda\colon A_\lambda \overset{h}{\otimes} A_\lambda \to A_\lambda$ which linearize the bilinear operators of multiplication in A and A_λ respectively, we have

$$\pi^\lambda(\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(u) = \tau^\lambda\pi_A(u)$$

for all $u \in A \overset{h}{\otimes} A$ (this is easily checked on elementary tensors). Therefore,

$$\pi^\lambda(d_\lambda) = \pi^\lambda(\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(d) = \tau^\lambda\pi_A(d) = \tau^\lambda(e) = e_\lambda.$$

Now we show that $\tilde{a} \cdot d_\lambda = d_\lambda \cdot \tilde{a}$ for all $\tilde{a} \in A_\lambda$. It is sufficient to verify this equality for elements $\tilde{a}$ of the form $\tau^\lambda(a)$ $(a \in A)$, because of the density of the set of such elements in A_λ and the complete continuity of the outer multiplications in the A_λ-bimodule $A_\lambda \overset{h}{\otimes} A_\lambda$. We have

$$\begin{aligned}\tau^\lambda(a) \cdot d_\lambda = \tau^\lambda(a) \cdot (\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(d) &= (\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(a \cdot d)\\ &= (\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(d \cdot a) = (\tau^\lambda \overset{h}{\otimes} \tau^\lambda)(d) \cdot \tau^\lambda(a) = d_\lambda \cdot \tau^\lambda(a).\end{aligned}$$

Thus d_λ is indeed a diagonal for A_λ.

Recall now [1, Theorem 2.3.2] that every unital Banach $\overset{h}{\otimes}$-algebra A is represented as a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H, i.e., there exists a completely isometric homomorphism $\varphi\colon A \to \mathcal{B}(H)$. Here $\mathcal{B}(H)$ is considered as a C^*-algebra with the standard quantization (see [7, Example 1.3.7]).

If A is a unital closed subalgebra of $\mathcal{B}(H)$, then it follows from [8, Corollary 2.4] that the existence of a diagonal in $A \overset{h}{\otimes} A$ implies that A is (completely) isomorphic to a finite-dimensional C^*-algebra or, equivalently, to the Cartesian product of finitely many full matrix C^*-algebras.

From this we get that, for each λ, $A_\lambda = \times_{k=1}^{n_\lambda} \mathbb{M}_{\lambda,k}$ (up to a complete isomorphism), where full matrix algebras $\mathbb{M}_{\lambda,k}$ are considered as C^*-algebras with the standard quantization.

Consider the semi-norms $\|\cdot\|_{\lambda,k}$, $k = 1, \dots, n_\lambda$ (which are easily seen to be quantum), defined as follows:

$$\|u\|_{\lambda,k} = \|(\tau^{\lambda,k}\tau^\lambda)_\infty(u)\|,$$

where $u \in \mathcal{F}A$, and $\tau^{\lambda,k}\colon A_\lambda \to \mathbb{M}_{\lambda,k}$ is the natural projection of A_λ onto the k-th factor $\mathbb{M}_{\lambda,k}$. We shall show that A is the Cartesian product of full matrix algebras $\mathbb{M}_{\lambda,k}$ chosen one from each equivalence class of semi-norms $\|\cdot\|_{\lambda,k}$.

As mentioned in Section 4, A is completely isomorphic to the subalgebra of $\times_\lambda A_\lambda$ consisting of all compatible families. Any element in $A_\lambda = \times_{k=1}^{n_\lambda} \mathbb{M}_{\lambda,k}$ is uniquely determined by its natural projections on $\mathbb{M}_{\lambda,k}$. Consequently, any compatible family $\{a_\lambda\}$ is uniquely determined by the collection

$$\{\tau^{\lambda,k}(a_\lambda) :\ \lambda \in \Lambda,\ k = 1, \dots, n_\lambda\}.$$

Therefore the operator taking a compatible family $\{a_\lambda\}$ to the element

$$\{\tau^{\lambda,k}(a_\lambda)\} \in \mathop{\times}_{\lambda,k} \mathbb{M}_{\lambda,k}$$

is injective. Its corestriction to the image is, clearly, an isomorphism of algebras. This isomorphism is completely continuous, because $\|\cdot\|_{\lambda,k} \le \|\cdot\|_\lambda$, and its inverse operator is also completely continuous, because

$$\|\cdot\|_\lambda \le \max_{1\le k\le n_\lambda}\{\|\cdot\|_{\lambda,k}\}.$$

Consequently, it is an isomorphism of Arens–Michael $\overset{h}{\otimes}$-algebras.

For any two equivalent semi-norms $\|\cdot\|_{\lambda,k}$ and $\|\cdot\|_{\mu,\ell}$, consider an isomorphism

$$\alpha^{\mu,\ell}_{\lambda,k}\colon \mathbb{M}_{\lambda,k} \to \mathbb{M}_{\mu,\ell}$$

such that $\alpha^{\mu,\ell}_{\lambda,k}\tau^{\lambda,k}\tau^\lambda = \tau^{\mu,\ell}\tau^\mu$. It can be constructed as follows: for $a_{\lambda,k} \in \mathbb{M}_{\lambda,k}$, we put

$$\alpha^{\mu,\ell}_{\lambda,k}(a_{\lambda,k}) = \tau^{\mu,\ell}\tau^\mu(a),$$

where a is some element of A such that $\tau^{\lambda,k}\tau^\lambda(a) = a_{\lambda,k}$. It follows from the equivalence of the semi-norms $\|\cdot\|_{\lambda,k}$ and $\|\cdot\|_{\mu,\ell}$ that $\operatorname{Ker}\tau^{\lambda,k}\tau^\lambda = \operatorname{Ker}\tau^{\mu,\ell}\tau^\mu$, and so the result does not depend on the choice of a. It is not hard to check that all $\alpha^{\lambda,k}_{\lambda,k}$ are the identity operators and

$$\alpha^{\eta,m}_{\mu,\ell}\alpha^{\mu,\ell}_{\lambda,k} = \alpha^{\eta,m}_{\lambda,k} \tag{4}$$

whenever these isomorphisms are defined.

Consider the set B of all elements $\{b_{\lambda,k}\} \in \times_{\lambda,k}\mathbb{M}_{\lambda,k}$ such that

$$\alpha^{\mu,\ell}_{\lambda,k}(b_{\lambda,k}) = b_{\mu,\ell}$$

whenever the semi-norms $\|\cdot\|_{\lambda,k}$ and $\|\cdot\|_{\mu,\ell}$ are equivalent. Obviously, $\{\tau^{\lambda,k}(a_\lambda)\} \in B$ for each compatible family $\{a_\lambda\}$. Let us show that, for any element $\{b_{\lambda,k}\} \in B$, the element $\{b_\lambda\} \in \times_\lambda A_\lambda$ for which

$$\tau^{\lambda,k}(b_\lambda) = b_{\lambda,k} \qquad (\lambda \in \Lambda,\ k = 1,\dots,n_\lambda)$$

is a compatible family.

Let $\mu \prec \lambda$, and the homomorphism $\tau^\mu_\lambda\colon A_\lambda \to A_\mu$ be defined as described in Section 4. Then, for each $\ell = 1,\dots,n_\mu$, the kernel of the composition $\tau^{\mu,\ell}\tau^\mu_\lambda$ is a two-sided ideal in A_λ, and hence is equal either to $\{0\}$ or to the sum of some of $\mathbb{M}_{\lambda,k}$. It is easy to see that the operator

$$a_\lambda + \operatorname{Ker}\tau^{\mu,\ell}\tau^\mu_\lambda \mapsto \tau^{\mu,\ell}\tau^\mu_\lambda(a_\lambda)$$

provides an isomorphism between the quotient algebra $A_\lambda/(\operatorname{Ker}\tau^{\mu,\ell}\tau^\mu_\lambda)$ and the algebra $\mathbb{M}_{\mu,\ell}$. Moreover, $A_\lambda/(\operatorname{Ker}\tau^{\mu,\ell}\tau^\mu_\lambda)$ is isomorphic to the sum of those $\mathbb{M}_{\lambda,k}$ which do not lie in $\operatorname{Ker}\tau^{\mu,\ell}\tau^\mu_\lambda$. A full matrix algebra $\mathbb{M}_{\mu,\ell}$ cannot be isomorphic to the sum of several matrix algebras. Therefore $\tau^{\mu,\ell}\tau^\mu_\lambda$ is the composition of the natural projection $\tau^{\lambda,k}$ onto one of the factors $\mathbb{M}_{\lambda,k}$ and an isomorphism between $\mathbb{M}_{\lambda,k}$ and $\mathbb{M}_{\mu,\ell}$ (a complete one, because we deal with finite-dimensional spaces). Hence the semi-norms $\|\cdot\|_{\lambda,k}$ and $\|\cdot\|_{\mu,\ell}$ are equivalent. For an element b_λ of our family $\{b_\lambda\} \in \times_\lambda A_\lambda$

$$\tau^{\mu,\ell}\tau^\mu_\lambda(b_\lambda) = \tau^{\mu,\ell}\tau^\mu_\lambda\tau^\lambda(a),$$

where a is an element of A for which $\tau^\lambda(a) = b_\lambda$, and we have

$$\tau^{\mu,\ell}\tau^\mu_\lambda\tau^\lambda(a) = \tau^{\mu,\ell}\tau^\mu(a) = \alpha^{\mu,\ell}_{\lambda,k}\tau^{\lambda,k}\tau^\lambda(a) = \\ = \alpha^{\mu,\ell}_{\lambda,k}\tau^{\lambda,k}(b_\lambda) = \alpha^{\mu,\ell}_{\lambda,k}(b_{\lambda,k}) = b_{\mu,\ell} = \tau^{\mu,\ell}(b_\mu).$$

Consequently, $\tau^\mu_\lambda(b_\lambda) = b_\mu$, and this means that the family $\{b_\lambda\}$ is compatible.

Thus A is completely isomorphic to the subalgebra $B \subseteq \times_{\lambda,k}\mathbb{M}_{\lambda,k}$. Let us choose one representative $\|\cdot\|_{\lambda^*,k^*}$ from each equivalence class of semi-norms $\|\cdot\|_{\lambda,k}$. Clearly, the natural projection of B onto $\times_{\lambda^*,k^*}\mathbb{M}_{\lambda^*,k^*}$ is a completely continuous homomorphism of algebras. Consider the operator taking an element

$$\{a_{\lambda^*,k^*}\} \in \underset{\lambda^*,k^*}{\times}\mathbb{M}_{\lambda^*,k^*}$$

to the element of $\times_{\lambda,k}\mathbb{M}_{\lambda,k}$ with coordinates $a_{\lambda,k} = \alpha^{\lambda,k}_{\lambda^*,k^*}(a_{\lambda^*,k^*})$, where the semi-norm $\|\cdot\|_{\lambda^*,k^*}$ is equivalent to $\|\cdot\|_{\lambda,k}$. It is easy to see, using (4), that the elements so defined belong to B. On the other hand, for any $\{b_{\lambda,k}\} \in B$, by virtue of the definition of B,

$$b_{\lambda,k} = \alpha^{\lambda,k}_{\lambda^*,k^*}(b_{\lambda^*,k^*})$$

if $\|\cdot\|_{\lambda,k}$ is equivalent to $\|\cdot\|_{\lambda^*,k^*}$. Consequently, our operator is a completely continuous homomorphism of algebras which is an inverse to the natural projection of B onto $\times_{\lambda^*,k^*}\mathbb{M}_{\lambda^*,k^*}$. It follows that the algebra B, and hence the initial algebra A, is completely isomorphic to the Cartesian product $\times_{\lambda^*,k^*}\mathbb{M}_{\lambda^*,k^*}$. ∎

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