

MULTIPLE EXISTENCE AND STABILITY OF STEADY-STATES FOR A PREY-PREDATOR SYSTEM WITH CROSS-DIFFUSION

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Abstract. This article discusses a prey-predator system with cross-diffusion. We obtain *multiple* positive steady-state solutions of this system. More precisely, we prove that the set of positive steady-states possibly contains an S or \supset -shaped branch with respect to a bifurcation parameter in the large cross-diffusion case. Next we give some criteria on the stability of these positive steady-states. Furthermore, we find the Hopf bifurcation point on the steady-state solution branch in a certain case. Our method of analysis uses the idea developed by Du and Lou [6] and is based on the bifurcation theory and the Lyapunov-Schmidt reduction technique.

1. Introduction. In this article, we are concerned with the following Lotka-Volterra prey-predator model with cross-diffusion:

$$(P) \begin{cases} u_t = \Delta u + u(a - u - cv) & \text{in } \Omega \times (0, \infty), \\ \sigma v_t = \Delta[(1 + \beta u)v] + v(b + du - v) & \text{in } \Omega \times (0, \infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 \geq 0, \quad v(\cdot, 0) = v_0 \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbf{R}^N ($N \geq 1$) with smooth boundary $\partial\Omega$; σ, a, b, c, d are positive constants and $\beta \geq 0$ is the *cross-diffusion* coefficient. In (P), unknown functions u and v represent the population densities of prey and predator species, respectively, which are interacting and migrating in the same habitat Ω . In a certain kind of prey-predator

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relationships, a great number of prey species form a huge group to protect themselves from the attack of predator. So we assume that the population pressure due to the high density of prey induces the diffusion of the form $\beta\Delta(uv)$ in the second equation. The boundary condition means that the habitat Ω is surrounded by a hostile environment. See also the monograph of Okubo and Levin [23] for the biological background. It should be noted that the local solvability of (P) has been established by Amann [1], where a wide class of quasilinear parabolic systems is discussed. According to his result, (P) has a unique local solution (u, v) provided $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ for $p > N$, and moreover, $u \geq 0, v \geq 0$ for all $(x, t) \in \Omega \times [0, T)$, where T is the maximal existence time of (u, v) . Recently, Kuiper and Le [9] have found the global attractor for a class of triangular cross diffusion systems involving (P).

System (P) originates from the competition population model with cross-diffusion proposed by Shigesada, Kawasaki and Teramoto [26]. Since their pioneer work, many mathematicians have discussed such cross-diffusion systems (primary on competition models) from various view-points, e.g., steady-state problems ([7, 15, 16, 17, 19, 20, 21, 22, 25]) and the global existence of time-dependent solutions ([3, 4, 9, 12, 13, 18, 27]). Nevertheless their all works, concerning cross-diffusion systems, many problems still remain open now. In particular, it is very difficult to know the detailed structure of the steady-state solution set (e.g., the number, the stability or the shape of steady-states) to cross-diffusion systems such as (P).

Our aim in this article is to obtain the global bifurcation structure of positive steady-state solutions to (P) in a special case when β is sufficiently large. Regarding a as a bifurcation parameter, we set

$$\mathcal{S} := \{(u, v, a) : (u, v) \text{ is a positive steady-state solution of (P)}\}.$$

Assuming that β is large and some coefficient conditions, we show that \mathcal{S} contains an S or \supset -shaped curve with respect to the bifurcation parameter a . Then (P) admits two or three positive steady-state solutions if a belongs to suitable ranges. This result implies a great contrast to the linear diffusion case ($\beta = 0$), where the uniqueness of positive steady-states is obtained by López-Gómez and Pardo [14] if the spatial dimension is one. Our method of analysis uses the idea developed by Du and Lou [6] and is based on the bifurcation theory and the Lyapunov-Schmidt reduction procedure. If β is large and both of $b - \lambda_1$ and $\lambda_1 - d/\beta$ are small positives, this reduction enables us to find a relationship to a suitable limiting problem. Further, we can get the solution set with an explicit expression of the limiting problem. Making use of the perturbation theory developed in [6], we will depict an S or \supset -shaped curve of \mathcal{S} near the limiting solution set.

In Section 2 we will discuss such multiple existence of steady-state solutions. In Section 3, we will give some criteria on the stability of the positive steady-states. Furthermore, we will find the Hopf bifurcation point on the S or \supset -shaped solution set if σ is sufficiently large.

Throughout the article, the usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\bar{\Omega})$ are defined by

$$\|u\|_p := \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p} \quad \text{and} \quad \|u\|_{\infty} := \max_{x \in \bar{\Omega}} |u(x)|.$$

In particular, we simply write $\|u\|$ instead of $\|u\|_2$. Furthermore, we will denote by Φ a unique positive solution of

$$-\Delta\Phi = \lambda_1\Phi \text{ in } \Omega, \quad \Phi = 0 \text{ on } \partial\Omega, \quad \|\Phi\| = 1,$$

where λ_1 is the least eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$.

2. Multiple existence of positive steady-state solutions.

2.1. Main result. It is well known that the problem

$$\Delta u + u(a - u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a unique positive solution θ_a if $a > \lambda_1$; moreover, $a \in [\lambda_1, \infty) \rightarrow \theta_a \in C(\bar{\Omega})$ is continuous and strictly increasing function. It is possible to show that (P) has two semitrivial steady-state solutions

$$(u, v) = (\theta_a, 0) \text{ for } a > \lambda_1 \quad \text{and} \quad (u, v) = (0, \theta_b) \text{ for } b > \lambda_1$$

in addition to the trivial solution $(u, v) = (0, 0)$.

Our result asserts that \mathcal{S} contains a bounded S or \supset -shaped branch, which connects the above two semitrivial solutions, in a certain case:

THEOREM 2.1. *Assume $\beta b > \beta\lambda_1 > d$. For any $c > 0$, there exist a large number M and an open set*

$$O = O(c) \subset \{(\beta, b, d) : \beta \geq M, 0 < \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that if $(\beta, b, d) \in O$, then \mathcal{S} contains a bounded smooth curve

$$\Gamma = \{(u(r), v(r), a(r)) \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \times (\lambda_1, \infty), r \in (0, C)\}$$

which possesses the following properties:

- (i) $(u(0), v(0)) = (0, \theta_b), \quad a(0) > \lambda_1, \quad a'(0) > 0$;
- (ii) $(u(C), v(C)) = (\theta_{a(C)}, 0), \quad a(C) > \lambda_1$;
- (iii) $a(r)$ attains a strict local maximum in $(0, C)$. Additionally, there exists an open set $O' \subset O$ such that, if $(\beta, b, d) \in O'$, then $a(r)$ attains a strict local minimum in $(0, C)$.

It is noted that we can find an unbounded S-shaped branch of \mathcal{S} , under another coefficient assumption [11, Theorem 1.2].

2.2. Sketch of the proof of Theorem 2.1. In (P), we employ the following change of variables:

$$a = \lambda_1 + \varepsilon a_1, \quad b = \lambda_1 + \varepsilon b_1, \quad d/\beta = \lambda_1 - \varepsilon\tau, \quad \beta = \gamma/\varepsilon, \quad u = \varepsilon w, \quad (1 + \beta u)v = \varepsilon z. \quad (2.1)$$

Here a_1, b_1, τ are positive constants. Furthermore, ε is a small positive constant, thus γ is also a positive constant. In what follows, we will mainly discuss the case when β is large and both of $b - \lambda_1$ and $\lambda_1 - d/\beta$ are small positives. We note that a_1 plays a role of a

bifurcation parameter. By (2.1), a pair of new unknown functions (w, z) satisfies

$$(PP) \begin{cases} w_t = \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) & \text{in } \Omega \times (0, \infty), \\ \sigma \left[-\frac{\gamma z}{(1 + \gamma w)^2} w_t + \frac{z_t}{1 + \gamma w} \right] = \Delta z + \lambda_1 z + \varepsilon g(w, z) & \text{in } \Omega \times (0, \infty), \\ w = z = 0 & \text{on } \partial\Omega \times (0, \infty), \\ w(\cdot, 0) = u_0/\varepsilon, \quad z(\cdot, 0) = (1 + \beta u_0)v_0/\varepsilon & \text{in } \Omega, \end{cases}$$

where

$$\begin{cases} f(w, z, a_1) := w \left(a_1 - w - \frac{cz}{1 + \gamma w} \right), \\ g(w, z) := \frac{z}{1 + \gamma w} \left(b_1 - \tau\gamma w - \frac{z}{1 + \gamma w} \right). \end{cases} \tag{2.2}$$

Here we note that positivity of solutions of (P) assures that of solutions of (PP). The steady-state problem associated with (PP) is reduced to the following semilinear elliptic equations:

$$\begin{cases} \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) = 0 & \text{in } \Omega, \\ \Delta z + \lambda_1 z + \varepsilon g(w, z) = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

By virtue of (2.1), it is easy to see that (2.3) has two semitrivial solutions

$$(w, z) = (\varepsilon^{-1}\theta_{\lambda_1 + \varepsilon a_1}, 0), \quad (w, z) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1})$$

in addition to the trivial solution. For the Lyapunov-Schmidt reduction, we will give a similar framework to that of Du and Lou [6]. For $p > N$, we define two Banach spaces

$$\begin{cases} X := [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)], \\ Y := L^p(\Omega) \times L^p(\Omega). \end{cases}$$

We note that $X \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ by the Sobolev embedding theorem. Define mappings $H : X \rightarrow Y$ and $B : X \times \mathbf{R} \rightarrow Y$ by

$$\begin{cases} H(w, z) := (\Delta w + \lambda_1 w, \Delta z + \lambda_1 z), \\ B(w, z, a_1) := (f(w, z, a_1), g(w, z)). \end{cases} \tag{2.4}$$

Then (2.3) is equivalent to the equation

$$H(w, z) + \varepsilon B(w, z, a_1) = 0. \tag{2.5}$$

Let X_1 and Y_1 be the L^2 -orthogonal complements of $\text{span}\{(\Phi, 0), (0, \Phi)\}$ in X and Y , respectively. Let $P : X \rightarrow X_1$ and $Q : Y \rightarrow Y_1$ represent L^2 -orthogonal projections. Thus a pair of unknown functions $(w, z) \in X$ is decomposed as

$$(w, z) = (r, s)\Phi + \mathbf{u}, \quad \mathbf{u} = P(w, z).$$

Since $H((r, s)\Phi) = 0$ and $(I - Q)H(X_1) = 0$, (2.5) is consequently reduced to

$$QH(\mathbf{u}) + \varepsilon QB((r, s)\Phi + \mathbf{u}, a_1) = 0 \tag{2.6}$$

and

$$(I - Q)B((r, s)\Phi + \mathbf{u}, a_1) = 0.$$

The Lyapunov-Schmidt reduction procedure leads us to the next lemma:

LEMMA 2.2. *For any $C > 0$, there exist a neighborhood N_0 of the set*

$$\{(w, z, a_1, \varepsilon) = (r\Phi, s\Phi, a_1, 0) \in X \times \mathbf{R}^2 : |r|, |s|, |a_1| \leq C\}$$

and a positive constant ε_0 such that all solutions of (2.6) in N_0 are given by

$$\{((r, s)\Phi + \varepsilon\mathbf{U}(r, s, a_1, \varepsilon), a_1, \varepsilon) : |r|, |s|, |a_1| \leq C + \varepsilon_0, |\varepsilon| \leq \varepsilon_0\}.$$

with a smooth X_1 -valued function \mathbf{U} . Then

$$(w, z, a_1, \varepsilon) = ((r, s)\Phi + \varepsilon\mathbf{U}(r, s, a_1, \varepsilon), a_1, \varepsilon)$$

becomes a solution of (2.5), or equivalently (2.3), in N_0 if and only if

$$F^\varepsilon(r, s, a_1)\Phi := (I - Q)B((r, s)\Phi + \varepsilon\mathbf{U}(r, s, a_1, \varepsilon), a_1) = 0.$$

See [11] for the proof of Lemma 2.2. Since $(I - Q)(u, v) = (\int_\Omega u\Phi dx, \int_\Omega v\Phi dx)\Phi$, it follows from (2.2) and (2.4) that

$$\begin{aligned} F^0(r, s, a_1) &= \left(\int_\Omega f(r\Phi, s\Phi, a_1)\Phi, \int_\Omega g(r\Phi, s\Phi)\Phi \right) \\ &= \begin{pmatrix} r \left(a_1 - r\|\Phi\|_3^3 - cs \int_\Omega \frac{\Phi^3}{1 + \gamma r\Phi} \right) \\ s \left\{ b_1 - (b_1 + \tau)\gamma r \int_\Omega \frac{\Phi^3}{1 + \gamma r\Phi} - s \int_\Omega \frac{\Phi^3}{(1 + \gamma r\Phi)^2} \right\} \end{pmatrix}. \end{aligned} \tag{2.7}$$

Thus $\text{Ker } F^0$ is the union of the following four sets:

$$\begin{aligned} \mathcal{L}_0 &= \{(0, 0, a_1) : a_1 \in \mathbf{R}\}, \\ \mathcal{L}_1 &= \{(a_1/\|\phi_1\|_3^3, 0, a_1) : a_1 \in \mathbf{R}\}, \\ \mathcal{L}_2 &= \{(0, b_1/\|\phi_1\|_3^3, a_1) : a_1 \in \mathbf{R}\}, \\ \mathcal{L}_p &= \{(r, \varphi(\gamma r), \psi(r)) : r \in \mathbf{R}\}, \end{aligned}$$

where

$$\begin{cases} \varphi(r) = \left[b_1 - (b_1 + \tau)r \int_\Omega \frac{\Phi^3}{1 + r\Phi} \right] \left(\int_\Omega \frac{\Phi^3}{(1 + r\Phi)^2} \right)^{-1}, \\ \psi(r) = r\|\Phi\|_3^3 + c\varphi(\gamma r) \int_\Omega \frac{\Phi^3}{1 + \gamma r\Phi}. \end{cases} \tag{2.8}$$

We note that $\mathcal{L}_p \cap \overline{\mathbf{R}_+^3}$ means the limiting set of positive solutions of (2.3) as $\varepsilon \rightarrow 0$. Indeed the following proposition holds true:

PROPOSITION 2.3. *For a sufficiently large $A_1 > 0$, there exist $\varepsilon_0 > 0$ and a family of smooth curves*

$$\{(r(\xi, \varepsilon), s(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \mathbf{R}_+^3 : (\xi, \varepsilon) \in (0, C_\varepsilon) \times (0, \varepsilon_0)\}$$

such that for each fixed $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of (2.3) with $a_1 \in (0, A_1]$ can be parametrized as

$$\begin{aligned} \Gamma^\varepsilon &= \{(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) = ((r, s)\Phi + \varepsilon\mathbf{U}(r, s, a_1, \varepsilon), a_1) : \\ &\quad (r, s, a_1) = (r(\xi, \varepsilon), s(\xi, \varepsilon), a_1(\xi, \varepsilon)) \text{ for } \xi \in (0, C_\varepsilon)\} \end{aligned}$$

and $(r(\xi, 0), s(\xi, 0), a_1(\xi, 0)) = (\xi, \varphi(\gamma\xi), \psi(\xi))$, $r(0, \varepsilon) = 0$. Here $C_\varepsilon > 0$ depends continuously on $\varepsilon \in [0, \varepsilon_0]$. Furthermore,

$$w(C_\varepsilon, \varepsilon) > 0 \text{ in } \Omega \text{ and } z(C_\varepsilon, \varepsilon) \equiv 0.$$

The above proposition implies that if $\varepsilon > 0$ is sufficiently small, then Γ^ε forms a positive solution branch near the curve $\{(r\Phi, \varphi(\gamma r)\Phi, \psi(r)) : 0 < r < C\}$. So it is important to study the profile of \mathcal{L}_p . By virtue of (2.8),

$$(0, \varphi(0), \psi(0)) = (0, b_1/\|\Phi\|_3^3, cb_1) \in \mathcal{L}_2.$$

It is easy to find a positive constant $r_0 = r_0(\tau/b_1)$ such that

$$\begin{cases} \varphi(r) > 0 & \text{for } r \in [0, r_0), \\ \varphi(r) < 0 & \text{for } r \in (r_0, \infty). \end{cases}$$

Thus it follows that

$$(r_0/\gamma, \varphi(r_0), \psi(r_0/\gamma)) = (r_0/\gamma, 0, r_0\|\Phi\|_3^3/\gamma) \in \mathcal{L}_1.$$

We note that C_ε stated in Proposition 2.3 satisfies $C_0 = r_0/\gamma$. Additionally the next lemma gives profiles of $\psi(r)$ in the interval of $\{r > 0 : \varphi(\gamma r) > 0\}$ when τ is close to 0 and γ is sufficiently large.

LEMMA 2.4. *There exist positive constants $\tilde{\tau} = \tilde{\tau}(c, b_1)$ and $\tilde{\gamma} = \tilde{\gamma}(c, b_1)$ such that if $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$, then $\psi'(0) > 0$ and $\psi(r)$ achieves a strict local maximum in $(0, r_0/\gamma)$. Furthermore, there exists a continuous function $\hat{\gamma}(\tau)$ in $(0, \tilde{\tau}]$ satisfying*

$$\tilde{\gamma} < \hat{\gamma}(\tau) \text{ for all } \tau \in (0, \tilde{\tau}] \text{ and } \lim_{\tau \downarrow 0} \hat{\gamma}(\tau) = \infty$$

and that, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau))$ for $\tau \in (0, \tilde{\tau}]$, then $\psi(r)$ attains a strict local minimum in $(0, r_0/\gamma)$.

From Proposition 2.3 and Lemma 2.4, one can see the following proposition.

PROPOSITION 2.5. *Suppose that $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and that $\varepsilon > 0$ is small enough. Then the positive solution set of (2.3) contains a bounded smooth curve*

$$\Gamma^\varepsilon = \{(w(\xi), z(\xi), a_1(\xi)) \in X \times \mathbf{R} : \xi \in (0, C_\varepsilon)\},$$

which possesses the following properties:

- (i) $(w(0), z(0)) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1})$, $a_1(0) > 0$, $a'_1(0) > 0$;
- (ii) $(w(C_\varepsilon), z(C_\varepsilon)) = (\varepsilon^{-1}\theta_{\lambda_1 + \varepsilon a_{1*}}, 0)$, $a_{1*} := a_1(C_\varepsilon) > 0$;
- (iii) $a_1(\xi)$ attains a strict local maximum in $(0, C_\varepsilon)$. In particular, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau))$ for $\tau \in (0, \tilde{\tau}]$, then $a_1(\xi)$ attains a strict local minimum in $(0, C_\varepsilon)$.

With use of (2.1), Theorem 2.1 immediately follows from Proposition 2.5. Actually, for small $\varepsilon > 0$, open sets stated in Theorem 2.1 are expressed as

$$\begin{aligned} O &= \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) : (\tau, \gamma) \in (0, \tilde{\tau}) \times (\tilde{\gamma}, \infty)\}, \\ O' &= \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) : (\tau, \gamma) \in (0, \tilde{\tau}) \times (\tilde{\gamma}, \hat{\gamma}(\tau))\}. \end{aligned}$$

We refer to [11] for the complete proofs.

3. Stability analysis

3.1. Main results. In this section, we will discuss the stability of steady-state solutions on Γ obtained in Theorem 2.1. Before stating our stability results, we need to divide Γ at every turning point with respect to a . In case $(\beta, b, d) \in O$, let

$$0 < r_1 < r_2 < \dots < r_{k-1} < C$$

be all strict local maximum or minimum points of $a(r)$. Since $a'(0) > 0$ (see Theorem 2.1), r_{2j-1} ($j = 1, 2, \dots, [k/2]$) are strict local maximum points, and r_{2j} ($j = 1, 2, \dots, [(k-1)/2]$) are strict local minimum points. For each $1 \leq i \leq k$, we set

$$\Gamma_i := \{(u(r), v(r), a(r)) \in \Gamma : r \in (r_{i-1}, r_i)\},$$

where $r_0 := 0$ and $r_k := C$.

We are ready to state stability results. In case when σ is sufficiently small, we can deduce that the stability of steady-states on Γ changes only at the *turning points*, and moreover, we can know whether each solution on Γ_i is asymptotically stable or not:

THEOREM 3.1. *For almost every $(\beta, b, d) \in O$, there exists a small positive constant δ such that if $\sigma \leq \delta$, then all steady-state solutions on Γ_{2j-1} ($j = 1, 2, \dots, [(k+1)/2]$) are asymptotically stable in the topology of X , while all steady-state solutions on Γ_{2j} ($j = 1, 2, \dots, [k/2]$) are unstable.*

In the above case, we remark that $(u(0), v(0)) = (0, \theta_b)$ and $(u(C), v(C)) = (\theta_{a(C)}, 0)$ by Theorem 2.1. So Theorem 3.1 implies that stable positive steady-states bifurcate from the semitrivial solution $(0, \theta_b)$, the stability on Γ changes at every turning point with respect to a , and moreover Γ connects the other semitrivial solution $(\theta_{a(C)}, 0)$. On the other hand, when σ becomes large enough, we can find the Hopf bifurcation point on Γ_1 ; so that, time-periodic solutions of (P) appear from the point:

THEOREM 3.2. *For any $(\beta, b, d) \in O$, there exists a large positive D such that if $\sigma \geq D$, then the Hopf bifurcation occurs at some point $(u(r^*), v(r^*), a(r^*)) \in \Gamma_1$. In this case, there exists a periodic solution of (P) if a lies in a neighborhood of $a(r^*)$ with $a > a(r^*)$.*

3.2. Sketch of the proofs of Theorems 3.1 and 3.2. By the regularity of (2.1), the stability of a steady-state (u^*, v^*) of (P) coincides with that of the steady-state $(w^*, z^*) = (u^*/\varepsilon, (1 + \beta u^*)z^*/\varepsilon)$ of (PP). So we will concentrate on the stability analysis for the steady-states on Γ^ε given in Proposition 2.5. By Proposition 2.3, all positive steady-states of (PP) with $a_1 \in (0, A_1)$ are parametrized as

$$\Gamma^\varepsilon = \{(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) : \xi \in (0, C_\varepsilon)\}$$

when $\varepsilon > 0$ is sufficiently small. For each $(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \Gamma^\varepsilon$, we define a linear operator $L(\xi, \varepsilon) : X \rightarrow Y$ by

$$L(\xi, \varepsilon) \begin{pmatrix} h \\ k \end{pmatrix} := -H \begin{pmatrix} h \\ k \end{pmatrix} - \varepsilon B_{(w,z)}(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \begin{pmatrix} h \\ k \end{pmatrix},$$

where H, B are mappings defined by (2.4) and $B_{(w,z)}$ denotes the Fréchet derivative of

B with respect to (w, z) . Furthermore, in view of the left hand side of (PP), we set

$$J(\xi, \varepsilon) := \begin{bmatrix} 1 & 0 \\ -\frac{\sigma\gamma z(\xi, \varepsilon)}{(1 + \gamma w(\xi, \varepsilon))^2} & \frac{\sigma}{1 + \gamma w(\xi, \varepsilon)} \end{bmatrix}.$$

Then the linearized eigenvalue problem associated with $(w(\xi, \varepsilon), z(\xi, \varepsilon))$ is given by

$$L(\xi, \varepsilon) \begin{pmatrix} h \\ k \end{pmatrix} = \mu J(\xi, \varepsilon) \begin{pmatrix} h \\ k \end{pmatrix}. \tag{3.1}$$

In this subsection, we study the linearized stability of steady-states on I^ε by the spectral analysis for (3.1). Put

$$\rho(\xi, \varepsilon) := \{\mu \in \mathbf{C} : (3.1) \text{ has no solution except for } h = k = 0\}.$$

We begin with the following lemma for proofs.

LEMMA 3.3. *Suppose that $\varepsilon > 0$ is sufficiently small. Then there exist positive constants κ_1, ω independent of (ξ, ε) such that $-\rho(\xi, \varepsilon) \supset \{z \in \mathbf{C} : |z| \geq \kappa_1 \text{ and } |\arg z| \leq \pi/2 + \omega\}$. On the other hand, all eigenvalues $\{\mu_i(\xi, \varepsilon)\}_{i=1}^\infty$ (counting multiplicity) of (3.1) satisfy*

$$\lim_{\varepsilon \downarrow 0} \mu_1(\xi, \varepsilon) = \lim_{\varepsilon \downarrow 0} \mu_2(\xi, \varepsilon) = 0 \tag{3.2}$$

and

$$\operatorname{Re} \mu_i(\xi, \varepsilon) > \kappa_2 \text{ for all } i \geq 3 \text{ and } \xi \in (0, C_\varepsilon)$$

for some positive constant κ_2 independent of (ξ, ε) .

Proof. It follows from Proposition 2.3 that for any fixed $\xi \in (0, C_\varepsilon)$

$$\lim_{\varepsilon \downarrow 0} (w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) = (\xi\Phi, \varphi(\gamma\xi)\Phi, \psi(\xi)) \text{ in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times \mathbf{R}.$$

Thus letting $\varepsilon \downarrow 0$ in (3.1), we have

$$\begin{cases} -\Delta h - \lambda_1 h = \mu h & \text{in } \Omega, \\ -\Delta k - \lambda_1 k = \sigma \mu \left[-\frac{\gamma\varphi(\gamma\xi)\Phi}{(1 + \gamma\xi\Phi)^2} h + \frac{k}{1 + \gamma\xi\Phi} \right] & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Clearly, $\mu = 0$ is a double eigenvalue of (3.3). If $h \not\equiv 0$, then each eigenvalue of (3.3) is real and nonnegative by the first equation. If $h \equiv 0$, we are led to the same result by the second equation. Consequently we see all eigenvalues of (3.3) are real and nonnegative. From this fact, we can obtain all assertions of Lemma 3.3 with the aid of the perturbation theory by T. Kato [8, Chapter 8]. ■

We note that all eigenvalues $\{\mu_i(\xi, \varepsilon)\}$ form a symmetric set with respect to the real axis in the complex space \mathbf{C} . Then $\mu_1(\xi, \varepsilon)$ and $\mu_2(\xi, \varepsilon)$ (with (3.2)) satisfy the following properties (i) or (ii);

- (i) both of $\mu_1(\xi, \varepsilon)$ and $\mu_2(\xi, \varepsilon)$ are real numbers;
- (ii) $\mu_1(\xi, \varepsilon)$ is a complex conjugate of $\mu_2(\xi, \varepsilon)$.

In what follows, we assume that $\mu_1(\xi, \varepsilon) \leq \mu_2(\xi, \varepsilon)$ in case (i), and $\operatorname{Im}\mu_1(\xi, \varepsilon) \geq \operatorname{Im}\mu_2(\xi, \varepsilon)$ in case (ii).

DEFINITION 3.1 (Linearized stability). A steady-state $(w(\xi, \varepsilon), z(\xi, \varepsilon))$ of (PP) is called *linearly stable* if $\text{Re } \mu_1(\xi, \varepsilon) > 0$. If $\text{Re } \mu_1(\xi, \varepsilon) < 0$, then it is called *linearly unstable*.

We define matrices $K(r)$ and $M(r)$ by

$$K(r) = \begin{bmatrix} 1 & 0 \\ -\sigma\gamma\varphi(\gamma r) \int_{\Omega} \frac{\Phi^3}{(1 + \gamma r\Phi)^2} & \sigma \int_{\Omega} \frac{\Phi^2}{1 + \gamma r\Phi} \end{bmatrix}, \tag{3.4}$$

$$M(r) = -K(r)^{-1}F^0_{(r,s)}(r, \varphi(\gamma r), \psi(r))$$

for the mapping F^0 defined by (2.7). To determine the sign of $\text{Re } \mu_1(\xi, \varepsilon)$, the following lemma plays an important role.

LEMMA 3.4. *Let $\nu_1(r)$ and $\nu_2(r)$ be eigenvalues of $M(r)$ and satisfy $\text{Re } \nu_1(r) \leq \text{Re } \nu_2(r)$, $\text{Im } \nu_1(r) \geq \text{Im } \nu_2(r)$. Then for any $r \in (0, C_0)$,*

$$\lim_{(\xi, \varepsilon) \rightarrow (r, 0)} \frac{\mu_i(\xi, \varepsilon)}{\varepsilon} = \nu_i(r) \text{ for } i = 1, 2. \tag{3.5}$$

Lemma 3.4 can be proved by taking L^2 -inner product of (3.1) with Φ and letting $\varepsilon \rightarrow 0$. See [10] for details.

LEMMA 3.5. *Suppose that $\varepsilon > 0$ is sufficiently small. Suppose further that $\xi \in (0, C_\varepsilon)$. Thus all zeros of $\mu_1(\xi, \varepsilon)$ coincide with all zeros of $\partial_\xi a_1(\xi, \varepsilon)$.*

The above lemma asserts that the degeneracy of steady-states on Γ^ε is equivalent to the criticality of $a_1(\xi, \varepsilon)$ with respect to ξ . The proof of Lemma 3.5 applies the perturbation theory for the Fredholm operator developed by Du and Lou [6, Theorem 3.13 and Appendix].

Since ψ is analytic, ψ' possesses at most a finite number of zeros in $(0, C_0)$. Furthermore, by (2.8), any zero of ψ' must be a strictly critical point of ψ for almost every $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$. For such $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and sufficiently small $\varepsilon > 0$, all zeros of $\partial_\xi a_1(\xi, \varepsilon)$ are denoted by

$$0 < \xi_1(\varepsilon) < \xi_2(\varepsilon) < \dots < \xi_{k-1}(\varepsilon) < C_\varepsilon.$$

That is,

$$(w_i, z_i, a_1^i) := (w(\xi_i(\varepsilon), \varepsilon), z(\xi_i(\varepsilon), \varepsilon), a_1(\xi_i(\varepsilon), \varepsilon)) \in \Gamma^\varepsilon \quad (i = 1, 2, \dots, k - 1)$$

are all turning points on Γ^ε with respect to a_1 . Here we remark that $\lim_{\varepsilon \downarrow 0} a_1(\cdot, \varepsilon) = \psi$ in $C^2([0, C_0])$ by Proposition 2.3 (see also the proof of [11, Lemma 5.3]). Additionally, for each $1 \leq i \leq k$ we set

$$\Gamma_i^\varepsilon := \{((w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) : \xi \in (\xi_{i-1}(\varepsilon), \xi_i(\varepsilon))\},$$

where $\xi_0(\varepsilon) := 0$ and $\xi_k(\varepsilon) = C_\varepsilon$. This implies $\bigcup_{i=1}^k \Gamma_i^\varepsilon = \Gamma^\varepsilon \setminus \bigcup_{i=1}^{k-1} \{(w_i, z_i, a_1^i)\}$.

LEMMA 3.6. *For almost every $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$, there exist small positive constants δ, ε_0 such that if $\sigma \leq \delta$ and $\varepsilon \leq \varepsilon_0$, then all steady-state solutions on $\Gamma_{2j-1}^\varepsilon$ ($j = 1, 2, \dots, [(k + 1)/2]$) are linearly stable, while all steady-state solutions on Γ_{2j}^ε ($j = 1, 2, \dots, [k/2]$) are linearly unstable.*

Proof. Taking the trace of $M(r)$, one can see

$$\begin{aligned} \nu_1(r) + \nu_2(r) &= \frac{\varphi(\gamma r)}{\sigma} \left[\int_{\Omega} \frac{\Phi^3}{(1 + \gamma r \Phi)^2} \left(\int_{\Omega} \frac{\Phi^2}{1 + \gamma r \Phi} \right)^{-1} - \sigma c \gamma r \int_{\Omega} \frac{\Phi^4}{(1 + \gamma r \Phi)^2} \right] \\ &\quad + r \|\Phi\|_3^3 + c \gamma r \varphi(\gamma r) \int_{\Omega} \frac{\Phi^3}{1 + \gamma r \Phi} \int_{\Omega} \frac{\Phi^3}{(1 + \gamma r \Phi)^2} \left(\int_{\Omega} \frac{\Phi^2}{1 + \gamma r \Phi} \right)^{-1}. \end{aligned} \tag{3.6}$$

We set $y_1(r) := \int_{\Omega} r \Phi^4 / (1 + r \Phi)^2$. Since $y_1(0) = 0$ and $y_1(r) = O(r^{-1})$ ($r \rightarrow \infty$), $y_1(\hat{r}) = \sup_{r>0} y_1(r)$ for some $\hat{r} > 0$. Then by (3.6), we obtain

$$\begin{aligned} \nu_1(r) + \nu_2(r) &\geq \frac{\varphi(\gamma r)}{\sigma} \left[\int_{\Omega} \frac{\Phi^3}{(1 + \gamma r \Phi)^2} \left(\int_{\Omega} \frac{\Phi^2}{1 + \gamma r \Phi} \right)^{-1} - \sigma c y_1(\gamma r) \right] + r \|\Phi\|_3^3 \\ &> \frac{\varphi(\gamma r)}{\sigma} \left[\int_{\Omega} \frac{\Phi^3}{(1 + \gamma C_0 \Phi)^2} - \sigma c y_1(\hat{r}) \right] + r \|\Phi\|_3^3 \end{aligned}$$

for all $r \in [0, C_0]$. Therefore, it follows from $\varphi(\gamma r) > 0$ ($r \in [0, C_0)$) that, if

$$\sigma < \frac{1}{2c y_1(\hat{r})} \int_{\Omega} \frac{\Phi^3}{(1 + \gamma C_0 \Phi)^2},$$

then $\nu_1(r) + \nu_2(r) > 0$ for all $r \in [0, C_0]$. Thus we can see by Lemma 3.4 that for sufficiently small $\varepsilon > 0$,

$$\mu_1(\xi, \varepsilon) + \mu_2(\xi, \varepsilon) > 0 \text{ for all } \xi \in [0, C_\varepsilon]. \tag{3.7}$$

Hence (3.7) also implies $\text{Re } \mu_2(\xi, \varepsilon) > 0$ for all $\xi \in [0, C_\varepsilon]$. On the other hand, in view of (3.4), (2.7) and (2.8), direct calculations enable us to obtain

$$\nu_1(r) \nu_2(r) = \det M(r) = \frac{r \varphi(\gamma r) \psi'(r)}{\sigma} \int_{\Omega} \frac{\Phi^3}{(1 + \gamma r \Phi)^2} \left(\int_{\Omega} \frac{\Phi^2}{1 + \gamma r \Phi} \right)^{-1}. \tag{3.8}$$

So $\text{sign } \nu_1(r) \nu_2(r) = \text{sign } \psi'(r)$ for all $r \in (0, C_0)$. Let $r_0 \in (0, C_0)$ be any fixed point. If $\psi'(r_0) > 0$, then Lemma 3.4 implies $\mu_1(\xi, \varepsilon) \mu_2(\xi, \varepsilon) > 0$ if (ξ, ε) is sufficiently near $(r_0, 0)$. Further, together with (3.7), we obtain $\text{Re } \mu_1(\xi, \varepsilon) > 0$. Similarly if $\psi'(r_0) < 0$ and (ξ, ε) is close to $(r_0, 0)$, then $\text{Re } \mu_1(\xi, \varepsilon) < 0$. Additionally it follows from Lemma 3.5 that $\mu_1(\xi, \varepsilon) = 0$ if and only if $\xi = \xi_i(\varepsilon)$ for some $1 \leq i \leq k - 1$ provided that $\varepsilon > 0$ is sufficiently small. Since $\text{Re } \mu_2(\xi, \varepsilon) > 0$ for all $\xi \in [0, C_\varepsilon]$, consequently $\text{Re } \mu_1(\xi, \varepsilon) = 0$ holds if and only if $\xi = \xi_i(\varepsilon)$ for some $1 \leq i \leq k - 1$. We now remark $\psi'(0) > 0$ if $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ (see [11, Lemma 4.1]). Therefore we obtain

$$\begin{cases} \text{Re } \mu_1(\xi, \varepsilon) > 0 & \text{if } (w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \Gamma_{2j-1}^\varepsilon, \\ \text{Re } \mu_1(\xi, \varepsilon) < 0 & \text{if } (w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \Gamma_{2j}^\varepsilon. \end{cases}$$

Thus the proof of Lemma 3.6 is complete. ■

By virtue of (2.1), we can derive the complete proof of Theorem 2.1 from Lemma 3.6. It should be noted that we use the linearized stability theory developed by Potier-Ferry [24] in the derivation. See [10] for details.

PROPOSITION 3.7. *For any $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$, there exist a large $D > 0$ and a small $\varepsilon_0 > 0$ such that if $\sigma \geq D$ and $\varepsilon \leq \varepsilon_0$, then the Hopf bifurcation occurs at a certain point on Γ_1^ε .*

Proof. It suffices to find small positive numbers ξ^* and ε such that $\mu_1(\xi^*, \varepsilon)$, $\mu_2(\xi^*, \varepsilon)$ form a pure imaginary pair and satisfy $\partial_\xi \text{Re}\mu_i(\xi^*, \varepsilon) < 0$ for $i = 1, 2$. We refer to Amann [2] for the abstract Hopf bifurcation theorem for strongly coupled parabolic equations.

Take $(\tau, \gamma) \in (0, \tilde{\tau}) \times [\tilde{\gamma}, \infty)$. Let $\nu_1(r)$ and $\nu_2(r)$ be eigenvalues of $M(r)$ defined by (3.4). We first remark that by (3.8) and $\psi'(0) > 0$,

$$\nu_1(r)\nu_2(r) > 0 \text{ for all } r \in (0, r_1) \tag{3.9}$$

with some $r_1 > 0$. If we set

$$y_2(r) := \int_\Omega \frac{\Phi^4}{(1 + \gamma r \Phi)^2} - \int_\Omega \frac{\Phi^3}{1 + \gamma r \Phi} \int_\Omega \frac{\Phi^3}{(1 + \gamma r \Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1 + \gamma r \Phi} \right)^{-1} - \frac{\|\Phi\|_3^3}{c\gamma\varphi(\gamma r)}$$

then (3.6) is rewritten as

$$\nu_1(r) + \nu_2(r) = \frac{\varphi(\gamma r)}{\sigma} \left[\int_\Omega \frac{\Phi^3}{(1 + \gamma r \Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1 + \gamma r \Phi} \right)^{-1} - \sigma c \gamma r y_2(r) \right].$$

Thus direct calculations imply

$$\nu_1(0) + \nu_2(0) = \frac{1}{\sigma}, \quad \nu'_1(0) + \nu'_2(0) = \frac{1}{\sigma} \left(\tilde{C} - \sigma c \gamma y_2(0) \right) \tag{3.10}$$

for some constant \tilde{C} independent of σ . By Schwarz' inequality and $\|\Phi\| = 1$, we see $\|\Phi\|_4^4 > \|\Phi\|_3^6$. Thus it turns out that $y_2(0) = \|\Phi\|_4^4 - \|\Phi\|_3^6 - \|\Phi\|_3^3 (cb_1\gamma)^{-1} > 0$ if γ is large enough. It follows from (3.10) that if σ is sufficiently large, we can find a small positive number $r_0 \in (0, r_1)$ such that

$$\begin{cases} \nu_1(r) + \nu_2(r) > 0 & \text{in } (0, r_0), \\ \nu_1(r_0) + \nu_2(r_0) = 0 & \text{and } \nu'_1(r_0) + \nu'_2(r_0) < 0. \end{cases} \tag{3.11}$$

We can find a certain (ξ^*, ε) near $(r_0, 0)$, such that eigenvalues $\mu_1(\xi^*, \varepsilon)$, $\mu_2(\xi^*, \varepsilon)$ are a pure imaginary pair and satisfy $\partial_\xi \text{Re}\mu_i(\xi^*, \varepsilon) < 0$ ($i = 1, 2$). In this part of the proof, we make use of Lemma 3.4 and Lyapunov-Schmidt reduction technique (see [10]). Therefore the Hopf bifurcation occurs at $(w(\xi^*, \varepsilon), z(\xi^*, \varepsilon), a_1(\xi^*, \varepsilon))$, which belongs to Γ_1^ε because ξ^* is sufficiently small. ■

By (2.1), Proposition 3.7 immediately yields Theorem 3.2.

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