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EQUILIBRIUM STATES FOR THE LANDAU-FERMI-DIRAC EQUATION

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Abstract. A kinetic collision operator of Landau type for Fermi-Dirac particles is considered. Equilibrium states are rigorously determined under minimal assumptions on the distribution function of the particles. The particular structure of the considered operator (strong non-linearity and degeneracy) requires a special investigation compared to the classical Boltzmann or Landau operator.

1. Introduction. The Landau or Landau-Fokker-Planck equation is a kinetic collision model used to describe the evolution of charged particles in a plasma [2, 3, 4, 11]. When quantum effects such as the Pauli exclusion principle come into play, this collision operator has to be modified and leads to the so-called Landau-Fermi-Dirac (LFD) operator [4, 6, 11]. Besides, a Landau equation with Fermi statistics also arises in the modelling of stellar systems [5, 9]. In this paper, we consider the LFD equation in the spatially homogeneous case. It reads:

$$\partial_t f(t,v) = Q_L(f)(t,v), \qquad t \in \mathbb{R}_+,\ v \in \mathbb{R}^3, \tag{1}$$

where

$$Q_L(f)(t,v) = \nabla \cdot \int \Psi(v-v_*)\Pi(v-v_*)\{f_*(1-f_*)\nabla f - f(1-f)\nabla f_*\}\, dv_*, \tag{2}$$

with $f = f(t,v)$, $f_* = f(t,v_*)$, $\Pi(z)$ denotes the orthogonal projection on $(\mathbb{R}z)^\perp$,

$$\Pi_{i,j}(z) = \delta_{i,j} - \frac{z_i z_j}{|z|^2}, \qquad 1 \le i,j \le 3,$$

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and Ψ is a function such as $\Psi(z) = |z|^{2+\gamma}$, $-3 \leq \gamma \leq 1$. Here as in the rest of this paper, ∇ denotes the gradient operator with respect to the v variable. The choice $\Psi(z) = |z|^{2+\gamma}$ corresponds to inverse power law potentials. According to the value of γ, we distinguish the Coulomb potential ($\gamma = -3$), soft potentials ($-3 < \gamma < 0$), the Maxwellian potential ($\gamma = 0$) and hard potentials ($0 < \gamma \leq 1$). We recall here that the Coulomb potential is however the only one to have a physical relevance.

Equilibrium states and trend to equilibrium for the classical Boltzmann and Landau equations have been considered in several papers, see [3, 7, 14, 15] for the Boltzmann equation and [8, 16, 17] for the Landau equation, and the references therein. For the Boltzmann-Fermi-Dirac (BFD) equation, Lu [12] has shown the existence of two classes of equilibria, which are the class of Fermi-Dirac distributions and the class of characteristic functions of the euclidean balls. Large time behaviour for the BFD equation has been studied in [13]. To our knowledge, there are few works on the Landau-Fermi-Dirac equation ([6, 10, 1]). In particular, the determination of its equilibrium states have not been yet considered at a rigorous level. We point out that the Pauli exclusion principle implies that both a solution to the LFD and BFD equations must satisfy $0 \leq f \leq 1$ as soon as this is satisfied by the initial data. Similarly to the BFD equation, there should be two classes of equilibria for the LFD equation, namely the class of Fermi-Dirac distributions and a class of degenerated equilibria. Our purpose in this present work is to clarify this claim. In particular, we rigorously determine the expressions of the equilibrium states (i.e. the solutions to $Q_L(f) = 0$) under minimal and 'natural' assumption on the distribution function f. The strong non-linearity in (2) (term $f(1-f)$) and its degeneracy for $f \sim 1$ give rise to additional difficulties compared to the classical case and a special treatment is required.

We now describe the contents of the paper. We set notations and state our main result in the next section. The proof is given in Section 3.

2. Main results. The usual *a priori* estimates are available for (1)-(2). Indeed, one can formally check that solutions preserve mass and energy, namely

$$\forall t \geq 0, \qquad \int f(t,v)\,dv = \int f_{in}\,dv \qquad \text{and} \qquad \int f(t,v)\,|v|^2\,dv = \int f_{in}\,|v|^2\,dv.$$

Moreover, considering the entropy for Fermi-Dirac particles defined by

$$S(f) = -\int [f\ln f + (1-f)\ln(1-f)]\,dv \geq 0,$$

one can see, still formally, that $t \longmapsto S(f)(t)$ is a non-decreasing function. More generally, the dissipation term reads

$$\int Q_L(f)[\ln(1-f) - \ln f]\,dv = \frac{1}{2}\iint \Pi(v-v_*)|v-v_*|^{\gamma+2}$$
$$(f_*(1-f_*)\nabla f - f(1-f)\nabla f_*)\left(\frac{\nabla f}{f(1-f)} - \frac{\nabla f_*}{f_*(1-f_*)}\right)dv_*\,dv.$$

The conservation of mass and energy and the fact that the entropy is a non-decreasing function have been rigorously proved in [1] for solutions to (1)-(2) with $0 \leq \gamma \leq 1$.

Equilibrium states are usually defined thanks to the cancellation of the dissipation term. The problem here is to give a meaning to this expression. Noting that

$$2\,\nabla\Big[\;\text{Arctan}\,\sqrt{\frac{f}{1-f}}\;\Big] = \frac{\nabla f}{\sqrt{f(1-f)}}$$

and that Π is a projector and thus satisfies $\Pi = \Pi^2$, we infer that

$$\int Q_L(f)[\ln(1-f) - \ln f]\,dv$$
$$= 2\iint |\Pi(v-v_*)|v-v_*|^{(2+\gamma)/2}[g_*\nabla(p(f)) - g\nabla_*(p(f_*))]|^2\,dv_*\,dv,$$

where $g = \sqrt{f(1-f)}$, $p(f) = \text{ Arctan }(\sqrt{f/(1-f)})$ and ∇_* denotes the gradient operator with respect to the v_* variable.

If f is a measurable function satisfying $0 \le f \le 1$ a.e. then $p(f)$ belongs to $L^\infty(\mathbb{R}^3)$. Consequently, $\nabla p(f) \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$. We may now define what we mean by equilibrium states. We consider

$$\Omega = \{(v, v_*) \in (\mathbb{R}^3)^2; v \neq v_*\}.$$

Definition 1. A function $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is said to be an equilibrium state for the LFD equation if it satisfies $0 \le f \le 1$ a.e. and

$$\Pi(v-v_*)|v-v_*|^{(2+\gamma)/2}[g_*\nabla(p(f)) - g\nabla_*(p(f_*))] = 0, \qquad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^3). \tag{3}$$

Formally, if f is a smooth function that satisfies $0 \le f \le 1$ a.e. and (3), then

$$f(v) = \frac{ae^{-b|v-V_0|^2}}{1 + ae^{-b|v-V_0|^2}},$$

with $a, b > 0$ and $V_0 \in \mathbb{R}^3$. Our aim is to give a rigorous proof for this statement, under 'minimal' assumptions for f.

Remark 2. Any function $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that $0 \le f \le 1$ a.e. and $f(1-f) = 0$ a.e. satisfies (3), that is, any characteristic function of a measurable set with a finite measure is a solution to (3). We thus recover a class of degenerate equilibria as for the BFD equation (see [12]). However, this new class strictly includes the one concerning the BFD equation.

Owing to the previous remark, we restrict ourselves to the functions that satisfy (3) and

$$\text{meas}(\{v \in \mathbb{R}^3; 0 < f(v) < 1\}) \neq 0. \tag{4}$$

Our main result is the following.

Theorem 3. *The equilibrium states of the LFD equation satisfying (4) are the Fermi-Dirac distributions, that is, the functions of the following form:*

$$f(v) = \frac{ae^{-b|v-V_0|^2}}{1 + ae^{-b|v-V_0|^2}},$$

with $V_0 \in \mathbb{R}^3$ *and* $a, b > 0$.

3. Proof of Theorem 3. Let $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfying (3), (4) and $0 \le f \le 1$ a.e. on $\mathbb{R}^3$. We set

$$T = g_* \nabla(p(f)) - g \nabla_*(p(f_*)).$$

Then, (3) implies that

$$\Pi(v - v_*) T = 0 \qquad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^3). \tag{5}$$

LEMMA 4. *If (5) holds, there exists a real-valued distribution $\Lambda_{v,v_*} \in \mathcal{D}'(\Omega, \mathbb{R})$ such that*

$$T = (v - v_*) \Lambda_{v,v_*}, \qquad \textit{in } \mathcal{D}'(\Omega, \mathbb{R}^3). \tag{6}$$

Proof. The proof of this lemma is similar to that of the classical case [17]. Let $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^3)$. Since $\Pi(z)$ is the orthogonal projection on $(\mathbb{R}z)^\perp$,

$$\varphi(v, v_*) = \lambda(v, v_*)(v - v_*) + \zeta(v, v_*),$$

with

$$\begin{aligned} \zeta(v, v_*) &= \Pi(v - v_*)\zeta(v, v_*) = \Pi(v - v_*)\varphi(v, v_*), \\ \lambda(v, v_*) &= \frac{\varphi(v, v_*) \cdot (v - v_*)}{|v - v_*|^2}. \end{aligned}$$

Then,

$$\begin{aligned} \langle T, \varphi(v, v_*) \rangle &= \langle (v - v_*) \cdot T \,,\, \lambda(v, v_*) \rangle + \langle T \,,\, \Pi(v - v_*)\zeta(v, v_*) \rangle \\ &= \langle (v - v_*) \cdot T \,,\, \frac{\varphi(v, v_*) \cdot (v - v_*)}{|v - v_*|^2} \rangle + \langle \Pi(v - v_*)\, T,\, \zeta(v, v_*) \rangle, \end{aligned}$$

where $\langle \ ,\ \rangle$ denotes the dual product. Owing to (5), equation (6) holds for

$$\Lambda_{v,v_*} = \frac{(v - v_*) \cdot T}{|v - v_*|^2}. \quad \blacksquare$$

LEMMA 5. *Let $\mathcal{P}$ be a measurable set with a positive measure. Then, there exist distinct points $u_i \in \mathbb{R}^3$, $i = 1, 2, 3$ such that, for $i = 1, 2, 3$, we have*

$$\forall r > 0, \quad meas(B(u_i, r) \cap \mathcal{P}) > 0, \tag{7}$$

where $B(u_i, r)$ denotes the ball with center u_i and radius r of $\mathbb{R}^3$.

Moreover, there exist $r_i > 0$, $i = 1, 2, 3$ such that

$$B_i \cap B_j = \emptyset, \quad \textit{if} \quad i \neq j, \tag{8}$$

where $B_i := B(u_i, r_i)$, $i = 1, 2, 3$.

Proof.
Step 1. We first prove that there exists $u_1 \in \mathbb{R}^3$ that satisfies (7). Suppose, contrary to our claim, that for every $w \in \mathbb{R}^3$ there exists $r(w) > 0$ such that $\text{meas}(B(w, r(w)) \cap \mathcal{P}) = 0$. Then, for $n \in \mathbb{N}$,

$$B(0, n) \subset \bigcup_{w \in B(0,n)} B(w, r(w)).$$

Since $B(0, n)$ is relatively compact in $\mathbb{R}^3$, there exist some w_i, $i = 1, \dots, N$, such that

$$B(0, n) \subset \bigcup_{i=1}^{N} B(w_i, r(w_i)).$$

Hence,

$$\text{meas}(B(0,n)\cap\mathcal{P})\le\sum_{i=1}^{N}\text{meas}(B(w_i,r(w_i))\cap\mathcal{P})=0$$

and $\text{meas}(\mathcal{P})=\lim_{n\to\infty}\text{meas}(B(0,n)\cap\mathcal{P})=0$, which contradicts our assumption on $\mathcal{P}$. Consequently, there exists $u_1\in\mathbb{R}^3$ that satisfies (7).

Step 2. The function τ defined by $\tau(r)=\text{meas}(B(u_1,r)\cap\mathcal{P})$ is continuous and satisfies $\tau(0)=0$ and $\lim_{r\to+\infty}\tau(r)=\text{meas}(\mathcal{P})$. Therefore, there exists $r_1>0$ such that

$$\text{meas}(B(u_1,2\,r_1)\cap\mathcal{P})\le\frac{\text{meas}(\mathcal{P})}{4}.\tag{9}$$

We set $\mathcal{P}_1:=\mathcal{P}\backslash B(u_1,2r_1)$. From (9) follows that $\text{meas}(\mathcal{P}_1)\ge 3\,\text{meas}(\mathcal{P})/4>0$. Similarly to the first step, we infer that there exists $u_2\in\mathbb{R}^3\backslash B(u_1,2r_1)$ such that

$$\forall r>0,\quad \text{meas}(B(u_2,r)\cap\mathcal{P}_1)>0.$$

Since $\mathcal{P}_1\subset\mathcal{P}$, u_2 also satisfies (7). As previously, there exists $\overline{r}_2>0$ such that

$$\text{meas}(B(u_2,2\,\overline{r}_2)\cap\mathcal{P})\le\frac{\text{meas}(\mathcal{P})}{4}.$$

We choose $r_2:=\min(\overline{r}_2,d(u_2,B(u_1,r_1)))$, where $d(u_2,B(u_1,r_1))$ denotes the distance between u_2 and $B(u_1,r_1)$.

We now set $\mathcal{P}_2:=\mathcal{P}\backslash(B(u_1,2r_1)\cup B(u_2,2r_2))$. Then, $\text{meas}(\mathcal{P}_2)\ge\text{meas}(\mathcal{P})/2>0$. Similarly to the first step, it implies that there exists $u_3\in\mathbb{R}^3\backslash(B(u_1,2r_1)\cup B(u_2,2r_2))$ such that

$$\forall r>0,\quad \text{meas}(B(u_3,r)\cap\mathcal{P}_2)>0.$$

Since $\mathcal{P}_2\subset\mathcal{P}$, u_3 satisfies (7). We set $r_3:=\min(d(u_3,B(u_1,r_1)),d(u_3,B(u_2,r_2)))$. ∎

PROPOSITION 6. *Let $f\in L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ satisfying $0\le f\le 1$ a.e., (4) and (5). Then $f\in\mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R})$ and $p(f)\in\mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R})$.*

Proof. We consider

$$U=\{(v_1,v_2,v_3)\in(\mathbb{R}^3)^3\ ;\ v_1\ne v_2,\ v_1\ne v_3,\ v_2\ne v_3\},$$

and for $(v_1,v_2,v_3)\in U$, we set $f_i=f(v_i)$, $g_i=\sqrt{f_i(1-f_i)}$ and $\Lambda_{i,j}=\Lambda_{v_i,v_j}$, $i,j=1,2,3$. We deduce from Lemma 4 that

$$\begin{aligned}g_2\,g_3\,\nabla p(f_1)-g_1\,g_3\,\nabla p(f_2)&=(v_1-v_2)g_3\Lambda_{1,2},\\ g_3\,g_1\,\nabla p(f_2)-g_2\,g_1\,\nabla p(f_3)&=(v_2-v_3)g_1\Lambda_{2,3},\\ g_1\,g_2\,\nabla p(f_3)-g_3\,g_2\,\nabla p(f_1)&=(v_3-v_1)g_2\Lambda_{3,1},\end{aligned}$$

in $\mathcal{D}'(U,\mathbb{R}^3)$. Summing these three equations leads to

$$0=(v_1-v_2)g_3\Lambda_{1,2}+(v_2-v_3)g_1\Lambda_{2,3}+(v_3-v_1)g_2\Lambda_{3,1},\qquad\text{in }\mathcal{D}'(U,\mathbb{R}^3).$$

Since $v_3-v_1=v_3-v_2+v_2-v_1$, we get

$$(v_1-v_2)[g_3\Lambda_{1,2}-\Lambda_{3,1}g_2]+(v_2-v_3)[g_1\Lambda_{2,3}-g_2\Lambda_{3,1}]=0,\qquad\text{in }\mathcal{D}'(U,\mathbb{R}^3).\tag{10}$$

For $(v_1, v_2, v_3) \in (\mathbb{R}^3)^3$, we set $V = (v_1 - v_2)|v_2 - v_3|^2 - \big[(v_1 - v_2)\cdot(v_2 - v_3)\big](v_2 - v_3)$ and

$$\begin{aligned} d(v_1 - v_2, v_2 - v_3) &= V \cdot (v_1 - v_2) \\ &= |v_1 - v_2|^2 |v_2 - v_3|^2 - \big[(v_1 - v_2)\cdot(v_2 - v_3)\big]^2. \end{aligned}$$

Easy calculations lead to the following properties of d:

LEMMA 7. *For every $(v_1, v_2, v_3) \in (\mathbb{R}^3)^3$, the function d satisfies*

- $d(v_1 - v_2, v_2 - v_3) = d(v_1 - v_2, v_1 - v_3) = d(v_1 - v_3, v_1 - v_2)$,
- $d(v_1 - v_2, v_2 - v_3) \geq 0$,
- $d(v_1 - v_2, v_2 - v_3) = 0 \Longleftrightarrow v_1 - v_2$ *and* $v_2 - v_3$ *colinear,*
  $\Longleftrightarrow v_1, v_2$ *and* v_3 *are aligned points in* $\mathbb{R}^3$.

In particular, if $v_1 \neq v_2$, meas $\{v_3 \in \mathbb{R}^3; d(v_1 - v_2, v_2 - v_3) = 0\} = 0$.

Taking test functions of the form $V\varphi$ with $\varphi \in \mathcal{D}(U, \mathbb{R})$ in (10), we deduce from $V \cdot (v_2 - v_3) = 0$ that

$$d(v_1 - v_2, v_2 - v_3)[\Lambda_{1,2} g_3 - \Lambda_{3,1} g_2] = 0, \qquad \text{in } \mathcal{D}'(U, \mathbb{R}). \tag{11}$$

We set $\mathcal{P} := \{v \in \mathbb{R}^3 \;/\; f(v)(1 - f(v)) > 0\}$. By (4) and Lemma 5, there exists $u_i \in \mathbb{R}^3$ and $r_i > 0$, $i = 1, 2, 3$ such that (7) and (8) hold. We first show that $f \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_3, \mathbb{R})$ and $p(f) \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_3, \mathbb{R})$. For $i = 1, 2, 3$, there exists a nonnegative function $\psi_i \in \mathcal{D}(\mathbb{R}^3, \mathbb{R})$ such that

$$B\left(u_i, \frac{r_i}{2}\right) \subset \text{ supp } (\psi_i) \subset B_i. \tag{12}$$

By (7) and the definition of $\mathcal{P}$, we have $\int g_3 \psi_3(v_3)\, dv_3 > 0$. Owing to Lemma 7,

$$\int d(v_1 - v_2, v_2 - v_3) g_3 \psi_3(v_3)\, dv_3 > 0, \qquad \forall (v_1, v_2) \in \Omega.$$

Moreover, the function

$$(v_1, v_2) \mapsto \int d(v_1 - v_2, v_2 - v_3) g_3 \psi_3(v_3)\, dv_3$$

belongs to $\mathcal{C}^\infty(\Omega, \mathbb{R})$. Taking test functions of the form $(v_1 - v_2)\cdot \varphi(v_1, v_2)\, \psi_3(v_3)$ with $\varphi \in \mathcal{D}(\Omega \backslash (B_3)^2, \mathbb{R}^3)$ in (11) leads to

$$g_1 \nabla p(f_2) - g_2 \nabla p(f_1) = (v_1 - v_2)\, G_{v_1}(v_2)\, g_2 \quad \text{ in } \quad \mathcal{D}'(\Omega \backslash (B_3)^2, \mathbb{R}^3), \tag{13}$$

where

$$G_{v_1}(v_2) = \frac{\langle d(v_1 - v_2, v_2 - v_3)\Lambda_{3,1}, \psi_3(v_3)\rangle_{v_3}}{\int d(v_1 - v_2, v_2 - v_3) g_3 \psi_3(v_3)\, dv_3}.$$

We denote here by $\langle , \rangle_{v_3}$ the dual product with respect to the v_3 variable. By (7), (12) and the definition of $\mathcal{P}$, we have $\int g_1\, \psi_1(v_1)\, dv_1 > 0$. Thus, taking test functions of the form $\theta(v_2)\psi_1(v_1)$ with $\theta \in \mathcal{D}(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R}^3)$ in (13), we get

$$\nabla p(f_2) = \xi(v_2)\, g_2 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R}^3), \tag{14}$$

where the function ξ is defined on $\mathbb{R}^3 \backslash (B_1 \cup B_3)$ by

$$\xi(v_2) = \frac{1}{\langle g_1, \psi_1(v_1)\rangle_{v_1}} \left[\langle \nabla p(f_1), \psi_1(v_1)\rangle_{v_1} + \langle (v_1 - v_2) G_{v_1}(v_2), \psi_1(v_1)\rangle_{v_1}\right].$$

Since $\xi \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R})$ and $g \in L^\infty(\mathbb{R}^3)$, we deduce that $p(f) \in W^{1,\infty}_{loc}(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R})$. From Sobolev embeddings follows that $p(f) \in \mathcal{C}(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R})$. We now consider $h = \sqrt{f/(1-f)}$. Since $p(f) = \operatorname{Arctan}(h)$, we deduce that $h \in \mathcal{C}(\mathbb{R}^3 \backslash (B_1 \cup B_3), \overline{\mathbb{R}})$. Moreover, (14) reads

$$\nabla \left(\operatorname{Arctan}(h)\right) = \xi \, \frac{h}{1+h^2} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R}^3).$$

Consequently, $\operatorname{Arctan}(h) \in \mathcal{C}^1(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R})$ and $h \in \mathcal{C}^1(\mathbb{R}^3 \backslash (B_1 \cup B_3), \overline{\mathbb{R}})$. By bootstrap, it follows that $h \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash (B_1 \cup B_3), \overline{\mathbb{R}})$. Thus,

$$f \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R}) \qquad \text{and} \qquad p(f) \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash (B_1 \cup B_3), \mathbb{R}).$$

The same calculations with ψ_2 instead of ψ_1 lead to $f \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash (B_2 \cup B_3), \mathbb{R})$ and $p(f) \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash (B_2 \cup B_3), \mathbb{R})$. From (8) follows that $f \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_3, \mathbb{R})$ and $p(f) \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_3, \mathbb{R})$. The same proof with B_2 instead of B_3 implies that $f \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_2, \mathbb{R})$ and $p(f) \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_2, \mathbb{R})$. By (8), the proof of Proposition 6 is now complete. ∎

Proof of Theorem 3. Owing to Proposition 6, $T \in \mathcal{C}^\infty((\mathbb{R}^3)^2, \mathbb{R}^3)$. We define the real function $\overline{\Lambda}$ by

$$\overline{\Lambda}(v, v_*) = \begin{cases} (v - v_*) \cdot T/|v - v_*|^2 & \text{if } v \neq v_* \\ 0 & \text{else.} \end{cases}$$

Then, it follows from Lemma 4 that

$$T = (v - v_*)\overline{\Lambda}(v, v_*), \qquad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^3).$$

Since T and $\overline{\Lambda}$ belong respectively to $\mathcal{C}^\infty((\mathbb{R}^3)^2, \mathbb{R}^3)$ and $\mathcal{C}^\infty(\Omega, \mathbb{R})$, this equality holds in fact a.e. on $(\mathbb{R}^3)^2$. Therefore,

$$\begin{aligned} g_3\, g_2\, \nabla p(f_1) - g_3\, g_1\, \nabla p(f_2) &= (v_1 - v_2)\overline{\Lambda}(v_1, v_2)\, g_3, \\ g_1\, g_3\, \nabla p(f_2) - g_1\, g_2\, \nabla p(f_3) &= (v_2 - v_3)\overline{\Lambda}(v_2, v_3)\, g_1, \qquad \text{a.e. on } (\mathbb{R}^3)^3. \\ g_2\, g_1\, \nabla p(f_3) - g_2\, g_3\, \nabla p(f_1) &= (v_3 - v_1)\overline{\Lambda}(v_3, v_1)\, g_2, \end{aligned}$$

As previously, we deduce that

$$(v_1 - v_2)\overline{\Lambda}(v_1, v_2) g_3 + (v_2 - v_3)\overline{\Lambda}(v_2, v_3) g_1 + (v_3 - v_1)\overline{\Lambda}(v_3, v_1) g_2 = 0,$$

a.e. on $(\mathbb{R}^3)^3$. Consequently, multiplying by $v_2 \times v_3$ leads to

$$\det(v_1, v_2, v_3)\left[\overline{\Lambda}(v_1, v_2) g_3 - \overline{\Lambda}(v_3, v_1) g_2\right] = 0 \qquad \text{a.e. on } (\mathbb{R}^3)^3.$$

Since $\operatorname{meas}\big\{(v_1, v_2, v_3) \in (\mathbb{R}^3)^3; \det(v_1, v_2, v_3) = 0\big\} = 0$, we get

$$\overline{\Lambda}(v_1, v_2) g_3 - \overline{\Lambda}(v_3, v_1) g_2 = 0 \qquad \text{a.e. on } (\mathbb{R}^3)^3.$$

Let θ be a nonnegative function from $\mathcal{D}(\mathbb{R}^3, \mathbb{R})$ such that $\int_{\mathbb{R}^3} g_2\, \theta(v_2)\, dv_2 > 0$. Then, $\overline{\Lambda}(v_3, v_1) = \mu_1\, g_3$. By symmetry, we deduce that

$$\overline{\Lambda}(v_3, v_1) = \lambda\, g_1\, g_3 \qquad \text{a.e. on } (\mathbb{R}^3)^2,$$

with $\lambda \in \mathbb{R}$. From (4) and Proposition 6 follows the existence of $u_0 \in \mathbb{R}^3$ and $r > 0$ such that $f(1-f) > 0$ on $B(u_0, r)$. Therefore,

$$f_*(1 - f_*)\nabla f - f(1-f)\nabla f_* = \lambda\, f f_*(1-f)(1-f_*)\,(v - v_*), \qquad \text{a.e. on } (B(u_0, r))^2.$$

Let ψ be a nonnegative function from $\mathcal{D}(B(u_0,r),\mathbb{R})$. Then, $\int_{\mathbb{R}^3} f_*(1-f_*)\,\psi(v_*)\,dv_* > 0$. We set, if $\lambda \neq 0$,

$$\lambda\, V_0 = \frac{1}{\int_{\mathbb{R}^3} f_*\,(1-f_*)\,\psi(v_*)\,dv_*}\left[-\langle \nabla f_*\,,\,\psi(v_*)\rangle + \lambda\langle f_*(1-f_*)v_*, \psi(v_*)\rangle\right] \in \mathbb{R}^3.$$

Then,

$$\nabla f = \lambda\, f\,(1-f)\,(v - V_0), \qquad \text{a.e. on } B(u_0,r).$$

Since $f(1-f) > 0$ on $B(u_0,r)$, we have

$$\nabla\left[\sqrt{\frac{f}{1-f}}\,e^{-\lambda\frac{|v-V_0|^2}{4}}\right] = 0, \qquad \text{on } B(u_0,r).$$

Hence,

$$f(v) = \frac{Ce^{\lambda\frac{|v-V_0|^2}{2}}}{1+Ce^{\lambda\frac{|v-V_0|^2}{2}}} \qquad \text{on } B(u_0,r), \tag{15}$$

where $\lambda < 0$ because $f \in L^1(\mathbb{R}^3)$. Owing to Proposition 6, we deduce that (15) holds on $\mathbb{R}^3$. Similar calculations for $\lambda = 0$ lead to a nonintegrable function. ■

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