

INTERFACE DYNAMICS FOR AN ANISOTROPIC ALLEN-CAHN EQUATION

MICHAL BENEŠ

*Department of Mathematics, Czech Technical University of Prague
Trojanova 13, 120 00 Prague, Czech Republic
E-mail: Benes@km1.fjfi.cvut.cz*

DANIELLE HILHORST

*Laboratoire de Mathématique, Université de Paris Sud
91405 Orsay Cedex, France
E-mail: Danielle.Hilhorst@math.u-psud.fr*

RÉMI WEIDENFELD

*Ecole Centrale de Lyon, Département Math-Info
36, Avenue Guy de Collongue, 69134 Ecully Cedex, France
E-mail: Remi.Weidenfeld@ec-lyon.fr*

1. Introduction. Main results. The purpose of this work is to study the limiting behavior as $\epsilon \rightarrow 0$ of the solution u^ϵ of Problem (P^ϵ) ,

$$(P^\epsilon) \quad \begin{cases} u_t = \operatorname{div} \left(\Phi^0(\nabla u) \Phi_\xi^0(\nabla u) \right) - F \tilde{\Phi}^0(\nabla u) + \frac{1}{\epsilon^2} f(u) & \text{in } \Omega \times (0, \infty), \\ \nabla_\Phi u \cdot \nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for all } x \in \Omega, \end{cases}$$

where Ω is a smooth bounded domain of \mathbb{R}^N ($N \geq 2$), ν is the Euclidean unit normal vector exterior to $\partial\Omega$, $f(s) = 2s(1 - s^2)$, $F = F(x, t)$ is a smooth function on $\bar{\Omega} \times (0, \infty)$, Φ and $\tilde{\Phi}$ are two Finsler metrics with dual functions Φ^0 and $\tilde{\Phi}^0$ respectively and where Φ_ξ^0 denotes the gradient of Φ^0 . For a precise definition of Finsler metric and Finsler geometry, we refer to [2] or to [3].

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Problem (P^ϵ) has been proposed by M. Beneš; the partial differential equation in (P^ϵ) represents an anisotropic version of the equation

$$u_t = \Delta u - F|\nabla u| + \frac{1}{\epsilon^2}f(u), \quad (1)$$

which has been analyzed by M. Beneš and K. Mikula in the context of mean curvature flow, and by M. Beneš in the context of microstructure formation during solidification (we refer to [3] for the references). In particular, they have remarked that the gradient term in (1) allows to construct a better numerical scheme than the usual form of the Allen-Cahn equation

$$u_t = \Delta u + \frac{1}{\epsilon^2} \left(f(u) + \frac{\epsilon}{c'(0)} F \right), \quad (2)$$

where c is the velocity of a related travelling wave.

In our work, we make the following hypotheses on Φ^0 and $\tilde{\Phi}^0$:

$$H_\Phi \quad \begin{cases} \Phi^0 \in C^{3+\alpha}(\mathbb{R}^N \setminus \{0\}), \\ \tilde{\Phi}^0 \in C^{1+\alpha}(\mathbb{R}^N \setminus \{0\}), \end{cases}$$

for some $\alpha \in (0, 1)$. Setting $\Gamma_0 := \{x \in \Omega, u_0(x) = 0\}$, we assume that

$$H_0 \quad \begin{cases} u_0 \in C^2(\bar{\Omega}) \text{ satisfies the compatibility condition } \nabla_\Phi u_0 \cdot \nu = 0 \text{ on } \partial\Omega, \\ \Gamma_0 \text{ is a } C^{4+\alpha} \text{ hypersurface without boundary such that } \Gamma_0 \subset\subset \Omega, \\ \nabla u_0 \neq 0 \text{ on } \Gamma_0, u_0 \text{ is negative inside } \Gamma_0 \text{ and positive outside.} \end{cases}$$

Moreover, since Φ^0 and $\tilde{\Phi}^0$ are Finsler metrics, they satisfy the following regularity properties,

$$\begin{cases} \Phi^0 \in C^{0,1}(\mathbb{R}^N), (\Phi^0)^2 \in C^1(\mathbb{R}^N), \\ (\Phi^0)^2 \text{ is strictly convex,} \\ \tilde{\Phi}^0 \in C^{0,1}(\mathbb{R}^N). \end{cases}$$

We associate to Problem (P^ϵ) the moving boundary problem

$$(P^0) \quad \begin{cases} V_{n,\Phi} = -(N-1)\kappa_\Phi + F \frac{\tilde{\Phi}^0(\vec{n})}{\Phi^0(\vec{n})} & \text{on } \Gamma_t, t \in (0, T], \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where $V_{n,\Phi}$ is the velocity in the anisotropic normal direction, κ_Φ is the anisotropic mean curvature and \vec{n} is the Euclidean normal vector to Γ_t .

For each $t \in [0, T]$, we define Ω_t^- as the interior of Γ_t , Ω_t^+ as the exterior of Γ_t and we set

$$u(., t) = \begin{cases} +1 & \text{in } \Omega_t^+ \\ -1 & \text{in } \Omega_t^- \end{cases} \quad \text{for } t \in [0, T].$$

To begin with we prove the existence and uniqueness of a smooth solution of Problem (P^0) locally in time.

THEOREM 1. *Let hypotheses H_Φ and H_0 be fulfilled. Then there exists $T > 0$ such that Problem (P^0) has a unique solution $\Gamma = \cup\{\Gamma_t \times \{t\}, 0 \leq t \leq T\} \in C^{3+\alpha, (3+\alpha)/2}$.*

Our second main result deals with the asymptotic behavior of the solution u^ϵ of Problem (P^ϵ) as ϵ tends to zero.

THEOREM 2. *Suppose that the hypotheses H_Φ and H_0 are satisfied and let T be the maximum time of existence of a smooth solution of Problem (P^0) . Then as $\epsilon \rightarrow 0$, u^ϵ converges to u almost everywhere in $\cup\{\Omega_t^\pm \times \{t\}, t \in (0, T]\}$.*

Convergence results to viscosity solutions have been obtained by [6] and [7] but they hold for a very limited class of initial conditions.

Sketches of the proofs of Theorems 1 and 2 are presented in Section 2; we refer the reader to [3] for the detailed proofs. In Section 3 we discuss the special case of space dimension 2.

2. Sketches of the proofs. The proofs of the theorems 1 and 2 follow and extend similar proofs presented in [4] and [5] in the isotropic case.

2.1. Sketch of the proof of Theorem 1. The idea is to rewrite Problem (P^0) as a parabolic partial differential equation on a fixed \mathbb{R}^{N-1} manifold.

To that purpose, we choose a closed $N - 1$ manifold \mathcal{M} without boundary that is diffeomorphic to Γ_0 . Let $X^0 : \mathcal{M} \rightarrow \Gamma_0$ be a smooth diffeomorphism and denote by $s' = (s_1, \dots, s_{N-1})$ the coordinates in \mathcal{M} . Then each point x in a tubular neighborhood of Γ_0 may be uniquely written in the form $x = X^0(s') + s_N \vec{n}$ where \vec{n} is the Euclidean unit normal vector to Γ_0 at the point $X^0(s')$ and $s_N \in \mathbb{R}$. In a neighborhood of Γ_0 , we perform a coordinate transformation from the usual coordinates (x_1, \dots, x_N) to the local coordinates (s_1, \dots, s_N) .

Therefore, solving Problem (P^0) for small time amounts to finding a function $\Lambda : \mathcal{M} \times [0, T] \rightarrow \mathbb{R}$ such that

$$\Gamma_t = \{x = X^0(s') + s_N \vec{n}, \quad s_N = \Lambda(s', t)\}.$$

The function $\psi = s_N - \Lambda$ is such that $\psi < 0$ on Ω_t^- , $\psi = 0$ on Γ_t and $\psi > 0$ on Ω_t^+ . It turns out that Λ satisfies the partial differential equation,

$$\Lambda_t = \sum_{i,j=1}^{N-1} a_{ij} \frac{\partial^2 \Lambda}{\partial s_i \partial s_j} + \Pi(s', t, \Lambda, \nabla \Lambda) \quad (loc),$$

where Π is a smooth function and a_{ij} is defined by

$$a_{ij} = \vec{\alpha}_i(s') \cdot (\Phi^0 \Phi_{\xi\xi}^0)(\vec{n} + \vec{G}(\nabla \Lambda)) \vec{\alpha}_j(s');$$

here $\{\vec{\alpha}_i\}_{1 \leq i \leq N-1}$ are smooth vector fields given by the partial derivatives of the diffeomorphism X^0 and such that $\vec{\alpha}_i \neq 0$, $\vec{\alpha}_i \perp \vec{n}$, and \vec{G} is a smooth vector field such that $\vec{G} \perp \vec{n}$.

The last step consists in proving that the equation (loc) is parabolic. To that purpose, we use convexity properties of Φ^0 to describe the eigenvalues and the corresponding eigenfunctions of the matrix $\Phi^0 \Phi_{\xi\xi}^0$. We prove that there exists $\bar{\lambda} > 0$ such that for all $\xi, \zeta \in \mathbb{R}^N \setminus \{0\}$

$$\zeta \cdot (\Phi^0 \Phi_{\xi\xi}^0)(\xi) \zeta \geq \bar{\lambda} \left(|\zeta|^2 - \frac{(\zeta \cdot \xi)^2}{|\xi|^2} \right).$$

Note that if Φ^0 is the Euclidean metric then $(\Phi^0\Phi_{\xi\xi}^0)(\xi)\zeta = \zeta - \frac{\zeta \cdot \xi}{|\xi|^2}\xi$ which is the projection of ζ on $(\xi)^\perp$. It follows from the orthogonality of $\vec{\alpha}_i$ and \vec{G} with \vec{n} that for all $\zeta \in \mathbb{R}^{N-1} \setminus \{0\}$,

$$\begin{aligned} \sum_{i,j=1}^{N-1} a_{ij}\zeta_i\zeta_j &= \left(\sum_{i=1}^{N-1} \zeta_i \vec{\alpha}_i \right) \cdot (\Phi^0\Phi_{\xi\xi}^0)(\vec{n} + \vec{G}(\nabla\Lambda)) \left(\sum_{i=1}^{N-1} \zeta_i \vec{\alpha}_i \right) \\ &\geq \bar{\lambda} \left(\left| \sum_{i=1}^{N-1} \zeta_i \vec{\alpha}_i \right|^2 - \frac{(\sum_{i=1}^{N-1} \zeta_i \vec{\alpha}_i \cdot \vec{G}(\nabla\Lambda))^2}{1 + |\vec{G}(\nabla\Lambda)|^2} \right) \\ &\geq \bar{\lambda} \frac{|\sum_{i=1}^{N-1} \zeta_i \vec{\alpha}_i|^2}{1 + |\vec{G}(\nabla\Lambda)|^2} \geq \bar{\lambda}_0 \frac{|\zeta|^2}{1 + |\vec{G}(\nabla\Lambda)|^2}, \end{aligned}$$

where the last inequality can be deduced from the fact that X^0 is a diffeomorphism. Thus the equation (*loc*) is parabolic as long as $\nabla\Lambda$ remains bounded. Since $\Lambda|_{t=0} = 0$ it follows that $\nabla\Lambda|_{t=0} = 0$ and we may apply an existence and uniqueness theorem for solutions of parabolic equations such as in [8] to deduce that there exists $T > 0$ and a unique solution of Problem (P^0) on $[0, T]$. The regularity of the solution $\Gamma \in C^{3+\alpha, (3+\alpha)/2}$ is obtained by a careful analysis of the regularity of the functions which appear in the proof.

2.2. Sketch of the proof of Theorem 2. The proof of Theorem 2 is divided into two steps. To begin with we state a **generation of interface** result: the function u^ϵ becomes close to ± 1 outside of a neighborhood of the level set zero of u_0 at a time of order $\mathcal{O}(\epsilon^2 |\ln \epsilon|)$. Then we present a result about the **propagation of interface**, namely we prove that the solution u^ϵ remains close to the function u on the time interval $[\tau^\epsilon, T]$.

These results are proved by constructing suitable upper and lower solutions. A difficulty comes from the lack of regularity of the second order term, $\text{div}(\Phi^0(\nabla u)\Phi_\xi^0(\nabla u))$. More precisely, if $u \in C^2(\Omega)$, the hypotheses H_Φ insure that this second order term is well defined and is continuous at each point $x \in \Omega$ where $\nabla u(x) \neq 0$. On the other hand, at a point $x \in \Omega$ where $\nabla u(x) = 0$, the application $x \mapsto \Phi^0(\nabla u)\Phi_\xi^0(\nabla u)$ is of class C^1 for a very limited class of Finsler metrics Φ^0 (such as the Euclidean metric). Due to this fact, we have to work with weak solutions and then with weak upper and lower solutions. But it is quite hard to check if a function is a lower or upper solution with the weak definition. We use homogeneity properties of Φ^0 and Φ_ξ^0 ,

$$\begin{aligned} \Phi^0(s\xi) &= |s|\Phi^0(\xi) \text{ for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N, \text{ and} \\ \Phi_\xi^0(s\xi) &= \frac{s}{|s|}\Phi_\xi^0(\xi) \text{ for all } s \neq 0 \text{ and } \xi \neq 0, \end{aligned} \tag{3}$$

to deduce that the application $x \mapsto \Phi^0(\nabla u)\Phi_\xi^0(\nabla u)$ admits partial derivatives in any direction even for $x \in \Omega$ such that $\nabla u(x) = 0$. More precisely,

LEMMA. *Let $u \in C^2(\bar{\Omega})$; then the application $x \mapsto \text{div}(\Phi^0(\nabla u)\Phi_\xi^0(\nabla u))$ is well defined and belongs to $L^\infty(\Omega)$.*

Then proving that a smooth function $u \in C^{2,1}(\bar{\Omega} \times [0, +\infty))$ is an upper or lower solution amounts to check the sign of the expression,

$$\mathcal{L}^\epsilon(u) := u_t - \text{div}(\Phi^0(\nabla u)\Phi_\xi^0(\nabla u)) + F\bar{\Phi}^0(\nabla u) - \frac{2}{\epsilon^2}u(1 - u^2).$$

Note that the solution u^ϵ of Problem (P^ϵ) is not necessarily smooth so that $\mathcal{L}^\epsilon(u^\epsilon)$ is not defined. The above criterion is valid only for smooth enough upper and lower solutions. To perform the computations, we extensively use the equality $\Phi_\xi^0(\xi) \cdot \xi = \Phi^0(\xi)$ for $\xi \neq 0$ which is obtained by differentiating the first equality in equation (3).

We construct upper and lower solutions in the following way.

For the generation of interface property: we slightly change the nonlinear function $f(u) = 2u(1 - u^2)$ around its unstable zero and denote the resulting function by f_ϵ . A basic ingredient is the initial value problem,

$$\begin{cases} \frac{\partial w}{\partial \tau}(\zeta, \tau) = f_\epsilon(w) & \text{for } \tau \in (0, \infty), \\ w(\zeta, 0) = \zeta, \end{cases}$$

where, more precisely, the function f_ϵ satisfies

(i) $f_\epsilon(s) = f(s)$ for all $s \in (-\infty, -\frac{\epsilon}{4}] \cup [3\epsilon|\ln \epsilon|, +\infty)$,

(ii) $f_\epsilon(s) \leq f(s)$ for all $s \in \mathbb{R}$,

(iii) $f_\epsilon(s) = \frac{s - \epsilon|\ln \epsilon|}{|\ln \epsilon|}$ for all $s \in [0, 2\epsilon|\ln \epsilon|]$,

(iv) f_ϵ has exactly three zeroes, namely -1 , $\epsilon|\ln \epsilon|$ and 1 .

Then for all $\tau_0 > 0$ there exists a constant $C = C(\tau_0) > 0$ such that the function $\underline{u}^\epsilon(x, t) = w(u_0 - Ct/\epsilon, t/\epsilon^2)$ is a lower solution of Problem (P^ϵ) on $[0, \tau_\epsilon]$ where $\tau_\epsilon = \tau_0 \epsilon^2 |\ln \epsilon|$. Moreover, for all $k > 0$, there exist $\tau_0, c > 0$ such that

$$\begin{cases} \underline{u}^\epsilon(x, \tau_\epsilon) \geq -1 - \epsilon^k & \text{for all } x \in \Omega, \\ \underline{u}^\epsilon(x, \tau_\epsilon) \geq 1 - \epsilon^k & \text{for all } x \in \Omega \text{ such that } u_0(x) \geq c\epsilon|\ln \epsilon|. \end{cases} \quad (4)$$

Similarly, we construct an upper solution which satisfies corresponding inequalities. Then at the time τ_ϵ the solution u^ϵ of Problem (P^ϵ) is close to ± 1 outside a tubular neighborhood of the level set $\{u_0 = 0\}$ of thickness $\mathcal{O}(\epsilon|\ln \epsilon|)$.

For the propagation of interface property: we construct upper and lower solutions on the time interval $[\tau_\epsilon, T + \tau_\epsilon]$ where T is the maximum time of existence of a solution of Problem (P^0) . The essential ingredients are :

The modified signed anisotropic distance function d_Φ to the interface Γ_t which is smooth, positive in Ω_t^+ , negative in Ω_t^- and such that $|d_\Phi(x, t)|$ is the anisotropic distance to Γ_t in a tubular neighborhood with thickness $d_0 > 0$ of Γ and is a constant outside a tubular neighborhood with thickness $2d_0$ of Γ . Two important properties of d_Φ are

$$\begin{cases} \Phi^0(\nabla d_\Phi) = 1 & \text{in a neighborhood of } \Gamma, \\ (d_\Phi)_t = \Delta_\Phi d_\Phi - F\tilde{\Phi}^0(\nabla d_\Phi) & \text{on } \Gamma. \end{cases}$$

The travelling wave solutions $U(z, \delta)$ of a family of related one dimensional parabolic equations indexed by δ ($|\delta|$ is small). More precisely, $U(\cdot, \delta)$ is the solution of

$$(Q) \quad \begin{cases} U_{zz} + c(\delta)U_z + 2U(1 - U^2) + \delta = 0 & z \in \mathbb{R}, \\ U(-\infty) = h_-(\delta), U(\infty) = h_+(\delta), \end{cases}$$

where $h_-(\delta)$, $h_0(\delta)$ and $h_+(\delta)$ are the three roots of the polynomial $2u(1 - u^2) + \delta = 0$ in increasing order. Since Problem (Q) is of bistable type, there exists a unique $c = c(\delta)$

and a unique solution $U = U(z, \delta)$ up to translations. Note that $c(0) = 0$ and that the function $\delta \mapsto c(\delta)$ is smooth. Moreover, U is increasing in \mathbb{R} and $U - h_{\pm}$, as well as its derivatives, decay exponentially fast at infinity.

Using these properties, we show that the functions \bar{u}^ϵ and \underline{u}^ϵ defined by

$$\begin{aligned}\bar{u}^\epsilon(x, t + \tau_\epsilon) &= U\left(\frac{d_\Phi(x, t) + m_1\epsilon|\ln \epsilon|e^{m_2t}}{\epsilon}, \epsilon^2 m_3 e^{m_2t}\right), \\ \underline{u}^\epsilon(x, t + \tau_\epsilon) &= U\left(\frac{d_\Phi(x, t) - m_1\epsilon|\ln \epsilon|e^{m_2t}}{\epsilon}, -\epsilon^2 m_3 e^{m_2t}\right),\end{aligned}$$

for constants $m_1, m_2, m_3 > 0$, are upper and lower solutions for Problem (P^ϵ) on $[\tau_\epsilon, \tau_\epsilon + T]$. Note that \bar{u}^ϵ and \underline{u}^ϵ tend to u as $\epsilon \rightarrow 0$.

3. The case of space dimension 2. An essential application is a generic anisotropy in space dimension 2. It has the form

$$\Phi^0(\xi) = |\xi| \Psi_0\left(\frac{\xi}{|\xi|}\right),$$

where Ψ_0 is a smooth function. If $\Psi(\theta) := \Psi_0\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}\right) = \Phi^0\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}\right)$, the convexity of Φ^0 implies that $\Psi + \Psi'' > 0$ and the corresponding motion law (P^0) has the form

$$(P^0) \quad \begin{cases} V_n = -(N-1)\Psi(\theta)(\Psi(\theta) + \Psi''(\theta))\kappa + F\tilde{\Psi}(\theta) & \text{on } \Gamma_t, t \in (0, T] \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the velocity in the Euclidean normal direction, κ is the usual mean curvature of Γ_t , $\theta = \theta(x, t)$ is the angle of the Euclidean normal vector to Γ_t at the point x with the x_1 -axis and $\tilde{\Psi}(\theta) := \tilde{\Phi}^0\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}\right)$.

Problem (P^0) , which arises in the description of phase transitions between liquid and solid phases in the context of Stefan problems, has been extensively studied by [1]. A consequence of our results is that Problem (P^0) coincides with the singular limit as ϵ tends to zero of Problem (P^ϵ)

$$(P^\epsilon) \quad \begin{cases} u_t = \Delta_\Phi u - F|\nabla u|\tilde{\Psi}_0\left(\frac{\nabla u}{|\nabla u|}\right) + \frac{1}{\epsilon^2}f(u) & \text{in } \Omega \times (0, \infty), \\ \nabla_\Phi u \cdot \nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for all } x \in \Omega, \end{cases}$$

where $\tilde{\Psi}_0\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}\right) := \tilde{\Psi}(\theta)$ and where the operator Δ_Φ is given by

$$\Delta_\Phi u := \operatorname{div} \nabla_\Phi u,$$

with

$$\nabla_\Phi u := \Psi_0^2\left(\frac{\nabla u}{|\nabla u|}\right) \nabla u - \Psi_0\left(\frac{\nabla u}{|\nabla u|}\right) \Psi_0'\left(\frac{\nabla u}{|\nabla u|}\right) (\nabla u)^\perp,$$

and $\xi^\perp = (\xi_2, -\xi_1)$.

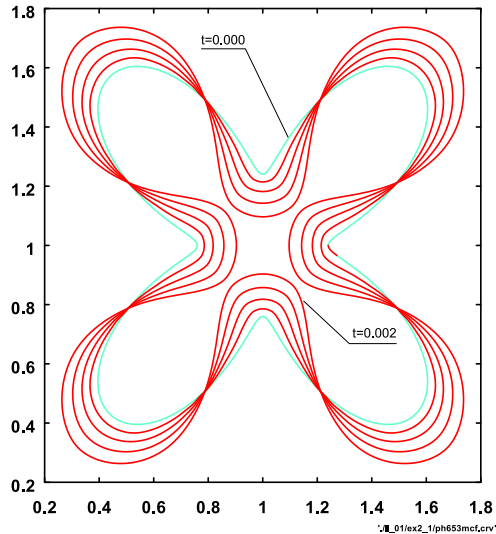


Fig. 1. Motion of an interface satisfying $V_n = -\Psi(\theta)(\Psi(\theta) + \Psi''(\theta))\kappa + F|_{\Gamma}$, obtained by solving the reaction-diffusion equation in Problem (\mathcal{P}^ϵ) .

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