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STEADY STATES FOR A FRAGMENTATION EQUATION WITH SIZE DIFFUSION

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Abstract. The existence of a one-parameter family of stationary solutions to a fragmentation equation with size diffusion is established. The proof combines a fixed point argument and compactness techniques.

1. Introduction. We study the existence and uniqueness of stationary solutions to a model of particle growth in which the size of the particles is altered by diffusion and fragmentation. A typical situation where there is an interplay between these two mechanisms is the growth of ice crystals: indeed, ice crystals grow or shrink in a way which looks like diffusion and are subjected to stresses which could lead to fragmentation [6]. An interesting feature of such a model is that diffusion and fragmentation have somehow opposite effects on the size distribution of the particles: while the fragmentation mechanism "moves" the size distribution towards small sizes, the diffusion mechanism "pushes" the size distribution towards large sizes. Owing to this competition, one might expect a balance to occur between these two effects leading to stationary size distributions. The existence of steady states has been observed in [6] in a particular case and the aim of this paper is to identify a class of data for which steady states exist.

From a mathematical viewpoint, the fragmentation equation with size diffusion describes the evolution of the size distribution function $f(t,x) \geq 0$ of particles of size $x \in (0,\infty)$ at time $t \geq 0$ and reads

$$\partial_t f(t,x) - D\,\partial_x^2 f(t,x) = -a(x)\,f(t,x) + \int_x^\infty a(x_*)\,b(x,x_*)\,f(t,x_*)\,dx_*. \tag{1}$$

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Here, $D > 0$ is the diffusion coefficient which is assumed to be a positive constant, a is the fragmentation rate and b the daughter size distribution. In the absence of diffusion, the equation (1) is the well-known fragmentation equation with multiple fragmentation [7, 11] which has been studied extensively. We refer to [4, 7, 9, 12] (and the references therein) for the computation of explicit solutions and to [1, 3, 5, 7, 8, 10, 11] (and the references therein) for analytical studies relying on either a deterministic or a stochastic approach. When size diffusion is taken into account, the solution to (1) has been computed explicitly in [6] when $a(x) = x$, $b(x, x_*) = 2$, the equation (1) being supplemented by the homogeneous Dirichlet boundary condition $f(t, 0) = 0$ and the requirement that f decays to zero at infinity. These boundary conditions and the choice of b ensure that there is no exchange of matter between the system of particles and the outer medium, that is,

$$\int_0^\infty x\, f(t,x)\, dx = \text{ const.} \tag{2}$$

through time evolution provided this quantity is initially finite. In addition, a one-parameter family of stationary solutions is exhibited in that case, and our purpose is to extend this result to a larger class of data a and b.

We thus consider the boundary-value problem

$$-\partial_x^2 f(x) = -a(x)\, f(x) + \int_x^\infty a(x_*)\ b(x,x_*)\, f(x_*)\, dx_*, \quad x \in (0,\infty), \tag{3}$$

$$f(0) = 0, \tag{4}$$

where we have taken $D = 1$ for simplicity. We assume that the fragmentation rate a and the daughter distribution function b fulfil the following conditions:

$$a(x) = x^\alpha, \quad x \in (0,\infty), \quad \alpha \geq 0, \tag{5}$$

$$b \quad \text{is measurable and} \quad b > 0 \ \text{ a.e.}, \tag{6}$$

$$\int_0^x x_*\, b(x_*,x)\, dx_* = x, \quad x \in (0,\infty), \tag{7}$$

and, for each $\beta \geq 0$, there is a positive constant B_β such that

$$\int_0^x x_*^\beta\, b(x_*,x)\, dx_* \leq B_\beta\, x^\beta, \quad x \in (0,\infty), \tag{8}$$

with the additional requirement that

$$B_\beta < 1 \quad \text{when} \quad \beta > 1. \tag{9}$$

A typical example of daughter distribution function b satisfying (6)-(9) is

$$b(x_*,x) = (\nu+2)\, x_*^\nu\, x^{-\nu-1}, \quad 0 < x_* < x, \tag{10}$$

for $\nu > -1$ (with $B_\beta = (\nu+2)/(\beta+\nu+1)$) [9]. Observe that the case studied in [6] corresponds to the choice $\alpha = 1$ in (6) and $\nu = 0$ in (10).

We may now state our main result.

THEOREM 1. *Assume that a and b fulfil the conditions (6)-(9). Given $\varrho \in [0,\infty)$, there is a unique non-negative solution*

$$f_\varrho \in W_0^{1,1}(0,\infty) \cap W^{2,1}(0,\infty) \cap L^1(0,\infty; x^{1+\alpha}\, dx)$$

to (3), (4) satisfying

$$\int_0^\infty x\, f_\varrho(x)\, dx = \varrho. \tag{11}$$

In addition, $f_\varrho \in L^1(0,\infty; x^\beta\, dx)$ *for each* $\beta \geq 0$.

Since (3), (4) is a linear boundary-value problem, we obviously have $f_\varrho = \varrho\, f_1$ for $\varrho \geq 0$. We will thus only consider the case $\varrho = 1$ in the sequel.

Observe that, under the assumptions (6)-(9), the boundary-value problem (3), (4) has a solution with a fast decay at infinity. Such a property is expected since the assumptions on a and b are somehow reminiscent of the strong fragmentation assumption which is used in the study of coagulation-fragmentation equations with binary fragmentation [2, 5]. More precisely, under the strong fragmentation assumption, it is known that the $L^1(0,\infty; x^\beta\, dx)$-norm of solutions to coagulation-fragmentation equations becomes instantaneously finite for positive times for every $\beta > 1$ [2, 5].

REMARK 2. Let us point out that the conservation of matter (2) discussed previously may fail for the fragmentation equation (1) without diffusion ($D = 0$) [9]. This phenomenon is the so-called "shattering" transition and corresponds to the appearance of dust (i.e. particles of size zero). It however only occurs when a is singular for $x = 0$ (typically $a(x) = x^\alpha$ with $\alpha < 0$) and is thus excluded from our analysis. We refer to [1, 4, 7, 8, 9] for a more detailed account on the "shattering" transition.

Owing to the unboundedness of a and b, the proof of Theorem 1 is split in two parts: we first consider a "truncated" boundary-value problem on a bounded interval $(0, N)$, $N \geq 1$, for which we establish an existence result by the Schauder fixed point theorem. The second step is to deduce from (8) and (9) a bound in $L^1(0, N; x^\beta\, dx)$ which is independent on N. A compactness argument then allows us to complete the existence part of Theorem 1. The uniqueness part of Theorem 1 relies on the contractivity properties in $L^1(0,\infty; x\, dx)$ of the fragmentation and diffusive terms.

2. A truncated boundary-value problem.

Let $N \geq 1$ be a fixed integer. In this section, we use the Schauder fixed point theorem to show the existence of a non-negative solution f to

$$-\partial_x^2 f(x) = -a(x)\, f(x) + \int_x^N a(x_*)\; b(x, x_*)\; f(x_*)\, dx_*, \quad x \in (0, N), \tag{12}$$

$$f(0) = N\, \partial_x f(N) - f(N) = 0, \tag{13}$$

such that

$$\int_0^N x\, f(x)\, dx = 1. \tag{14}$$

PROPOSITION 3. *There is a unique non-negative solution* $f \in W^{2,1}(0, N)$ *to (12), (13) which satisfies (14).*

Proof. (a) Uniqueness. Let f and $\hat{f}$ be two non-negative solutions in $W^{2,1}(0, N)$ to (12), (13) which fulfil (14), and put $F = f - \hat{f}$. Then $F \in W^{2,1}(0, N)$ is also a solution to (12),

(13) and satisfies

$$\int_0^N x\, F(x)\, dx = 0. \tag{15}$$

We put $\sigma(x) = \text{ sign } (F(x))$, $x \in (0,N)$. On the one hand, we infer from the Kato inequality and (13) that

$$-\int_0^N x\,\sigma(x)\,\partial_x^2 F(x)\,dx \le -N\,\sigma(N)\ \partial_x F(N) + |F(N)| - |F(0)| = 0.$$

On the other hand, (7) and the Fubini theorem yield

$$\begin{aligned}
&\int_0^N x\,\sigma(x)\,(-a(x)\,F(x) + \int_x^N a(x_*)\,b(x,x_*)\,F(x_*)\,dx_*)\,dx \\
&= -\int_0^N x\,a(x)\,|F(x)|\,dx + \int_0^N a(x)\,F(x)\,\int_0^x x_*\,b(x_*,x)\,\sigma(x_*)\,dx_* dx \\
&= -\int_0^N a(x)\,|F(x)|\,\int_0^x x_*\,b(x_*,x)\,(1-\sigma(x)\,\sigma(x_*))\,dx_* dx.
\end{aligned}$$

We now multiply (12) for F by $x\,\sigma(x)$, integrate over $(0,N)$ and use the previous two inequalities to conclude that

$$\int_0^N a(x)\,|F(x)|\,\int_0^x x_*\,b(x_*,x)\,(1-\sigma(x)\,\sigma(x_*))\,dx_* dx \le 0.$$

Since $\sigma(x)\,\sigma(x_*) \le 1$, the positivity of a and b entail that

$$|F(x)|\,(1-\sigma(x)\,\sigma(x_*))\,\mathbf{1}_{(0,x)}(x_*) = 0 \quad \text{a.e. in} \quad (0,N)\times(0,N). \tag{16}$$

Introducing

$$\mathcal{P}_k = \left\{ x \in (0,N), \quad F(x) \ge \frac{1}{k} \right\} \quad \text{for} \quad k \ge 1 \quad \text{and} \quad \mathcal{P} = \bigcup_{k\ge 1} \mathcal{P}_k,$$

$$\mathcal{N}_k = \left\{ x \in (0,N), \quad F(x) \le -\frac{1}{k} \right\} \quad \text{for} \quad k \ge 1 \quad \text{and} \quad \mathcal{N} = \bigcup_{k\ge 1} \mathcal{N}_k,$$

we deduce from (16) that

$$\begin{aligned}
\frac{2}{k}\,|\mathcal{P}_k|\ |\mathcal{N}_k| &= \frac{2}{k}\int_0^N \int_0^x \mathbf{1}_{\mathcal{P}_k}(x)\ \mathbf{1}_{\mathcal{N}_k}(x_*)\,dx_* dx \\
&+ \frac{2}{k}\int_0^N \int_0^{x_*} \mathbf{1}_{\mathcal{P}_k}(x)\ \mathbf{1}_{\mathcal{N}_k}(x_*)\,dx_* dx \\
&= \frac{1}{k}\int_0^N \int_0^x \mathbf{1}_{\mathcal{P}_k}(x)\ \mathbf{1}_{\mathcal{N}_k}(x_*)\,(1-\sigma(x)\,\sigma(x_*))\,dx_* dx \\
&+ \frac{1}{k}\int_0^N \int_0^{x_*} \mathbf{1}_{\mathcal{P}_k}(x)\ \mathbf{1}_{\mathcal{N}_k}(x_*)\,(1-\sigma(x)\,\sigma(x_*))\,dx_* dx \\
&\le \int_0^N \int_0^x |F(x)|\,(1-\sigma(x)\,\sigma(x_*))\,dx_* dx \\
&+ \int_0^N \int_0^{x_*} |F(x_*)|\,(1-\sigma(x)\,\sigma(x_*))\,dx_* dx \\
&\le 0.
\end{aligned}$$

Consequently, $|\mathcal{P}_k|\,|\mathcal{N}_k| = 0$ for every $k \geq 1$ and letting $k \to \infty$ entail that $|\mathcal{P}|\ |\mathcal{N}| = 0$. We have thus shown that either $F \geq 0$ a.e. or $F \leq 0$ a.e., which, together with (15), implies that $F = 0$ and completes the uniqueness proof.

(b) Existence. We put $\omega = N^{\alpha/2} = \|a\|_{L^\infty(0,N)}^{1/2}$. For $g \in L^1(0,N)$, we denote by u_g the unique solution in $W^{2,1}(0,N)$ to the boundary-value problem

$$-\partial_x^2 u_g + \omega^2\, u_g = g, \qquad x \in (0,N), \tag{17}$$

$$u_g(0) = N\,\partial_x u_g(N) - u_g(N) = 0. \tag{18}$$

We recall that u_g is given by

$$u_g(x) = \left(\lambda - \int_0^x e^{-\omega x_*}\, g(x_*)\, dx_*\right) \frac{e^{\omega x}}{2\omega} - \left(\lambda - \int_0^x e^{\omega x_*}\, g(x_*)\, dx_*\right) \frac{e^{-\omega x}}{2\omega}$$

with

$$\begin{aligned} \lambda &= \vartheta \int_0^N e^{-\omega x_*}\, g(x_*)\, dx_* + (1-\vartheta) \int_0^N e^{\omega x_*}\, g(x_*)\, dx_*, \\ \vartheta &= \frac{N\,\omega - 1}{N\,\omega - 1 + (N\,\omega + 1)\ e^{-2\omega N}}. \end{aligned}$$

In particular, there is a constant Γ_N depending on N such that

$$\|u_g\|_{W^{1,\infty}(0,N)} \leq \Gamma_N\, \|g\|_{L^1(0,N)}. \tag{19}$$

Next, for $f \in L^1(0,N)$, we define

$$\mathcal{L}(f)(x) = (\omega^2 - a(x))\, f(x) + \int_x^N a(x_*)\, b(x,x_*)\, f(x_*)\, dx_*, \quad x \in (0,N).$$

Owing to the choice of ω, we have $\mathcal{L}(f) \geq 0$ whenever $f \geq 0$ and straightforward computations, (7) and (8) yield

$$\int_0^N x\, \mathcal{L}(f)(x)\, dx = \omega^2 \int_0^N x\, f(x)\, dx, \tag{20}$$

$$\|\mathcal{L}(f)\|_{L^1(0,N)} \leq \omega^2\, (2 + B_0)\ \|f\|_{L^1(0,N)}. \tag{21}$$

We finally introduce the set

$$\mathcal{C} = \left\{ f \in L^1(0,N),\ \ f \geq 0,\ \ \int_0^N x\ f(x)\, dx = 1,\ \ \|f\|_{L^1(0,N)} \leq R_\star \right\},$$

with

$$R_\star = N\,\omega^2 + \frac{N}{2}\,(3 + \omega^2),$$

which is a bounded closed and convex subset of $L^1(0,N)$. For $f \in \mathcal{C}$, we define $\Lambda(f) = u_{\mathcal{L}(f)}$ and first check that $\Lambda(f)$ also belongs to $\mathcal{C}$. Indeed, it is clear that $u_{\mathcal{L}(f)} \in W^{2,1}(0,N) \subset L^1(0,N)$. Next, since $f \geq 0$, we have also $\mathcal{L}(f) \geq 0$ and the maximum principle implies that $u_{\mathcal{L}(f)} \geq 0$. It also readily follows from (17), (18) and (20) that

$$\int_0^N x\, u_{\mathcal{L}(f)}(x)\, dx = 1. \tag{22}$$

Now, we multiply (17) for $\Lambda(f)$ by x^3, integrate over $(0,N)$ and use (7), (18) and (22) to obtain

$$-N^3\,\partial_x\Lambda(f)(N)+3\,N^2\,\Lambda(f)(N)-6\int_0^N x\,\Lambda(f)(x)\,dx+\omega^2\int_0^N x^3\,\Lambda(f)(x)\,dx$$
$$=\int_0^N x^3\,(\omega^2-a(x))\,f(x)\,dx+\int_0^N a(x_*)\,f(x_*)\int_0^{x_*} x^3\,b(x,x_*)\,dxdx_*$$

$$\begin{aligned}2\,N^2\,\Lambda(f)(N)-6 &\le \omega^2\int_0^N x^3\,f(x)\,dx+N^2\int_0^N a(x_*)\,f(x_*)\,x_*\,dx_*\\ &\le N^2\,\omega^2\int_0^N x\,f(x)\,dx+N^2\,\omega^2\int_0^N x\,f(x)\,dx\\ &\le 2\,N^2\,\omega^2,\end{aligned}$$

whence

$$\Lambda(f)(N)\le 3+\omega^2. \tag{23}$$

We then multiply (17) for $\Lambda(f)$ by x^2, integrate over $(0,N)$ and use (18) and the nonnegativity of f, a and b to obtain

$$N\,\Lambda(f)(N)-2\int_0^N \Lambda(f)(x)\,dx+\omega^2\int_0^N x^2\,\Lambda(f)(x)\,dx\ge-\int_0^N x^2\,a(x)\,f(x)\,dx$$
$$N\,\Lambda(f)(N)+\omega^2\int_0^N x^2\,\Lambda(f)(x)\,dx+\int_0^N x^2\,a(x)\,f(x)\,dx\ge 2\int_0^N \Lambda(f)(x)\,dx,$$

whence, thanks to (22) and (23),

$$2\int_0^N \Lambda(f)(x)\,dx\le N\,(3+\omega^2)+N\,\omega^2+N\,\omega^2=2\,R_\star.$$

Consequently, Λ maps $\mathcal{C}$ into itself. In addition, we infer from (19) and (21) that, for $f\in\mathcal{C}$ and $\hat{f}\in\mathcal{C}$, we have

$$\|u_{\mathcal{L}(f)}-u_{\mathcal{L}(\hat{f})}\|_{L^1(0,N)}=\|u_{\mathcal{L}(f)-\mathcal{L}(\hat{f})}\|_{L^1(0,N)}=\|u_{\mathcal{L}(f-\hat{f})}\|_{L^1(0,N)}\le C(N)\,\|f-\hat{f}\|_{L^1(0,N)}.$$

Therefore, Λ maps continuously $\mathcal{C}$ into itself and $\Lambda(\mathcal{C})$ is compact in $L^1(0,N)$ by the compactness of the embedding of $W^{1,\infty}(0,N)$ in $L^1(0,N)$. We are then in a position to employ the Schauder fixed point theorem and conclude that there exists $f\in\mathcal{C}$ such that $\Lambda(f)=f$. In other words, f satisfies (13) and (14), and

$$-\partial_x^2 f(x)+\omega^2\,f(x)=(\omega^2-a(x))\,f(x)+\int_x^N a(x_*)\,b(x,x_*)\,f(x_*)\,dx_*$$

for $x\in(0,N)$, i.e. f satisfies (12). ∎

3. Proof of Theorem 1. For $N\ge 1$, we denote by f_N the solution to (12), (13) satisfying (14) given by Proposition 3, and we still denote by f_N its extension by zero to $(0,\infty)$. In particular, we have

$$f_N\ge 0,\qquad \int_0^\infty x\,f_N(x)\,dx=1. \tag{24}$$

LEMMA 4. *For each $\beta \geq 3$, there is a constant C_β depending only on α and β such that*

$$N^{\beta-1} f_N(N) + \int_0^N x^{\alpha+\beta} f_N(x)\, dx \leq C_\beta. \tag{25}$$

There is a positive constant C_0 depending only on α such that

$$\int_0^N f_N(x)\, dx \leq C_0. \tag{26}$$

Proof. We multiply (12) by x^β, integrate over $(0,N)$ and use (8), (9) and (13) to obtain

$$-N^\beta\, \partial_x f_N(N) + \beta\, N^{\beta-1} f_N(N) - \beta\,(\beta-1) \int_0^N x^{\beta-2} f_N(x)\, dx$$
$$= \int_0^N a(x)\, f_N(x) \int_0^x x_*^\beta\, b(x_*,x)\, dx_* dx - \int_0^N a(x)\, x^\beta f_N(x)\, dx,$$

$$(\beta-1)\, N^{\beta-1} f_N(N) - \beta\,(\beta-1) \int_0^N x^{\beta-2} f_N(x)\, dx \leq (B_\beta - 1) \int_0^N a(x)\, x^\beta\, f_N(x)\, dx,$$

$$N^{\beta-1} f_N(N) + (1-B_\beta) \int_0^N x^{\alpha+\beta} f_N(x)\, dx \leq \beta\,(\beta-1) \int_0^N x^{\beta-2} f_N(x)\, dx. \tag{27}$$

Since $\beta \geq 3$, the Young inequality and (24) ensure that there is a positive constant C_β depending on α and β such that

$$\int_0^N x^{\beta-2} f_N(x)\, dx \leq \frac{1-B_\beta}{2} \int_0^N x^{\alpha+\beta} f_N(x)\, dx + C_\beta \int_0^N x\, f_N(x)\, dx$$
$$\leq \frac{1-B_\beta}{2} \int_0^N x^{\alpha+\beta} f_N(x)\, dx + C_\beta.$$

Combining (27) and the previous estimate yields (25).

We next multiply (12) by x^2, integrate over $(0,N)$ and use the non-negativity of f_N, a and b to obtain

$$N\, f_N(N) - 2 \int_0^N f_N(x)\, dx \geq - \int_0^N x^{2+\alpha} f_N(x)\, dx.$$

It then follows from (24), (25) and the Hölder inequality that

$$2 \int_0^N f_N(x)\, dx \leq \frac{C_3}{N} + C_3^{(\alpha+1)/(\alpha+2)},$$

which completes the proof of Lemma 4. ∎

REMARK 5. We point out here that Lemma 4 illustrates the opposite effects of the fragmentation and diffusion on the size distribution. Indeed, on the one hand, the bound (25) on moments of f_N of high order results from the assumptions on the fragmentation and prevents the diffusion to expand the size distribution to the right. On the other hand, the L^1-bound (26) follows from the diffusive term and shows that diffusion prevents the concentration of the size distribution near $x = 0$.

LEMMA 6. *There is a constant C depending only on α such that*

$$\|f_N\|_{W^{1,\infty}(0,N)} \leq C. \tag{28}$$

Proof. It follows from (12) and (13) that, for $x \in (0,N)$,

$$\begin{aligned}\partial_x f_N(x) &= \partial_x f_N(N) - \int_x^N \partial_x^2 f_N(x')\,dx' \\ &= \frac{f_N(N)}{N} - \int_x^N a(x')\,f_N(x')\,dx' + \int_x^N \int_{x'}^N a(x_*)\,b(x',x_*)\,f_N(x_*)\,dx_* dx'.\end{aligned}$$

Since $\alpha \geq 0$, we infer from (8), (25), (26) and the Hölder inequality that

$$\begin{aligned}|\partial_x f_N(x)| &= \frac{f_N(N)}{N} + \int_0^N a(x)\,f_N(x)\,dx + \int_0^N a(x_*)\,f_N(x_*) \int_0^{x_*} b(x,x_*)\,dx dx_* \\ &\leq \frac{C_3}{N^3} + (1+B_0) \int_0^N x^\alpha\,f_N(x)\,dx \\ &\leq C_3 + (1+B_0)\,\|f_N\|_{L^1(0,N)}^{3/(\alpha+3)} \left(\int_0^N x^{\alpha+3}\,f_N(x)\,dx \right)^{\alpha/(\alpha+3)}, \\ |\partial_x f_N(x)| &\leq C.\end{aligned} \tag{29}$$

Now, since

$$0 \leq f_N(x) = \int_0^x \partial_x f_N(x_*)\,dx_*$$

by (13), Lemma 6 readily follows from (29). ∎

Proof of Theorem 1. It readily follows from (25), (28) and the Vitali theorem that (f_N) is relatively compact in $L^1(0,\infty; x^\beta\,dx)$ for each $\beta \geq 0$, while (28) and the Ascoli theorem entail the relative compactness of (f_N) in $\mathcal{C}([0,R])$ for each $R > 0$. Consequently, there are a subsequence of (f_N) (not relabeled) and

$$f \in \mathcal{C}([0,\infty)) \cap \bigcap_{\beta \geq 0} L^1(0,\infty; x^\beta\,dx)$$

such that (f_N) converges towards f uniformly on compact subsets of $[0,\infty)$ and

$$\lim_{N\to\infty} \int_0^\infty x^\beta\,|f_N(x) - f(x)|\,dx = 0 \tag{30}$$

for each $\beta \geq 0$. In particular, f is non-negative. Also, (30) allows us to let $N \to \infty$ in (24) and obtain that

$$\int_0^\infty x\,f(x)\,dx = 1.$$

In addition, we may pass to the limit as $N \to \infty$ in (12) and (13) and use classical arguments to conclude that $f \in W^{2,1}(0,\infty) \cap W_0^{1,1}(0,\infty)$ is a solution to (3), (4).

Finally, owing to the integrability properties required in Theorem 1, the proof of the uniqueness part of Theorem 1 is similar to that of Proposition 3. ∎

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