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A NONLOCAL ELLIPTIC EQUATION IN A BOUNDED DOMAIN

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Abstract. The existence of a positive solution to the Dirichlet boundary value problem for the second order elliptic equation in divergence form

$$-\sum_{i,j=1}^{n} D_i(a_{ij}D_j u) = f\left(u, \int_\Omega g(u^p)\right),$$

in a bounded domain Ω in $\mathbb{R}^n$ with some growth assumptions on the nonlinear terms f and g is proved. The method based on the Krasnosel'skiĭ Fixed Point Theorem enables us to find many solutions as well.

1. Introduction. We study the following boundary value problem:

$$(1)\qquad \begin{aligned} &-\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) = f\left(u, \int_\Omega g(u^p)\right),\\ &u|_{\partial\Omega} = 0, \end{aligned}$$

and look for its positive solutions. Ω is assumed to be an open, bounded domain in $\mathbb{R}^n$, where $n \geq 3$, with boundary $\partial\Omega$ sufficiently smooth, i.e. belonging to the Hölder space $C^{1,\varepsilon}$ for $\varepsilon > 0$. The coefficients a_{ij} with its first order derivatives are supposed to be Hölder continuous functions (i.e. $a_{ij} \in C^{1,\varepsilon}$), such that the matrix $(a_{ij})_{i,j=1}^n$ is symmetric

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and uniformly positive definite in Ω, i.e. there exists a positive constant $A > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq A|\xi|^2$ for any $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

The study of such problems is motivated by numerous physical applications including: fully turbulent behaviour of a real flow [3], thermal runaway in Ohmic heating [2] and systems of particles in thermodynamical equilibrium interacting via gravitational (Coulomb) potential [11]. The nonlinearities f and g in applications always have a specific form and the differential operator is $-\Delta$. In this paper we consider f of at most linear growth with respect to the first local variable but f can be either sublinear or superlinear with respect to the second variable.

To our best knowledge, this is a first application of the Cone-Compression (Cone-Expansion) Theorem to nonradial elliptic boundary value problems. This is possible due to the presence of the nonlocal term in the nonlinearity f. Otherwise, it seems that the unboundedness of the Green function is an obstacle to an application of the aforementioned theorem.

The method we use is typical for local boundary value problems. We shall formulate an equivalent fixed point problem and look for its solution in the cone of nonnegative functions in $L^p(\Omega)$. We shall apply the following Cone-Compression and Cone-Expansion Theorem due to M. Krasnosel'skiĭ [10], in the form taken from [9]:

THEOREM 1. *Let P be a cone in a Banach space, U_1 and U_2 two bounded open neighbourhoods of zero such that $\overline{U_1} \subset U_2$, let $T : P \to P$ be a compact operator (nonlinear). If*

$$\begin{aligned} \|T(u)\| &\leq \|u\| \quad \text{for} \quad u \in \partial U_1 \cap P, \\ \|T(u)\| &\geq \|u\| \quad \text{for} \quad u \in \partial U_2 \cap P, \end{aligned} \qquad (\textit{expansion})$$

or vice versa

$$\begin{aligned} \|T(u)\| &\geq \|u\| \quad \text{for} \quad u \in \partial U_1 \cap P, \\ \|T(u)\| &\leq \|u\| \quad \text{for} \quad u \in \partial U_2 \cap P, \end{aligned} \qquad (\textit{compression})$$

then the operator T has a fixed point in $\overline{U_2} \setminus U_1$.

If one has several U_i's forming an increasing sequence with the opposite inequalities on the boundaries of U_i and U_{i+1}, then one gets several solutions. If the sequence of neighbourhoods is infinite, then there are infinitely many solutions. A similar method has been used in [13], with the nonlinear term of a specific form, to obtain radial solutions in the space of continuous functions with the supremum norm. For a reference on application of this method to some local BVP one can see [12].

2. General setting. Suppose that the function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, \infty)$) is Lipschitz with respect to the first variable. We look for a classical solution of BVP (1), i.e. $u \in C^2$.

To prove the existence of solutions to BVP (1), we consider the following integral equation:

$$u(x) = \int_\Omega G(x,y) f\Big(u(y), \int_\Omega g(u^p)\Big)\, dy, \tag{2}$$

in the space $X = L^p(\Omega)$, with $p > 1$. To simplify notation in the above formula and throughout this paper, we shall omit the variable of integration in $\int_\Omega g(u^p)$. Here the function G is the Green function for BVP (1). The existence, positivity and symmetry of the Green function G follow from [6], due to assumptions on regularity of $\partial\Omega$ and a_{ij}. Moreover, G enjoys the following estimates:

$$\begin{aligned} |G(x,y)| &\le C|x-y|^{2-n}, \\ |\nabla_x G(x,y)| &\le C|x-y|^{1-n}, \end{aligned} \tag{3}$$

for some constant $C > 0$ and any $x, y \in \Omega$. Let P denote the cone in X of all almost everywhere nonnegative functions and let $T : P \to P$ be defined by the formula

$$T(u)(x) = \int_\Omega G(x,y) f\Big(u(y), \int_\Omega g(u^p)\Big) dy. \tag{4}$$

The operator T can be also written as a composition of a linear integral operator K (called also the Green operator) and the nonlinear superposition operator N (called Nemytski operator), which are defined by the formulas:

$$\begin{aligned} K(u)(x) &= \int_\Omega G(x,y)h(y)dy, \\ N(u)(x) &= f\Big(u(x), \int_\Omega g(u^p)\Big). \end{aligned} \tag{5}$$

The operator N maps $L^p(\Omega)$ into itself provided function g satisfies the Lipschitz condition, the function f is Lipschitz continuous with respect to the first variable, and enjoys the following estimate:

$$f(u,\lambda) \le A(\lambda)u + B(\lambda), \qquad u, \lambda \in \mathbb{R}_+, \tag{6}$$

where A and B are continuous functions (cf. [1], Thm. 2.2, p. 16). Now, we explicitly show that $K : L^p(\Omega) \to W^{1,p}(\Omega)$, since we need these estimates in the sequel. Therefore we recall the following version of the well known Young inequality (cf. also [7]), with a similar proof as for convolutions:

LEMMA 1. *Let $H : \Omega \times \Omega \to \mathbb{R}_+$ be symmetric, nonnegative function integrable with respect to any of its variables and $h \in L^p(\Omega)$, $p \ge 1$, and $(Fh)(x) \overset{\mathrm{df}}{=} \int_\Omega H(x,y)h(y)dy$. Then $F : L^p(\Omega) \to L^p(\Omega)$ and*

$$\|Fh\|_{L^p(\Omega)} \le \sup_{x\in\Omega} \|H(x,\cdot)\|_{L^1(\Omega)} \|h\|_{L^p(\Omega)}. \tag{7}$$

Proof. Applying the Hölder inequality we obtain, for any $x \in \Omega$,

$$\begin{aligned} \Big|\int_\Omega H(x,y)h(y)dy\Big|^p &\le \Big(\int_\Omega (H(x,y))^{1-\frac{1}{p}} \,(H(x,y)^{\frac{1}{p}}|h(y)|)dy\Big)^p \\ &\le \Big(\int_\Omega H(x,y)dy\Big)^{p-1} \int_\Omega H(x,y)|h(y)|^p dy. \end{aligned} \tag{8}$$

Integrating this inequality with respect to $x \in \Omega$, applying the Fubini theorem, and using symmetry of H one obtains:

$$(9)\quad \int_\Omega \left| \int_\Omega H(x,y)h(y)dy \right|^p dx \le \int_\Omega \left(\int_\Omega H(x,y)dy \right)^{p-1} \int_\Omega H(x,y)|h(y)|^p dydx$$
$$\le \sup_{x\in\Omega} \left(\int_\Omega H(x,y)dy \right)^{p-1} \int_\Omega \int_\Omega H(x,y)dx|h(y)|^p dy$$
$$\le \sup_{x\in\Omega} \left(\int_\Omega H(x,y)dy \right)^{p} \int_\Omega |h(y)|^p dy,$$

which yields the inequality (7). ■

We apply the above lemma for $F = K$ (to establish that $K : L^p(\Omega) \to W^{1,p}(\Omega)$). Indeed, it suffices to set $H(x,y) = C|x-y|^{2-n}$ and use the first inequality from (3) to get that K maps $L^p(\Omega)$ into itself. Moreover, we can set $H(x,y) = C|x-y|^{1-n}$, and use the second of the estimates (3) to prove the claim.

Now, since the inclusion $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ is compact by the Rellich-Kondrashov Theorem if $p \le n$ (cf. [8], Thm. 5.8.1, p. 291) and by Morrey estimates if $p > n$ (cf. [8], Thm. 5.8.3, p. 293), we have thus proved the compactness of the operator $K : L^p(\Omega) \to L^p(\Omega)$ and thus of $T : L^p(\Omega) \to L^p(\Omega)$.

We look for fixed points of T, since they will prove to be positive solutions of BVP (1). Notice that, since the Green function for (1) is positive, the operator T maps the cone P of almost everywhere nonnegative functions from $X \stackrel{\text{df}}{=} L^p(\Omega)$ into itself.

3. Main results.

Now, we are ready to prove the main result.

THEOREM 2. *Let the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the Lipschitz condition and suppose that there exist two positive constants b_1 and b_2 such that, for any $u \ge 0$,*

$$(10)\qquad b_1 u \le g(u) \le b_2 u.$$

Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be Lipschitz continuous and satisfy (6), and there exist two positive constants c_1 and c_2 such that for any $u \ge 0$,

$$(11)\qquad f(u,c_1)^p \le \frac{c_1}{b_2\gamma^p}, \qquad f(u,c_2)^p \ge \frac{c_2}{b_1\gamma^p},$$

where

$$\gamma = \left(\int_\Omega \left(\int_\Omega G(x,y)dy \right)^p dx \right)^{\frac{1}{p}}.$$

Then BVP (1) has a classical positive solution with $\int_\Omega g(u^p)$ lying between c_1 and c_2.

Proof. The constant γ is, by inequality (7), well defined. Indeed, by estimate (3)

$$(12)\qquad \gamma \le |\Omega|^{\frac{1}{p}} \sup_{x\in\Omega} \|G(x,\cdot)\|_{L^1(\Omega)} \le |\Omega|^{\frac{1}{p}} \sup_{x\in\Omega} \int_\Omega C|x-y|^{2-n}dy,$$

and the latter is finite since the exponent $n-2$ is less than n. Here the $|\Omega|$ denotes Lebesgue measure of set Ω.

To apply Theorem 1 let us define sets $U_1 \stackrel{\text{df}}{=} \{u \in L^p(\Omega) : \int_\Omega g(|u|^p) < c_1\}$ and $U_2 \stackrel{\text{df}}{=} \{u \in L^p(\Omega) : \int_\Omega g(|u|^p) < c_2\}$.

Now, we shall prove the boundedness of the set U_1:

$$c_1 > \int_\Omega g(|u|^p) \geq b_1 \int_\Omega |u|^p = b_1 \|u\|^p, \tag{13}$$

which yields the boundedness of the set U_1, and similarly for U_2.

The sets U_1 and U_2 are also open. Indeed, fix $u \in U_1$ and put $\varepsilon := c_1 - \int g(|u|^p)$. If L denotes a Lipschitz constant for g, one can find a positive number r such that

$$Lp(\|u\|^{p/q} + (\|u\| + r)^{p/q})r < \varepsilon$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Take $v \in L^p$ satisfying $\|v - u\| < r$. We have

$$\int_\Omega g(|v|^p) \leq \int_\Omega g(|u|^p) + L \int_\Omega \||v|^p - |u|^p|.$$

Consider the set $\Omega_+ := \{x : |v(x)| \geq |u(x)|\}$. The second integral over Ω_+ can be estimated by the Mean Value Theorem and the Hölder inequality:

$$\int_{\Omega_+} \||v|^p - |u|^p| \leq p \int_{\Omega_+} |v|^{p-1}(|v| - |u|) \leq p\|v\|^{p/q}\|v - u\|$$

and similarly on $\Omega \setminus \Omega_+$. Thus,

$$\int_\Omega g(|v|^p) \leq c_1 - \varepsilon + Lp(\|u\|^{p/q} + \|v\|^{p/q})\|v - u\| < c_1,$$

which asserts the claim.

Moreover, we shall show that $\partial U_1 \subset \{u \in L^p : \int_\Omega g(|u|^p) = c_1\}$. Take $u_0 = \lim_{k\to\infty} u_k$ (in $L^p(\Omega)$) for some sequence $u_k \in U_1$ for $k \in N$. Then $\int_\Omega g(|u_k|^p) < c_1$ and from Fatou lemma, and by estimate $g(|u|) \leq b_2|u|$, we can pass to the limit obtaining $\int_\Omega g(|u|^p) \leq c_1$. It means that $\overline{U_1} \subset \{u \in L^p : \int_\Omega g(u^p) \leq c_1\}$, and together with the openness of U_1 finishes the proof.

Now, we are ready to verify conditions of the Krasnosel'skiĭ Theorem 1. Obviously, $0 \in U_i$, $i = 1, 2$. Let $u \geq 0$ and $\int_\Omega g(u^p) = c_1$. Then

$$\begin{aligned} \|Tu\|^p_{L^p(\Omega)} &= \int_\Omega \left(\int_\Omega G(x,y) f(u(y), c_1) dy \right)^p dx \\ &\leq \frac{c_1}{b_2 \gamma^p} \int_\Omega \left(\int_\Omega G(x,y) dy \right)^p dx \leq \frac{c_1}{b_2} \leq \int_\Omega u^p \leq \|u\|^p_{L^p(\Omega)}, \end{aligned} \tag{14}$$

and similarly one gets $\|Tu\|_{L^p(\Omega)} \geq c_2$ for $\|u\|_{L^p(\Omega)} = c_2$. Hence, the assertion follows from the Krasnosel'skiĭ Theorem. If $c_1 < c_2$, we use the cone expansion case, otherwise – the cone compression one.

The last part of the proof is to show that a solution u of the integral equation (2) is, in fact, a classical solution of BVP (1). We have obtained that $u \in W^{1,p}(\Omega)$. By continuous inclusion of this Sobolev space, [8] Thm. 5.7.7, Thm. 5.7.8, p. 287, we have $u \in L^q(\Omega)$ with some $q > p$. By a standard bootstrap argument, using Lemma 1 we can get that the right hand side of equation (1) is in fact in $L^q(\Omega)$ with $q > n$, whence $u \in W^{1,q}(\Omega)$. However, once again, since $W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ where $\alpha \overset{\text{df}}{=} 1 - \frac{n}{q}$ ([8], Thm. 5.7.8 , p. 287) we have $u \in C^{0,\alpha}(\Omega)$. By the Lipschitz continuity of f with respect to the first variable one gets that the right hand side of (1) is $C^{0,\alpha}$, so by the Schauder estimates ([5], Thm. 6.13, p. 106) that u is of class $C^{2,\alpha}$ and therefore it is a classical solution to (1). ■

REMARK 1. Assumption (11) can be satisfied only in the presence of nonlocal nonlinearity, unless f is bounded. Therefore, the case of local nonlinearity f depending only on u is excluded herein.

The following result is a simple corollary:

THEOREM 3. *Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfy (6). If there exist positive numbers $c_0 < c_1 < \ldots < c_{2N+1}$ such that*

$$f(u, c_{2i})^p \le \frac{c_{2i}}{b_2 \gamma^p},$$
$$f(u, c_{2i+1})^p \ge \frac{c_{2i+1}}{b_1 \gamma^p},$$

for any $u > 0$ and $i = 0, 1, \ldots, N$, then BVP (1) has at least $N + 1$ positive solutions. If the sequence of c_i is infinite, then there exist infinitely many solutions, and they constitute an unbounded set in L^p. The same is valid when the reverse inequalities hold true.

4. Examples. The above results have especially clear form when nonlinearity f is a product of functions of u and λ respectively). Moreover, let for simplicity $g = id_{\mathbb{R}_+}$ or more generally $g(u) = u \cdot g_1(u)$, where g_1 is Lipschitz and bounded by positive constants both from below and from above. Suppose that

$$f(u, \lambda) = l(u) \cdot h(\lambda),$$

where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is Hölder continuous and bounded, $\inf l > 0$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Conditions (6) and (10) hold, and the operator T is completely continuous. The inequalities from the assumption of Theorem 2 have the form:

$$h(c_1) \le (\gamma \sup l)^{-1} c_1^{1/p} =: r c_1^{1/p}, \qquad h(c_2) \ge (\gamma \inf l)^{-1} c_2^{1/p} =: R c_2^{1/p}.$$

The constants c_i can be found if the following conditions (typical for several results for similar local BVP's) are satisfied:

$$\lim_{\lambda \to 0^+} \frac{h(\lambda)^p}{\lambda} = 0, \qquad \lim_{\lambda \to \infty} \frac{h(\lambda)^p}{\lambda} = \infty,$$

or reversely

$$\lim_{\lambda \to 0^+} \frac{h(\lambda)^p}{\lambda} = \infty, \qquad \lim_{\lambda \to \infty} \frac{h(\lambda)^p}{\lambda} = 0.$$

The lower bound for the number of solutions of the problem is given by the number of cross-sections of the strip $\mathbb{R}_+ \times [r, R]$ by the graph of the function $\lambda \mapsto h(\lambda)\lambda^{-1/p}$.

Similar results can be obtained for $f(u, \lambda) = l(u) + h(\lambda)$ with the above properties of l and h :

$$h(c_1) \le \frac{c_1^{1/p}}{\gamma} - \sup l, \qquad h(c_2) \ge \frac{c_2^{1/p}}{\gamma} - \inf l.$$

The lower bound for the number of solutions is the number of cross-sections of the strip $\mathbb{R}_+ \times [\inf l, \sup l]$ by the graph of the function $\lambda \mapsto h(\lambda) - \lambda^{1/p}/\gamma$.

References

[1] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, C. U. P., Cambridge, 1993.

[2] J. W. Bebernes and A. A. Lacey, *Global existence and finite-time blow-up for a class of nonlocal parabolic problems*, Adv. Differential Equations 2 (1997), 927–953.

[3] E. Caglioti, P. L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*, Comm. Math. Phys. 143 (1992), 501–525.

[4] L. C. Evans, *Partial Differential Equations*, AMS, Providence, Rhode Island, 1998.

[5] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[6] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. 37 (1982), 303–342.

[7] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959 (in Russian; German translation in 1978).

[8] A. Kufner, O. John and S. Fučik, *Function Spaces*, Akademia, Prague, 1977.

[9] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, 1988.

[10] M. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[11] A. Krzywicki and T. Nadzieja, *Nonlocal elliptic problems*, in: Banach Center Publ. 52, Polish Acad. Sci., Warsaw, 2000, 147–152.

[12] B. Przeradzki and R. Stańczy, *Positive solutions for sublinear elliptic equations*, Colloq. Math. 92 (2002), 141–151.

[13] R. Stańczy, *Nonlocal elliptic equations*, Nonlinear Anal. 47 (2001), 3579–3585.