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BEHAVIOUR OF A NON-LOCAL REACTIVE-CONVECTIVE PROBLEM WITH VARIABLE VELOCITY IN OHMIC HEATING OF FOOD

NIKOS I. KAVALLARIS and DIMITRIOS E. TZANETIS

Department of Mathematics, Faculty of Applied Sciences
National Technical University of Athens
157 80 Athens, Greece
E-mails: {nkaval,dtzan}@math.ntua.gr

Abstract. We consider a non-local equation with initial and boundary conditions of the form $\rho(u)u_t + (1 - \int_0^x \rho'(u)u_t\,dy)u_x = \lambda f(u)/(\int_0^1 f(u)\,dx)^2$, which models the temperature when an electric current flows through a moving material with negligible thermal conductivity and time-dependent velocity. The potential difference across the material is fixed but the electrical resistivity $f(u)$ varies significantly with temperature. It is found that for $f(u)$ decreasing with $\int_0^\infty f(s)\,ds < \infty$, blow-up occurs if λ is too large for a steady state to exist or if the initial condition is large enough. On the other hand, if $f(u)$ is decreasing with $\int_0^\infty f(s)\,ds = \infty$, then it is proved that $u(x,t)$ is a global-in-time and unbounded solution.

1. Introduction and mathematical model. In this work we study the asymptotic behaviour and blow-up of solutions for the non-local problem

$$\rho(u)u_t + \left(1 - \int_0^x \rho'(u)u_t dy\right)u_x = \frac{\lambda f(u)}{(\int_0^1 f(u)dx)^2}, \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{3}$$

where $u = u(x,t)$; the functions $f(u)$, $\rho(u)$ are positive and decreasing. The initial data $u_0(x)$ is a non-negative bounded function with bounded first derivative.

Mathematical problem (1)-(3) describes a process appearing in food industry. More precisely, Ohmic heating is a novel commercial process in which an electric current is

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applied to a flowing food stream. The passage of current generates heat, which is used to sterilise the food; it is thus possible to sterilise particles as fast as liquids containing particulates up to 1-inch cube. The controlling parameter is the electrical conductivity of the solid and liquid phases. More background of this process can be found in [1, 2, 3, 7, 8, 9, 10].

The physical process is described by two equations; the first one, expressing the conservation of electrical charge, is the elliptic equation

$$\nabla \cdot (\sigma \nabla \varphi) = 0, \tag{4}$$

where φ is the electrical potential and σ the electrical conductivity of the food. The heat flow is described by the parabolic equation

$$\rho c \left[\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right] = \nabla \cdot (k \nabla T) + \sigma |\nabla \varphi|^2, \tag{5}$$

where k and c stand for the thermal conductivity and the specific heat of the food respectively, see [7]. The term $\sigma|\nabla\varphi|^2$ in (5) represents Ohmic heating. It is noted that c and k are almost constant over the range of expected operating conditions and that for all the materials considered, they are very close to the values of water. Hence for the purposes of modelling, these parameters are taken to be constants. The electrical conductivity σ, on the other hand, can vary substantially, depending on the type of food material and its temperature, and this variation provides an important coupling between the current flow and heat flow in the system. If it is also assumed that the density ρ and the velocity $\vec{v}$ of the food vary significantly with temperature T, then one has to take the change of mass of the food into account as well, so we additionally have the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{6}$$

expressing the conservation of mass. System (4)-(6) can be simplified assuming the following:

(i) the heating of the food is rapid, so the heat diffusion both in the direction of flow and normal to it, can be neglected ($0 < k \ll 1$);

(ii) the food enters the heater with temperature T_0 independent of its position across the channel;

(iii) end effects for the problem can be neglected so that the potential $\varphi = 0$ and $|\vec{v}| = \rho = 1$ at the inlet of the heater $x = 0$ and $\phi = V$ at the far (downstream) end $x = L$.

Here V is known if the potential difference across the device is specified (but has to be determined if the process is controlled in some other way); x is the distance along the channel, that is, in the direction of $\vec{v}$. The channel has parallel sides.

It is convenient to scale the problem so that the length and the specific heat become 1. The problem, taking also into account that the potential φ and temperature T vary only with distance x and time t (it is a consequence of (ii)), can be written as

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial \varphi}{\partial x} \right) = 0, \quad 0 < x < 1, \quad t > 0, \tag{7}$$

$$\rho \frac{\partial T}{\partial t} + \rho v \frac{\partial T}{\partial x} = \sigma \left(\frac{\partial \varphi}{\partial x} \right)^2, \quad 0 < x < 1, \quad t > 0, \tag{8}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \quad 0 < x < 1, \quad t > 0. \tag{9}$$

Integrating (7) we get

$$\sigma \frac{\partial \varphi}{\partial x} = I(t) \quad \text{implying} \quad V = I \int_0^1 \frac{dx}{\sigma},$$

where $I = I(t)$ is the electrical current density (along the channel). Hence (8) takes the form

$$\rho \frac{\partial T}{\partial t} + \rho v \frac{\partial T}{\partial x} = \frac{I^2}{\sigma} = \frac{V^2}{\sigma} \left(\int_0^1 \frac{dx}{\sigma} \right)^{-2}.$$

Integrating (9) with respect to x and substituting in the previous relation we get the non-local equation

$$\rho(T) \frac{\partial T}{\partial t} + \left(1 - \int_0^x \rho'(T) \frac{\partial T}{\partial t} dy \right) \frac{\partial T}{\partial x} = \frac{\lambda f(T)}{(\int_0^1 f(T) dx)^2}, \quad 0 < x < 1, \quad t > 0,$$

where $\lambda = V^2$ and $f = 1/\sigma$ is the electrical resistivity of the food. It is convenient to change the temperature variable so it becomes 0 at the inlet $x = 0$. Also note that with $0 < k \ll 1$ (no diffusion) the problem for (dimensionless) temperature $u = u(x, t)$ is strictly advective and so any boundary condition at $x = 1$ can be dropped arriving at problem (1)-(3).

Depending upon the substance undergoing the heating, the electrical resistivity $f(u)$ might be an increasing, a decreasing, or a non-monotonic function of temperature. For most food products conductivity increases with temperature so $f(u)$ is decreasing. It should be noted that if one were to consider an alternative processing technique, one in which there was a flow in a channel whose walls were the electrodes and across which a prescribed current flowed, a very similar model would apply. This model would differ, however, in that conductivity would replace resistivity for f. It therefore makes sense to consider both decreasing and increasing functions; however in this work we deal only with the decreasing case.

If the heater forms part of a circuit, so that it can be connected in series with a constant resistance, and a fixed EMF is applied across the two ends of the circuit, the scaled non-local equation is replaced by

$$\rho(u) u_t + \left(1 - \int_0^x \rho'(u) u_t dy \right) u_x = \frac{f(u)}{(a + b \int_0^1 f(u) dx)^2}, \ 0 < x < 1, \ t > 0. \tag{10}$$

A similar model, but with constant velocity, was considered in [6]. There, both problems, with $f(u)$ a decreasing or an increasing function were discussed. For decreasing f, it was proved, using comparison techniques, that if either λ or $u_0(x)$ is large enough then blow-up for u (thermal-runaway, food burning) occurs across the channel. It was also proved that thermal runaway still occurs, but in a small region (single-point blow-up), for increasing $f(u)$ growing fast enough and for sufficiently large initial data $u_0(x)$ with a sharp maximum.

2. Existence and uniqueness. Since (1) is a first-order differential equation we can prove existence of a solution to problem (1)-(3) through the characteristic curves theory.

In fact, the characteristics are given as solutions of the following system of ordinary differential equations:

$$\frac{dt}{d\tau} = \rho(u), \tag{11}$$

$$\frac{dx}{d\tau} = 1 - \int_0^x \rho'(u) u_t dy, \tag{12}$$

$$\frac{du}{d\tau} = \frac{\lambda f(u)}{(\int_0^1 f(u)\, dx)^2}, \tag{13}$$

(τ varies along the characteristics). As a result, the behaviour of this problem is crucial. Although discontinuities of u_0 or a mismatch between u_0 and the boundary condition give rise to irregular behaviour of u, these are simply propagated along the characteristics and allow the existence of a (local) weak solution. So, in the following we will generally be thinking of u_0 being continuous (and normally, but not always, differentiable) with $u_0(0) = 0$.

Now if f is a Lipschitz continuous function and $\rho \in C^1((0,\infty))$, then Picard iteration arguments imply the existence of a solution to (11)-(13); also nonexistence can only come about through blow-up with u becoming infinite after some finite time t^*, see [6]. Especially, (11) and (13) together with Picard iteration arguments imply, since $\rho \in C^1((0,\infty))$, that u_t is bounded as far as u is bounded. Using the same arguments it is proved that u_x becomes unbounded only when u becomes unbounded.

Although (1)-(3) is a hyperbolic problem, in the case where f is a decreasing function, more information can be gained proving a comparison result; this will be used in the following section. In fact, if f is a decreasing and Lipschitz continuous function, then $0 \leq f(\beta) - f(\alpha) \leq K(\alpha - \beta)$, where $\beta \leq \alpha \leq M,\ \ M > \sup_{(0,1)} u_0(x)$ for some positive constant $K \equiv K(M)$. Then a lower solution $\underline{u}$ and a solution u to problem (1)-(3) satisfy the inequality $dv/d\tau \leq \lambda f(0)/f^2(M)$ on a characteristic curve as long as they lie under M. So $\underline{u} \leq M$ and $u \leq M$ while

$$\tau \leq \frac{(M - \sup u_0) f^2(M)}{\lambda f(0)}.$$

Considering now $v_0 = \underline{u},\ \{v_n\}$ for $n \geq 1$ can be defined iteratively by

$$\frac{dv_n}{d\tau} + \frac{\lambda K}{f^2(M)} v_n = \frac{\lambda f(v_{n-1})}{(\int_0^1 f(v_{n-1}) dx)^2} + \frac{\lambda K}{f^2(M)} v_{n-1}, \tag{14}$$

with $v_n = u_0$ at $\tau = 0$ and $v_n = 0$ for $s = 0$.

Problem (14) has a unique solution since it is linear and more precisely $v_n \leq M$ for $\tau \leq T \equiv (M - \sup u_0) f^2(M)/\lambda(f(0) + KM) \leq T_1$; note also that

$$\frac{dv_1}{d\tau} + \frac{\lambda K}{f^2(M)} v_1 = \frac{\lambda f(v_0)}{(\int_0^1 f(v_0) dx)^2} + \frac{\lambda K}{f^2(M)} v_0 \geq \frac{dv_0}{d\tau} + \frac{\lambda K}{f^2(M)} v_0,$$

since $v_0 = \underline{u}$ is a lower solution to equation (13) and $v_1 \geq v_0$ for $\tau = 0,\ s = 0$; thus $v_1 \geq v_0$ for $0 \leq \tau \leq T_1\ \ (0 \leq s \leq 1)$ and some $T_1 > 0$.

Moreover,

$$\frac{dv_1}{d\tau} - \frac{\lambda f(v_1)}{(\int_0^1 f(v_1)dx)^2} = \frac{\lambda f(v_1)\int_0^1 (f(v_1)+f(v_0))dx \int_0^1 (f(v_1)-f(v_0))dx}{(\int_0^1 f(v_1)dx)^2(\int_0^1 f(v_0)dx)^2} + \frac{\lambda(f(v_0)-f(v_1))}{(\int_0^1 f(v_0)dx)^2} + \frac{\lambda K}{f^2(M)}(v_0-v_1)$$
$$\le \frac{\lambda f(v_1)\int_0^1 (f(v_1)+f(v_0))dx \int_0^1 (f(v_1)-f(v_0))dx}{(\int_0^1 f(v_1)dx)^2(\int_0^1 f(v_0)dx)^2} + \lambda K(v_0-v_1)\left(\frac{1}{f(M)^2} - \frac{1}{(\int_0^1 f(v_0)dx)^2}\right) \le 0,$$

provided that f is Lipschitz continuous and decreasing. It follows, inductively, that $\underline{u} = v_0 \le v_1 \le v_2 \le \ldots \le v_n \le \ldots \le M$ and so $v_n \to u \ge \underline{u}$ for some solution $u \le M$ and $0 \le \tau \le T$. The uniqueness of the solution for $\tau \in [0,T]$ is proved similarly. Supposing there exist two solutions u_1, u_2 in $[0,T]$ then $0 \le u_1, u_2 \le M$ and using the Lipschitz continuity of f we get

$$\left|\frac{d}{d\tau}(u_1-u_2)\right| \le \frac{\lambda f(u_1)\int_0^1 (f(u_1)+f(u_2))dx \int_0^1 |f(u_1)-f(u_2)|dx}{(\int_0^1 f(u_1)dx)^2(\int_0^1 f(u_2)dx)^2} + \frac{\lambda|f(u_1)-f(u_2)|}{(\int_0^1 f(u_2)dx)^2} \le \Lambda|u_1-u_2|, \tag{15}$$

where $\Lambda = (2\lambda f^2(0) + \lambda f^2(M))K/f^4(M)$. Since for $0 \le \tau \le T$ we have $|u_1 - u_2| \le M$ due to (15) we get $|u_1 - u_2| \le \Lambda\, M\, T$ and inductively we obtain $|u_1 - u_2| \le M(\Lambda T)^n/n! \to 0$ as $n \to \infty$ resulting in $u_1 \equiv u_2$.

Using the same arguments but now starting at $\tau = T$ we deduce that $u \ge \underline{u}$ as long as they both exist. The proof that $u \le \overline{u}$, if $\overline{u}$ is an upper solution to (1)-(3), is similar.

3. General decreasing f

3.1. *Steady-state problem.* The steady-state problem to (1)-(3) is

$$w' = \mu f(w), \quad 0 < x < 1, \quad w(0) = 0, \tag{16}$$

with $\mu = \lambda/(\int_0^1 f(w)dx)^2$ referred to as the local parameter while λ as the non-local one.

Equation (16) can be written

$$\frac{dw}{f(w)} = \mu\, dx, \quad 0 < x < 1,$$

from which by integration over $(0,1)$ we obtain

$$\mu = \mu(M) = \int_0^M \frac{ds}{f(s)}, \quad M = \sup_x w(x) = w(1). \tag{17}$$

The latter implies that $\mu'(M) = 1/f(M) > 0$ leading to the response diagram appearing in Fig. 1. Also by integration of (16) over $(0,1)$ we get $\lambda = M^2/\mu$ and so $\lambda = \lambda(M) = M^2/\int_0^M ds/f(s)$. Since $\lim_{M\to\infty} \lambda(M) = 2\lim_{M\to\infty} Mf(M)$, we distinguish two cases:

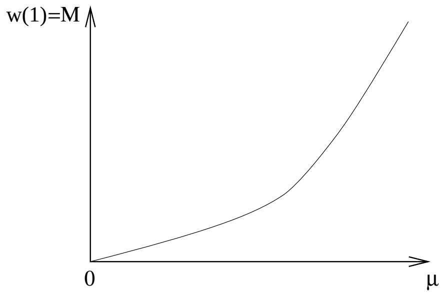

Fig. 1. The local response diagram to (16), where $M(\mu) = w(1;\mu) = w(1)$.

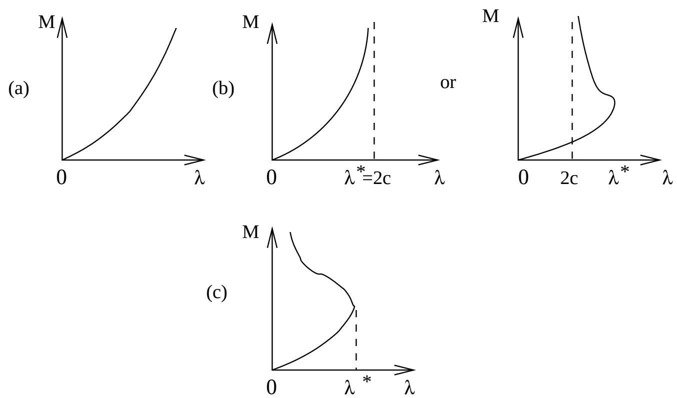

Fig. 2. Possible non-local response diagrams to (16). (a) $Mf(M) \to \infty$ as $M \to \infty$, (b) $Mf(M) \to c$, $0 < c < \infty$ as $M \to \infty$, (c) $Mf(M) \to 0$ as $M \to \infty$.

(i) $\int_0^\infty f(s)ds < \infty$, then $Mf(M) \le 2\int_{M/2}^M f(s)\,ds \to 0$ as $M \to \infty$, and so there exists a λ^* such that for $0 < \lambda < \lambda^*$ problem (16) has at least two steady-state solutions while for $\lambda > \lambda^*$ there is no steady-state solution, see Fig. 2(c).

(ii) $\int_0^\infty f(s)ds = \infty$, if $\lim_{M\to\infty} Mf(M)$ exists then two things might happen. Either $Mf(M) \to c$, $0 < c < \infty$ as $M \to \infty$ and so the spectrum of (16) is bounded (Fig. 2(b)), or $Mf(M) \to \infty$ as $M \to \infty$ and (16) has at least one steady state for any $\lambda > 0$ ($\lambda^* = \infty$, see Fig. 2(a)).

Moreover, if $\mu(M) = \int_0^M ds/f(s) > M/2f(M)$ for $M > 0$, then

$$\lambda'(M) = \frac{M}{\mu(M)}\left[2\mu(M) - \frac{M}{f(M)}\right] > 0,$$

thus there is a unique steady state to each $0 < \lambda < \lambda^*$. From the above analysis we get the possible non-local response diagrams of Fig. 2. Each diagram may contain more turning points than shown (so that for some λ there are more solutions).

3.2. *Stability and blow-up.* In this subsection we study the stability of steady states and the blow-up of solutions to (1)-(3), using the comparison result proved in Section 2. For this reason we consider comparison functions of the form $v(x,t) = w(x;\mu(t))$, so

$v_x = \mu f(v)$, and since $\mu x = \int_0^w ds/f(s)$ we get $v_t = \dot{\mu}(t)xf(v)$ with $v(0,t) = 0$. Thus

$$\begin{aligned}\mathcal{F}(v) &:= \rho(v)v_t + \left(1 - \int_0^x \rho'(v)v_t dy\right)v_x - \frac{\lambda f(v)}{(\int_0^1 f(v)dx)^2} \\ &= \dot{\mu}(t)x\, f(v)\rho(v) + \left(1 - \int_0^x \rho'(v)\dot{\mu}(t)y\, f(v)dy\right)\mu(t)f(v) - \frac{\lambda f(v)}{(\int_0^1 f(v)dx)^2} \\ &= f(v)\left[\dot{\mu}(t)x\rho(v) + \mu(t) - \dot{\mu}(t)\int_0^x \rho'(v)\mu(t)f(v)y\, dy - \lambda/\left(\int_0^1 f(v)dx\right)^2\right] \\ &= f(v)\left[\dot{\mu}(t)x\rho(v) + \mu(t) - \dot{\mu}(t)\int_0^v y\rho'(s)ds - \lambda/\left(\int_0^1 f(v)dx\right)^2\right].\end{aligned}$$

However $\int_0^1 f(v)dx = (1/\mu(t))\int_0^1 v_x dx = M(t)/\mu(t)$, where $M(t) = \sup_x v(x,t) = v(1,t)$, and choosing

$$\dot{\mu}(t) = \dot{\mu} = h(\mu(t)) \equiv \frac{1}{\rho(0)}\left(\frac{\lambda\mu^2}{M^2(\mu)} - \mu\right), \quad t > 0, \tag{18}$$

we obtain

$$\mathcal{F}(v) = f(v)\left[\frac{1}{\rho(0)}\left(\frac{\lambda\mu^2}{M^2(\mu)} - \mu\right)\left(\rho(v)x - \int_0^v y\rho'(s)ds\right) + \mu - \frac{\lambda\mu^2}{M^2(\mu)}\right].$$

If now $\lambda > M^2(\mu)/\mu$, using the fact that $\rho(s)$ is a positive decreasing function, we get $\mathcal{F}(v) \le 0$ for $0 < x < 1$. Thus in this case $v(x,t)$ is an increasing-in-time ($v_t = \dot{\mu}(t)xf(v) > 0$) lower solution to (1)-(3), provided that $v(x,0) = w(x;\mu(0)) \le u_0(x)$. Also for $\lambda < M^2(\mu)/\mu$ and $v(x,0) = w(x;\mu(0)) \ge u_0(x)$ we obtain that $v(x,t)$ is a decreasing-in-time upper solution to (1)-(3).

We start with the case that a unique steady state w exists (Fig. 2(a) (for $\lambda > 0$), or Fig. 2(b)-first case ($0 < \lambda < 2c$)). Then to each $0 < \lambda < \lambda^*$ ($\lambda^* = \infty$ when $\int_0^\infty f(s)ds = \infty$) there exists $\mu > 0$ such that $\lambda = \lambda(\mu) := \mu(\int_0^1 f(w)dx)^2$ and the function $\lambda(\mu)$ is increasing. For the case $u_0(x) \le w(x)$ we can choose $0 < \underline{\mu}(t) < \mu$, so $\lambda = \lambda(\mu) > \lambda(\underline{\mu}(t)) = \underline{M}^2(t)/\underline{\mu}(t)$, satisfying the equation (18). Then $\underline{\mu}(t)$ satisfies the transcendental equation

$$\int_{\underline{\mu}(0)}^{\underline{\mu}(t)} \frac{ds}{h(s)} = t, \quad t > 0, \qquad \text{where} \quad h(s) = \frac{1}{\rho(0)}\left(\frac{\lambda s^2}{M^2(s)} - s\right). \tag{19}$$

Equation (19) has a unique solution in $[\underline{\mu}(0),\mu)$, for any $\underline{\mu}(0) \ge 0$, since in this case $G : [\underline{\mu}(0),\mu) \to [0,\infty)$ with $G(\xi) = \int_{\underline{\mu}(0)}^{\xi} ds/h(s)$ is a C^1-diffeomorphism, [5]. Thus (18) has a unique solution $\underline{\mu}(t)$ and since $w_{\underline{\mu}} = x\, f(w) \ge 0$ we can choose $\underline{\mu}(0) \ge 0$ such that $w(x;\underline{\mu}(0)) \le u_0(x)$. Hence $v(x,t) = w(x;\underline{\mu}(t))$ is an increasing-in-time lower solution to (1)-(3), so $v(x,t) \le u(x,t) \le w(x)$ for $x \in [0,1]$ and $t > 0$. Moreover $\underline{\mu}(t) \to \mu-$ as $t \to \infty$ (otherwise there would be another steady state) and so $v(\cdot,t) \to w(\cdot)$ as $t \to \infty$ uniformly in x resulting in $u(\cdot,t) \to w(\cdot)$ as $t \to \infty$ uniformly in x.

When $u_0(x) \ge w(x)$, it is possible to choose $\bar{\mu}(t) > \mu$ (so $\lambda < \lambda(\bar{\mu}(t))$) to satisfy (18) and construct a decreasing-in-time upper solution $z(x,t)$ to (1)-(3), provided that $z(x,0) = w(x;\bar{\mu}(0)) \ge u_0(x)$ (the latter is possible since $u_0(x)$, $u_0'(x)$ are bounded and $w_{\bar{\mu}} > 0$). Thus we obtain $w(x) \le u(x,t) \le z(x,t)$ and finally $u(\cdot,t) \to w(\cdot)$ as $t \to \infty$

uniformly in x, since also $\bar{\mu}(t) \to \mu+$ as $t \to \infty$. Hence the unique steady state $w(x)$ is globally asymptotically stable and $u(x,t)$ is a global-in-time bounded solution.

We continue now our study with the case where (16) has two steady states $w_1 = w(x;\mu_1)$ and $w_2 = w(x;\mu_2)$ (Fig. 2(b)-second case, or Fig. 2(c)). Then for each $\lambda_* < \lambda < \lambda^*$ ($\lambda_* = 0$ in Fig. 2(c) and $\lambda_* = 2c$ in Fig. 2(b)—second case) there exist μ_1 and μ_2 such that $\lambda = \lambda(\mu_1) = \lambda(\mu_2)$ and the function $\lambda(\mu)$ is increasing for $0 < \mu < \mu^*$ and decreasing for $\mu > \mu^*$ (μ^*: $\lambda'(\mu^*) = 0$). For $0 < u_0(x) < w_1(x)$, on choosing $0 < \underline{\mu}(t) < \mu_1 < \mu^*$ to satisfy (18), we get as above a lower solution $v(x,t) = w(x;\underline{\mu}(t))$ with $v(\cdot,t) \to w_1(\cdot)$ as $t \to \infty$ uniformly in x. Whereas for $w_1(x) < u_0(x) < w_2(x)$, on choosing $\mu_1 < \bar{\mu}(t) < \mu^*$, we construct an upper solution $z(x,t) = w(x;\bar{\mu}(t))$ such that $z(\cdot,t) \to w_1(\cdot)$ as $t \to \infty$ uniformly in x. Hence for $\lambda_* < \lambda < \lambda^*$ the minimal steady state w_1 is asymptotically stable with a region of attraction $[0, w_2]$, while for $0 < \lambda < \lambda_*$, w_1 is globally asymptotically stable. This implies that $u(x,t)$ is a global-in-time bounded solution.

If we consider $u_0(x) > w_2(x)$ and choose $\underline{\mu}(t) > \mu_2$ satisfying (18) then an unbounded lower solution $v(x,t) = w(x;\underline{\mu}(t))$ can be constructed. More precisely $\underline{\mu}(t) \to \infty$ as $t \to T^* \le \infty$ (otherwise there would be a third steady state which is a contradiction). Hence $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^* \le T^* \le \infty$, which means that $u(x,t)$ is unbounded. The latter implies that the maximal steady state w_2 is unstable.

Moreover, it is easily seen that $w^*(x) = w(x;\lambda^*)$ is unstable. More precisely w^* is stable from below ($0 < u_0(x) < w^*(x)$) and unstable from above ($u_0(x) > w^*(x)$). If for each $\lambda_* < \lambda < \lambda^*$ more than two steady states exist, then using similar arguments as above it is proved that the minimal steady state is stable, the greater one unstable and so on.

We also note that problem (1)-(3) has unbounded solutions for $\lambda > \lambda^*$. In this case $\lambda > \lambda(\mu) = M^2/\mu$ for $\mu > 0$ and so we can construct a lower solution of the form $w(x;\mu(t))$. But for $\lambda > \lambda^*$ there is no steady state implying that $\mu(t) \to \infty$ as $t \to T^* \le \infty$, hence $u(x,t)$ becomes unbounded at $t^* \le T^* \le \infty$, i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*-$.

The behaviour of unbounded solutions to (1)-(3) depends upon the decreasing rate of $f(s)$. More precisely we have:

Proposition 1. *If $\int_0^\infty f(s)ds < \infty$ and $\rho(s) \ge \gamma > 0$ for $s > 0$ then the unbounded solutions to (1)-(3) blow up globally in finite time, i.e. $u(x,t) \to \infty$ as $t \to t^*- < \infty$ for any $x \in (0,1]$ and $u_x(0,t) \to \infty$ as $t \to t^*-$.*

Proof. As we have noted above, the unbounded solutions to problem (1)-(3) exist either for $\lambda > \lambda^*$ or for $u_0(x)$ sufficiently large and $\lambda \le \lambda^*$. In both cases we can construct a lower solution of the form $v(x,t) = w(x;\nu(t))$ with $\nu(t)$ satisfying (18). Now since $M(0) = 0$, applying Hardy's inequality, [4], we get $\int_0^\nu (M(\sigma)/\sigma)^2 d\sigma < 4\int_0^\nu (M'(\sigma))^2 d\sigma$, where $\nu = \nu(t)$. Also using that $M'(\nu) = f(M(\nu))$ the latter gives

$$\int_0^\nu \left(\frac{M(\sigma)}{\sigma}\right)^2 d\sigma < 4\int_0^\nu (M'(\sigma))^2 d\sigma = 4\int_0^{M(\nu)} f(s)ds < 4\int_0^\infty f(s)ds < \infty. \tag{20}$$

Recalling now that $\nu(t)$ satisfies (18) we obtain

$$t = \rho(0) \int_0^{\nu(t)} \frac{M^2(\sigma)}{\sigma^2(\lambda - M^2(\sigma)/\sigma)}\, d\sigma \quad \text{for any } t > 0, \tag{21}$$

and taking into account that $M^2(\mu)/\mu = M^2(\mu)/\int_0^M ds/f(s) \to 0$ as $\mu \to \infty$, since $\int_0^\infty f(s) < \infty$, we get

$$\int_\beta^\infty \frac{M^2(\sigma)}{\sigma^2(\lambda - M^2(\sigma)/\sigma)}\, d\sigma \sim \frac{1}{\lambda} \int_\beta^\infty \left(\frac{M(\sigma)}{\sigma}\right)^2 d\sigma \quad \text{for} \quad \beta \gg 1. \tag{22}$$

Finally, combining (20), (21) and (22) we get $\nu(t) \to \infty$ as $t \to T^*-$, where

$$T^* = \rho(0) \int_0^\infty \frac{M^2(\sigma)}{\sigma^2(\lambda - M^2(\sigma)/\sigma)}\, d\sigma < \infty.$$

Hence $u(x,t)$ blows up (in finite time), i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*- \le T^* < \infty$.

To prove global blow-up, we first note that $N(t) = max_{[0,1]} u(\cdot,t)$ satisfies

$$\frac{dN}{dt} = \frac{\lambda f(N)}{\rho(N)(\int_0^1 f(u)dx)^2} \le \frac{\lambda f(0)}{\gamma(\int_0^1 f(u)dx)^2} = h(t),$$

and since u blows up we take $N(t) - N(0) \le \int_0^t h(s)ds \to \infty$ as $t \to t^* -$. The latter implies $h(t) \to \infty$ as $t \to t^*-$ and so $\int_0^1 f(u)dx \to 0$ as $t \to t^*-$, giving that $u(x,t) \to \infty$ as $t \to t^*-$ for any $x \in (0,1]$ and $u_x(0,t) \ge w_x(0,\nu(t)) = \nu(t)\, f(0) \to \infty$ as $t \to t^* -$. ■

A complementary result to Proposition 1 is the following:

PROPOSITION 2. *If $\int_0^\infty f(s)ds = \infty$ and $\rho(s) \ge \gamma > 0$ for $s > 0$ then any unbounded solution $u(x,t)$ to (1)-(3) diverges globally, i.e. $u(x,t) \to \infty$ as $t \to \infty$ for any $x \in (0,1]$ and $u_x(0,t) \to \infty$ as $t \to \infty$.*

Proof. We consider the function $z(x,t) = v(t) > 0$ such that

$$\frac{dv}{dt} = \frac{\lambda}{\gamma f(v)} \quad \text{for any } 0 < x < 1. \tag{23}$$

Then

$$\begin{aligned} \mathcal{F}(z) &:= \rho(z)z_t + \left(1 - \int_0^x \rho'(z)z_t dy\right) z_x - \frac{\lambda f(z)}{(\int_0^1 f(z)dx)^2} \\ &= \rho(v)\frac{dv}{dt} - \frac{\lambda f(v)}{(\int_0^1 f(v)dx)^2} \\ &\ge \frac{\lambda}{f(v)} - \frac{\lambda}{f(v)} = 0, \end{aligned} \tag{24}$$

and choosing $v(0)$ such that $u_0(x) \le v(0)$ we see that $z(x,t)$ is an upper solution to (1)-(3). Also (23) implies $\int_{v(0)}^{v(t)} f(s)ds = \frac{\lambda}{\gamma} t$, leading, due to the hypothesis $\int_0^\infty f(s)\, ds = \infty$, to $v(t) \to \infty$ as $t \to \infty$. Hence, $z(x,t)$ is a global-in-time unbounded upper solution to (1)-(3). This implies that $u(x,t)$ diverges, i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to \infty$. Using similar arguments as in Proposition 1, it is proved that $\int_0^1 f(u)dx \to 0$ as $t \to \infty$. Thus $u(x,t) \to \infty$, for any $x \in (0,1]$ and $u_x(0,t) \to \infty$ as $t \to \infty$. This completes the proof of the proposition. ■

4. Discussion. In this work, the behaviour of the solutions to the non-local equation $\rho(u)u_t+(1-\int_0^x \rho'(u)u_t dy)u_x = \lambda f(u)/(\int_0^1 f(u)dx)^2$, associated with boundary and initial conditions, is studied in the case where $f(s)$, $\rho(s)$ are positive and decreasing functions. Using comparison methods, we study the stability of the corresponding steady states and investigate under which circumstances, the solution $u(x,t)$ is global in time or blow up in finite time. More precisely, it is found that the unbounded solutions of the problem which exist either for $\lambda > \lambda^*$ or for $\lambda \leq \lambda^*$ and sufficiently large initial data, blow up if $\int_0^\infty f(s)\,ds < \infty$, while exist for any time if $\int_0^\infty f(s)\,ds = \infty$.

The above analysis can be applied, without any changes, in the case of an increasing bounded function $\rho(s)$ ($\rho'(s) > 0$, $0 < \rho(s) \leq K$ for $s \geq 0$).

The mathematical problem has a physical meaning only in the one-dimensional space.

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