IDEAL INTERPOLATION:
MOURRAIN’S CONDITION VS. D-INvariance

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Abstract. Mourrain [Mo] characterizes those linear projectors on a finite-dimensional polynomial space that can be extended to an ideal projector, i.e., a projector on polynomials whose kernel is an ideal. This is important in the construction of normal form algorithms for a polynomial ideal. Mourrain’s characterization requires the polynomial space to be ‘connected to 1’, a condition that is implied by $D$-invariance in case the polynomial space is spanned by monomials. We give examples to show that, for more general polynomial spaces, $D$-invariance and being ‘connected at 1’ are unrelated, and that Mourrain’s characterization need not hold when his condition is replaced by $D$-invariance.

By definition (see [Bi]), ideal interpolation is provided by a linear projector whose kernel is an ideal in the ring $\Pi$ of polynomials (in $d$ real ($F = \mathbb{R}$) or complex ($F = \mathbb{C}$) variables). The standard example is Lagrange interpolation; the most general example has been called ‘Hermite interpolation’ (in [M] and [Bo]) since that is what it reduces to in the univariate case.

Ideal projectors also occur in computer algebra, as the maps that associate a polynomial with its normal form with respect to an ideal; see, e.g., [CLO]. It is in this latter context that Mourrain [Mo] poses and solves the following problem. Among all linear projectors $N$ on

$$\Pi_1(F) := \sum_{j=0}^{d}(\cdot)_{j}F$$

with range the linear space $F$, characterize those that are the restriction to $\Pi_1(F)$ of an ideal projector with range $F$. Here,

$$(\cdot)_{j} := (\cdot)^{\varepsilon_j}, \quad \varepsilon_j := (\delta_{jk} : k = 1:d), \quad j = 0:d,$$

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with
\[(\alpha)^d : \mathbb{F}^d \to \mathbb{F} : x \mapsto x^\alpha := \prod_{j=1}^{d} x(j) \alpha(j)\]
a handy if nonstandard notation for the monomial with exponent \(\alpha\), with
\[\alpha \in \mathbb{Z}_+^d := \{\alpha \in \mathbb{Z}^d : \alpha(j) \geq 0, j = 1:d\} .\]
I also use the corresponding notation
\[D_j\]
for the derivative with respect to the \(j\)th argument, and
\[D^\alpha := \prod_{j=1}^{d} D_j^{\alpha(j)}, \quad \alpha \in \mathbb{Z}_+^d .\]

To state Mourrain’s result, I also need the following, standard, notations. The (total) degree of the polynomial \(p \neq 0\) is the nonnegative integer
\[\deg p := \max\{|\alpha| : \hat{p}(\alpha) \neq 0\},\]
with
\[p =: \sum_{\alpha} (\alpha)^{\hat{p}(\alpha)} ,\]
and
\[|\alpha| := \sum_{j} \alpha(j) ,\]
while
\[\Pi_{<n} := \{p \in \Pi : \deg p < n\} .\]

**Theorem 1** ([Mo]). Let \(F\) be a finite-dimensional linear subspace of \(\Pi\) satisfying Mourrain’s condition:
\[(2) \quad f \in F \implies f \in \Pi_1(F \cap \Pi_{<\deg f}) ,\]
and let \(N\) be a linear projector on \(\Pi_1(F)\) with range \(F\). Then, the following are equivalent:
(a) \(N\) is the restriction to \(\Pi_1(F)\) of an ideal projector with range \(F\).
(b) The linear maps \(M_j : F \to F : f \mapsto N(\alpha_{j} f), j = 1:d\), commute.

For a second proof of this theorem and some unexpected use of it in the setting of ideal interpolation, see [Bo].

Mourrain’s condition (2) implies that, if \(F\) contains an element of degree \(k\), it must also contain an element of degree \(k - 1\). In particular, if \(F\) is nontrivial, then it must contain a constant polynomial. This explains why Mourrain [Mo] calls a linear subspace satisfying his condition connected to 1. Since the same argument can be made in case \(F\) is \(D\)-invariant, i.e., closed under differentiation, this raises the question what connection if any there might be between these two properties.

In particular, for the special case \(d = 1\), if \(F\) is a linear subspace of dimension \(n\) and either satisfying Mourrain’s condition or being \(D\)-invariant, then, necessarily, \(F = \Pi_{<n} .\)
More generally, if $F$ is an $n$-dimensional subspace in the subring generated by the linear polynomial 

$$
\langle \cdot, y \rangle : \mathbb{F}^d \to \mathbb{F} : x \mapsto \langle x, y \rangle := \sum_{j=1}^{d} x(j)y(j)
$$

for some $y \neq 0$, then, either way,

$$
F = \text{ran}[\langle \cdot, y \rangle^{j-1} : j = 1:n] := \left\{ \sum_{j=1}^{n} \langle \cdot, y \rangle^{j-1} a(j) : a \in \mathbb{F}^n \right\}.
$$

As a next example, assume that $F$ is a monomial space (meaning that it is spanned by monomials). If such $F$ is $D$-invariant, then, with each $(\cdot)^{\alpha}$ for which $\alpha - \varepsilon_j \in \mathbb{Z}_+^d$, it also contains $(\cdot)^{\alpha - \varepsilon_j}$ and therefore evidently satisfies Mourrain’s condition.

Slightly more generally, assume that $F$ is dilation-invariant, meaning that it contains $f(h \cdot)$ for every $h > 0$ if it contains $f$ or, equivalently, $F$ is spanned by homogeneous polynomials. Then every $f \in F$ is of the form

$$
f = f_\uparrow + f_{< \deg f},
$$

with $f_\uparrow$ the leading term of $f$, i.e., the unique homogeneous polynomial for which

$$
\deg(f - f_\uparrow) < \deg f,
$$

hence in $F$ by dilation-invariance, therefore also

$$
f_{< \deg f} \in F_{< \deg f} := F \cap \Pi_{< \deg f},
$$

while, by the homogeneity of $f_\uparrow$,

$$
\sum_{j=1}^{d} (\cdot)^j D_j(f_\uparrow) = (\deg f) f_\uparrow
$$

(this is Euler’s theorem for homogeneous functions; see, e.g., [Enc: p281] which gives the reference [E: §225 on p154]). If now $F$ is also $D$-invariant, then $D_j(f_\uparrow) \in F_{< \deg f}$, hence, altogether,

$$
f \in \Pi_1(F_{< \deg f}), \quad f \in F.
$$

In other words, if a dilation-invariant finite-dimensional subspace $F$ of $\Pi$ is $D$-invariant, then it also satisfies Mourrain’s condition.

On the other hand, the linear space

$$
\text{ran}[(\cdot)^0, (\cdot)^{1,0}, (\cdot)^{1,1}] = \{(\cdot)^0 a + (\cdot)^{1,0} b + (\cdot)^{1,1} c : a, b, c \in \mathbb{F}\}
$$

fails to be $D$-invariant even though it satisfies Mourrain’s condition and is monomial, hence dilation-invariant.

The final example, of a space that is $D$-invariant but does not satisfy Mourrain’s condition, is slightly more complicated. In its discussion, I find it convenient to refer to

$$
\text{supp} \hat{p}
$$

as the ‘support’ of the polynomial $p = \sum_{\alpha} (\cdot)\hat{p}(\alpha)$, with the quotation marks indicating that it is not actually the support of $p$ but, rather, the support of its coefficient sequence, $\hat{p}$.
The example is provided by the $D$-invariant space $F$ generated by the polynomial

$$p = (1,7) + (3,3) + (5,0),$$

hence the ‘support’ of $p$ is

$$\text{supp } \hat{p} = \{(1,7), (3,3), (5,0)\}$$

(see (4) below). Here are a first few elements of $F$:

$$D_1p = (0,7) + 3(2,3) + 5(4,0), \quad D_2p = 7(1,6) + 3(3,2),$$

hence

$$D_1D_2p = 7(0,6) + 9(2,2), \quad D_2^2p = 42(1,5) + 6(3,1),$$

also

$$D_1^2p = 6(1,3) + 20(3,0), \quad D_1D_2^2p = 42(0,5) + 18(2,1),$$

etc. This shows (see (4) below) that any $q \in \Pi_1(F_{\leq \deg p})$ having some ‘support’ in $\text{supp } \hat{p}$ is necessarily a weighted sum of $(1)D_1p$ and $(2)D_2p$ (and, perhaps, others not having any ‘support’ in $\text{supp } \hat{p}$), yet $(p, (1)D_1p, (2)D_2p)$ is linearly independent ‘on’ $\text{supp } \hat{p}$, as the matrix

$$\begin{bmatrix}
1 & 1 & 7 \\
1 & 3 & 3 \\
1 & 5 & 0
\end{bmatrix}$$

(of their coefficients indexed by $\alpha \in \text{supp } \hat{p}$) is evidently 1-1. Consequently,

$$p \notin \Pi_1(F_{\leq \deg p}),$$

i.e., this $F$ does not satisfy Mourrain’s condition (as also follows from Proposition 3 below, in view of Theorem 1).

This space also provides the proof that, in Theorem 1, one may not, in general, replace Mourrain’s condition by $D$-invariance.

**Proposition 3.** Let $F$ be the $D$-invariant space spanned by

$$p = (1,7) + (3,3) + (5,0).$$

Then there exists a linear projector, $N$, on $\Pi_1(F)$ with range $F$ for which (b) but not (a) of Theorem 1 is satisfied.

**Proof.** For $\alpha, \beta \in \mathbb{Z}_+^d$, set

$$[\alpha \ldots \beta] := \{\gamma \in \mathbb{Z}_+^d : \alpha \leq \gamma \leq \beta\},$$

with

$$\alpha \leq \gamma := \alpha(j) \leq \gamma(j), \quad j = 1:d.$$

With this, we determine a basis for $F$ as follows.

Since $D^{0,4}p$ is a positive scalar multiple of $(1,3)$, we know, by the $D$-invariance of $F$, that

$$\{(\zeta^\gamma : \zeta \in [(0,0) \ldots (1,3)]\} \subset F.$$

This implies, considering $D^{2,0}p$, that $(3,0)$, hence also $(2,0)$, is in $F$. Hence, altogether,

$$F = \Pi_{\Xi_0} \oplus \text{ran}[D^\alpha p : \alpha \in [(0,0) \ldots (1,3)]],$$
with
\[ \Pi_\Gamma := \text{ran}[\gamma : \gamma \in \Gamma] \]
and
\[ \Xi_0 := \{(0,0) \ldots (1,3)\} \cup \{(2,0), (3,0)\}. \]
This provides the convenient basis
\[ b_\Xi := \{\xi : \xi \in \Xi\} \]
for \( F \), indexed by
\[ \Xi := \Xi_0 \cup \Xi_1, \quad \Xi_1 := \{(0,4) \ldots (1,7)\}, \]
namely
\[ b_\xi := \begin{cases} \xi, & \xi \in \Xi_0; \\ D^{(1.7)} - \xi p, & \xi \in \Xi_1. \end{cases} \]

The following schema indicates the sets \( \text{supp} \hat{p}, \Xi_0, \) and \( \Xi_1 \), as well as the sets \( \partial \Xi_0 \) and \( \partial \Xi_1 \) defined below:

\[
\begin{array}{cccc}
1 & 1 & \times & \times \\
0 & 0 & + & \otimes \\
0 & 0 & + & + \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Now, let \( N \) be the linear projector on \( \Pi_1(F) \) with range \( F \) and kernel \( \text{ran}[b_\Xi] \), with \( b_\Xi \) obtained by thinning
\[ [b_\Xi, (1)b_\Xi, (2)b_\Xi] \]
to a basis \([b_\Xi, b_Z] \) for \( \Pi_1(F) \). This keeps the maps \( M_j : F \to F : f \mapsto N((j)f) \) very simple since, as we shall see, for many of the \( \xi \in \Xi \), \( (j)b_\xi \) is an element of the extended basis \([b_\Xi, b_Z] \), hence \( N \) either reproduces it or annihilates it.

Specifically, it is evident that the following are in \( F \), hence not part of \( b_Z \):
\[
(1)b_\xi, \quad \xi \in [(0,0) \ldots (0,2)],
(2)b_\xi, \quad \xi \in [(0,0) \ldots (1,3)],
\]
with \((2)b_\xi \in F\) for \( \xi = (0,3), (1,3)\) since \( D^{(1.6)} - \xi p \) and \((\xi + (2,-3)) \) are in \( F \). Further, for each
\[ \zeta \in \partial \Xi_0 \cup \partial \Xi_1, \]
with
\[ \partial \Xi_0 := \{(2,3), (2,2), (2,1), (3,1), (4,0)\}, \quad \partial \Xi_1 := \{(2,4) \ldots (2,7), (1,8), (0,8)\}, \]
there is \( \xi \in \Xi \) so that, for some \( j \), \( \zeta - \xi = \varepsilon_j \). Set, correspondingly,
\[ b_\xi := (j)b_\xi. \]
Then, none of these is in $F$, and, among them, each $b_\xi$ is the only one having some ‘support’ at $\xi$, hence they form a linearly independent sequence. Therefore, each such $b_\xi$ is in $b_Z$.

The remaining candidates for membership in $b_Z$ require a more detailed analysis. We start from the ‘top’, showing also along the way that (b) of Theorem 1 holds for this $F$ and $N$ by verifying that

\[(5)\quad M_1M_2 = M_2M_1 \text{ on } b_\xi\]

for every $\xi \in \Xi$.

$\xi = (1, 7)$: As already pointed out, both $(1)b_{1,7}$ and $(2)b_{1,7}$ are in $b_Z$, hence (5) holds trivially for $\xi = (1, 7)$.

$\xi = (0, 7), (1, 6)$: Both $(1)b_{0,7} = (1)^{1,7} + 3(3)^{3,3} + 5(6)^{5,0}$ and $(2)b_{1,6} = 7(1)^{1,7} + 3(3)^{3,3}$ have their ‘support’ in that of $p = b_{1,7} = (1)^{1,7} + (3)^{3,3} + (6)^{5,0}$, while, as pointed out and used earlier, the three are independent. Hence $(1)b_{0,7}, (2)b_{1,6} \in b_Z$, while we already pointed out that $(2)b_{0,7}, (1)b_{1,6} \in b_Z$, therefore (5) holds trivially.

$\xi = (0, 6), (1, 5)$: Both $(1)b_{0,6} = 7(1)^{1,6} + 9(6)^{3,2}$ and $(2)b_{1,5} = 42(1)^{1,6} + 6(6)^{3,2}$ have their ‘support’ in that of $b_{1,6} = 7(1)^{1,6} + 3(3)^{3,2}$, but neither is a scalar multiple of $b_{1,6}$. Hence, one is in $b_Z$ and the other is not. Which is which depends on the ordering of the columns of $[b_\Xi, (1)b_\Xi, (2)b_\Xi]$. Assume the ordering such that $(2)b_{1,5} \in b_Z$. Then, since we already know that $(1)b_{1,5} \in b_Z$, (5) holds trivially for $\xi = (1, 5)$. Further, $(1)b_{0,6} = 4b_{1,6} - (1/2)(2)b_{1,5}$, hence $M_1b_{0,6} = 4b_{1,6}$, while we already know that $(1)b_{1,6} \in b_Z$ therefore, $M_2M_1b_{0,6} = 0$. On the other hand, $(2)b_{0,6} = 7(1)^{0,7} + 3(3)^{3,3} + 5(4)^{4,0}$ has its ‘support’ in that of $b_{0,7} = (1)^{0,7} + 3(3)^{3,3} + 5(4)^{4,0}$, but is not a scalar multiple of it, hence is in $b_Z$, and therefore already $M_2b_{0,6} = 0$. Thus, (5) also holds for $\xi = (0, 6)$.

$\xi = (0, 5), (1, 4)$: Both $(1)b_{0,5} = 42(1)^{1,5} + 18(3)^{3,1}$ and $(2)b_{1,4} = 210(1)^{1,5} + 6(3)^{3,1}$ have their ‘support’ in that of $b_{1,5} = 42(1)^{1,5} + 6(3)^{3,1}$ but $(3)^{3,1} = b_{3,1}$ was already identified as an element of $b_Z$, hence neither $(1)b_{0,5}$ nor $(2)b_{1,4}$ is in $b_Z$. But, since $(3)^{3,1} \in b_Z$, and so $b_{1,5} = Nb_{1,5} = N(42(1)^{1,5})$, we have $M_1b_{0,5} = b_{1,5}$ and $M_2b_{1,4} = 5b_{1,5}$. Since we already know that $(1)b_{1,5} \in b_Z$, it follows that $M_1M_2b_{1,4} = 0$ while we already know that $(1)b_{1,4} \in b_Z$, hence already $M_1b_{1,4} = 0$. Therefore, (5) holds for $\xi = (1, 4)$. Further, we already know that $(2)b_{1,5} \in b_Z$, hence $M_2M_1b_{0,5} = 0$, while $(2)b_{0,5} = 42(0)^{0,6} + 18(2)^{2,2}$ has the same ‘support’ as $b_{0,6} = 7(1)^{0,6} + 9(2)^{2,2}$ but is not a scalar multiple of it, hence is in $b_Z$ and, therefore, already $M_2b_{0,5} = 0$, showing that (5) holds for $\xi = (0, 5)$.

$\xi = (0, 4)$: $(2)b_{0,4} = 210(1)^{0,5} + 18(2)^{2,1}$ is $5b_{0,5} - 72b_{2,1}$, with $b_{2,1} \in b_Z$, hence $(2)b_{0,4}$ is not in $b_Z$ and $M_2b_{0,4} = 5b_{0,5}$, therefore $M_1M_2b_{0,4} = 5M_1b_{0,5} = 5b_{1,5}$, the last equation from the preceding paragraph. On the other hand, $(1)b_{0,4} = 210(1)^{1,4} + 18(3)^{3,0}$ is $b_{1,4} + 12b_{3,0}$, with both $b_{1,4}$ and $b_{3,0}$ in $F$, hence $(1)b_{0,4}$ is not in $b_Z$, and $M_1b_{0,4} = b_{1,4} + 12b_{3,0}$, therefore, since $(2)b_{3,0} = b_{3,1} \in b_Z$, $M_2M_1b_{0,4} = M_2b_{1,4} = 5b_{1,5}$, the last equation from the preceding paragraph. Thus, (5) holds for $\xi = (0, 4)$.

$\xi = (1, 3)$: We already know that $(1)b_{1,3} = b_{2,3} \in b_Z$ and therefore already $M_1b_{1,3} = 0$, while $(2)b_{1,3} = (1)^{1,4} = (b_{1,4} - 6b_{3,0})/210 \in F$, therefore $210M_1M_2b_{1,3} = M_1b_{1,4} = 0$, thus (5) holds for $\xi = (1, 3)$. 

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For the remaining $\xi \in \Xi$, each $b_{\xi}$ is a monomial, hence $(\cdot)_{j}b_{\xi}$ is again a monomial, and either in $F$ or not and, if not, then its exponent is in
\[ \partial \Xi_0 := \{(2, 3), (2, 2), (2, 1), (3, 1), (4, 0)\}. \]
Moreover, $(\cdot)_{1}(\cdot)_{2}b_{\xi}$ is in $F$ iff $(\cdot)_{2}(\cdot)_{1}b_{\xi}$ is. Hence, (5) also holds for the remaining $\xi \in \Xi$. This finishes the proof that, for this $F$ and $N$, (b) of Theorem 1 holds.

It remains to show that, nevertheless, (a) of Theorem 1 does not hold. For this, observe that $(\cdot)^{2,1}$ and $(\cdot)^{4,0}$ are in ker $N$, as is, e.g., $(\cdot)_{2}b_{1,6} = 7(\cdot)^{1,7} + 3(\cdot)^{3,3}$, hence $p = (\cdot)^{1,7} + (\cdot)^{3,3} + (\cdot)^{5,0}$ is in the ideal generated by ker $N$, making it impossible for $N$ to be the restriction to $\Pi_1(F)$ of an ideal projector $P$ with range $F$ since this would place the nontrivial $p$ in both ker $P$ and ran $P$. ■

References


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