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TWO-WEIGHTED CRITERIA FOR INTEGRAL TRANSFORMS WITH MULTIPLE KERNELS

VAKHTANG KOKILASHVILI

International Black Sea University
2, Agmashenebeli Alley 13th km., Tbilisi 0131, Georgia
E-mail: kokil@rmi.acnet.ge

ALEXANDER MESKHI

A. Razmadze Mathematical Institute, Georgian Academy of Sciences
1, M. Aleksidze St., 0193 Tbilisi, Georgia
Current address:
Scuola Normale Superiore
Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail: a.meskhi@sns.it

Abstract. Necessary and sufficient conditions governing two-weight L^p norm estimates for multiple Hardy and potential operators are presented. Two-weight inequalities for potentials defined on nonhomogeneous spaces are also discussed. Sketches of the proofs for most of the results are given.

Introduction. The survey deals with the boundedness (compactness) criteria for a large class of classical integral operators in classical Lebesgue spaces. For the considerable achievement in this area we refer to [GR], [K2], [OK], [GGKK], [BK], [KP], [GM], [EKM]. The latter book focuses our attention on boundedness (compactness) criteria of integral operators arising naturally in boundary value problems for PDE, the spectral theory of differential operators, continuum and quantum mechanics, stochastic processes, etc.

The paper is organized as follows: in Section 1 we present some well-known two-weight results for the classical Hardy and potential operators; weighted boundedness criteria for

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Volterra-type integral operators with multiple kernel are discussed in Section 2; Section 3 is devoted to the new two-weight conditions guaranteeing the boundedness of the double Hardy operators; two-weight norm inequalities for Riesz potentials with multiple kernels are established in Section 4; an extension of Stein-Weiss theorem for fractional integrals defined on nonhomogeneous spaces is given in Section 5.

1. Preliminaries. Let X be a space with a complete measure μ. Suppose that w is a non-negative locally μ-integrable function on X. Such functions are called weight functions. By $L^p_w(X)$ $(1 \le p < \infty)$ we denote the set of all μ-measurable functions f for which the norm

$$\|f\|_{L^p_w(X)} = \bigg(\int_X |f(x)|^p w(x) d\mu(x) \bigg)^{1/p} < \infty.$$

The criterion for the two-weight inequality for the Riesz potential

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n,$$

has been given by the following (see [KK], [GGKK], [SWZ]).

Theorem A. *Let $1 < p < q < \infty$. Then I_α is bounded from $L^p_w(R^n)$ into $L^q_v(R^n)$ if and only if*

$$\sup_{\substack{x \in R^n \\ r>0}} (vB(x,2r))^{1/q} \bigg(\int_{|x-y|>r} |x-y|^{(\alpha-n)p'} w^{1-p'}(y) dy \bigg)^{1/p'} < \infty$$

and

$$\sup_{\substack{x \in R^n \\ r>0}} (w^{1-p'}B(x,2r))^{1/p'} \bigg(\int_{|x-y|>r} |x-y|^{(\alpha-n)p} v(y) dy \bigg)^{1/q} < \infty.$$

These criteria avoid the notion of capacity and can be easily verified.

The proof of Theorem A is based on the two-weight weak-type criterion for the Riesz potentials given by E. Sawyer [Sa1] and on the more transparent one established by M. Gabidzashvili [Ga], [GK] (see also [KK]).

For a solution of the two-weight problem for integral transforms with positive kernels we refer to [GGKK] and [SWZ]. The results mentioned above were essentially employed, for example, in the theory of the spaces of differentiable functions (see [B1]–[B3]).

In our opinion, one of the challenging problems in the weight theory currently is to solve the two-weight problem for integral transforms with the product kernels. It is worth mentioning that this problem remains still open.

Let us recall two-weight criteria for the one-sided potentials:

$$R_\alpha f(x) = \int_0^x f(t)(x-t)^{\alpha-1} dt; \;\; W_\alpha f(x) = \int_x^\infty f(t)(t-x)^{\alpha-1} dt,$$

where $\alpha > 0$. If $\alpha = 1$, then R_α is the classical Hardy operator defined by

$$Hf(x) = \int_0^x f(t) dt.$$

The two-weight problem for H have been solved by B. Muckenhoupt [Mu1] for $p = q$, and F. J. Bradly [Br], V. Kokilashvili [K1] for $p \le q$ (see also [Ma], Ch. 1), while necessary and sufficient conditions on weights guaranteeing the boundedness of R_α with $\alpha > 1$ from $L^p_w(R_+)$ to $L^q_v(R_+)$ have been established by V. Stepanov [St1] and J. Martin-Reyes and E. Sawyer [MS]. Namely, the following statement holds:

Theorem B. *Let $1 < p \le q < \infty$, $\alpha \ge 1$. Then the operator R_α acts boundedly from $L^p_w(R_+)$ to $L^q_v(R_+)$ if and only if the following two conditions*

$$A_1 := \sup_{t>0} \left(\int_t^\infty \frac{v(x)}{(x-t)^{(1-\alpha)q}} dx \right)^{1/q} \left(\int_0^t w^{1-p'}(x)dx \right)^{1/p'} < \infty;$$

$$A_2 := \sup_{t>0} \left(\int_t^\infty v(x)dx \right)^{1/q} \left(\int_0^t \frac{w^{1-p'}(x)}{(t-x)^{(1-\alpha)p'}} dx \right)^{1/p'} < \infty$$

hold. Moreover, there exist positive constants c_1 and c_2 depending only on α, p and q such that $c_1 \max\{A_1, A_2\} \le \|R_\alpha\| \le c_2 \max\{A_1, A_2\}$.

For a survey of the results on the boundedness and compactness problems for R_α in the case when the order of integration is not less than one see [St2]. For the same problems for one-sided potentials defined on the real line we refer to [KM1].

K. Andersen and E. Sawyer [AS] proved the following

Theorem C. *Let $1 < p < q = p/(1-\alpha p)$, $0 < \alpha < 1/p$. Then the inequality*

$$\|uR_\alpha f\|_{L^q(R_+)} \le c\|uf\|_{L^p(R_+)}$$

holds if and only if

$$\sup_{\substack{a,r \\ 0<r<a}} \frac{1}{r} \left(\int_a^{a+r} u^q(x)dx \right)^{1/q} \left(\frac{1}{r} \int_{a-r}^a u^{-p'}(x) \right)^{1/p'} < \infty.$$

Further, in the case of more general exponents p and q we have

Theorem D ([EKM], p. 131). *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$. Then R_α is bounded from $L^p(R_+)$ to $L^q_u(R_+)$ if and only if*

$$C_1 \equiv \sup_{\substack{h,a \\ 0 \le h \le a}} \left(\int_a^{a+h} u(x)dx \right)^{1/q} h^{\alpha-1/p} < \infty.$$

Moreover, there exist positive constants c_k, $k = 1, 2$ depending only on α, p and q such that $c_1 C_1 \le \|R_\alpha\| \le c_2 C_1$.

From the previous theorem we obtain the following

Corollary A. *Let $0 < \alpha < 1/p$ and let $q = p/(1-\alpha p)$. Then the following conditions are equivalent:*

(*i*) *R_α is bounded from $L^p(R_+)$ to $L^q_v(R_+)$;*

(*ii*) *$\|v\|_{L^\infty(R_+)} < \infty$.*

For the latter result see [PS].

Theorem E ([M1]). *Let $1 < p \le q < \infty$ and let $\alpha > 1/p$. Then the following conditions are equivalent:*

(*i*) *R_α is bounded from $L^p(R_+)$ to $L^q_v(R_+)$;*

(ii)

$$C_2 \equiv \sup_{a>0} C_2(a) \equiv \left(\int_a^\infty v(x) x^{(\alpha-1)q} dx \right)^{1/q} a^{1/p'} < \infty;$$

(iii)

$$C_3 \equiv \sup_{k\in Z} C_3(k) \equiv \sup_{k\in Z} \left(\int_{2^k}^{2^{k+1}} v(x) dx \right)^{1/q} 2^{k(\alpha-1/p)} < \infty.$$

Moreover, $\|R_\alpha\| \approx C_2 \approx C_3$.

The next statement gives the compactness criteria for R_α.

THEOREM F ([M1]). *Let* $1 < p < \infty$ *and let* $\alpha > 1/p$. *Then the following conditions are equivalent:*

i) R_α *is compact from* $L^p(R_+)$ *to* $L^q_v(R_+)$*;*
ii) $\lim_{a\to 0} C_2(a) = \lim_{a\to\infty} C_2(a) = 0$*;*
iii) $\lim_{k\to+\infty} C_3(k) = \lim_{k\to-\infty} C_3(k) = 0$.

For the last two statements see also [EKM] (Section 2) and [Pr]. When $p = q = 2$ the assertion was obtained in [NS].

In the diagonal case $p = q$ the two-weighted characterization for the Riesz potentials has been done in [MV], [VW].

For one-sided potentials we have

THEOREM G ([KM2]). *Let* $1 < p < \infty$ *and let* $0 < \alpha < 1/p$. *Then the operator* R_α *is bounded from* $L^p(R_+)$ *to* $L^p_v(R_+)$ *if and only if* $W_\alpha v \in L^{p'}_{loc}(R_+)$ *and*

$$C_4 \equiv \operatorname*{ess\,sup}_{x\in R_+} \left(\frac{W_\alpha [W_\alpha v]^{p'}(x)}{W_\alpha v(x)} \right)^{1/p'} < \infty.$$

Moreover, $\|R_\alpha\| \approx C_4$.

THEOREM H ([KM2]). *Let* $1 < p < \infty$ *and let* $0 < \alpha < 1/p$. *Then the operator* W_α *is bounded from* $L^p(R_+)$ *to* $L^p_v(R_+)$ *if and only if* $R_\alpha v \in L^{p'}_{loc}(R_+)$ *and*

$$C_5 \equiv \operatorname*{ess\,sup}_{x\in R_+} \left(\frac{R_\alpha [R_\alpha v]^{p'}(x)}{R_\alpha v(x)} \right)^{1/p'} < \infty.$$

Moreover, $\|W_\alpha\| \approx C_5$.

The Volterra-type integral operators with general kernels extending one-sided potentials mainly in the case when $\alpha > 1/p$ were investigated in [M2] (see also [EKM], Section 2.7).

For the applications of the results presented above to the solubility problems of Abel's integral equation and certain superlinear inhomogeneous integral equation we refer to [KM2] (see also [EKM], Sections 2.12, 2.13 and 2.14), where the criteria of the existence of positive solutions of some Volterra-type nonlinear integral equations are obtained.

2. One-sided potentials with multiple kernels. Now we consider the operator

$$R_{\alpha,\beta} f(x,y) = \int_0^x \int_0^y \frac{f(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} dt d\tau.$$

When $\alpha = \beta = 1$ this operator is the multiple Hardy transform which will be discussed in Section 3.

THEOREM 2.1 ([KM4]). *Let $1 < p \le q < \infty$, $\alpha, \beta > 1/p$. Then the following statements are equivalent:*

(i) $R_{\alpha,\beta}$ is bounded from $L^p(R^2_+)$ to $L^q_v(R^2_+)$;
(ii)

$$\sup_{a,b>0} \left(\int_a^\infty \int_b^\infty v(x,y) x^{(\alpha-1)q} y^{(\beta-1)q} dx dy \right)^{1/q} (ab)^{1/p'} < \infty;$$

(iii)

$$\sup_{k,j\in Z} \left(\int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} v(x,y) dx dy \right)^{1/q} 2^{k(\alpha-1/p)} 2^{j(\beta-1/p)} < \infty.$$

To prove this statement we represent the integral $R_{\alpha,\beta} f$ as follows:

$$\begin{aligned} R_{\alpha,\beta} f(x,y) &= \int_0^{x/2} \int_0^{y/2} (\cdots) dt d\tau + \int_0^{x/2} \int_{y/2}^{y} (\cdots) dt d\tau \\ &\quad + \int_{x/2}^{x} \int_0^{y/2} (\cdots) dt d\tau + \int_{x/2}^{x} \int_{y/2}^{y} (\cdots) dt d\tau \\ &\equiv R^{(1)}_{\alpha,\beta} f(x,y) + R^{(2)}_{\alpha,\beta} f(x,y) + R^{(3)}_{\alpha,\beta} f(x,y) + R^{(4)}_{\alpha,\beta} f(x,y). \end{aligned}$$

Further, the following statement holds.

LEMMA 2.1 ([KM4]). *Suppose that $1 < p \le q < \infty$. Then the operator H_2 is bounded from $L^p(R^2_+)$ to $L^q_u(R^2_+)$ if and only if*

$$A \equiv \sup_{a,b>0} ((H_2' u)(a,b))^{1/q} (ab)^{1/p'} < \infty.$$

Moreover, $\|H_2\| \approx A$.

This lemma yields

$$\|R^{(1)}_{\alpha,\beta} f\|_{L^q_v(R^2_+)} \le c \|f\|_{L^p(R^2_+)},$$

while due to the classical Hardy's inequality (see [HLP], p. 240 for $p = q$ and, e.g., [SKM], Ch. 2, for $p \le q$):

$$\left(\int_0^\infty (Hf(x))^q x^{-q/p'-1} dx \right)^{1/q} \le c \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \tag{2.1}$$

we have the boundedness of $R^{(i)}$, $i = 1, 2$, from $L^p(R^2_+)$ to $L^q_v(R^2_+)$. Hölder's inequality and simple calculations of integrals guarantee

$$\|R^{(4)}_{\alpha,\beta} f\|_{L^q_v(R^2_+)} \le c \|f\|_{L^p(R^2_+)}.$$

This completes the proof of sufficiency. For details see [KM4].

It should be mentioned that the condition

$$\sup_j \left(\int_{2^j}^{2^{j+1}} \left[\sup_k \left(\int_{2^k}^{2^{k+1}} v(x,y)dx \right) 2^{k(\alpha p-1)} \right] dy \right) 2^{j(\beta p-1)} < \infty \tag{$*$}$$

is sufficient but not necessary for the boundedness of $R_{\alpha,\beta}$ from $L^p(R^2_+)$ to $L^p_v(R^2_+)$.

Example. Let

$$v(x,y) = \begin{cases} m2^{-m\alpha p} & \text{if } (x,y) \in [2^m, 2^{m+1}] \times (1, 1+1/m], \\ 2^{-m\alpha p} & \text{for } (x,y) \in [2^m, 2^{m+1}] \times [1+1/m, 2], \\ x^{-\alpha p} y^{-\beta p} & \text{otherwise} \end{cases}$$

For such v the condition $(*)$ fails but (i) of Theorem 2.1 holds.

The remaining statements of this section are published in [KM4]–[KM5].

Theorem 2.2. *Let $1 < p < \infty$. Suppose that $0 < \alpha < 1/p$ and $\beta > 1/p$. Then the operator $R_{\alpha,\beta}$ is bounded from $L^p(R^2_+)$ to $L^p_v(R^2_+)$ if and only if $W_\alpha V_j \in L^{p'}_{loc}(R_+)$ for all $j \in Z$ and*

$$W_\alpha [W_\alpha V_j]^{p'}(x) \le c W_\alpha [V_j](x) \tag{2.2}$$

for all $x > 0$ and $j \in Z$, where

$$V_j(x) \equiv \int_{2^j}^{2^{j+1}} v(x,y) y^{\beta p-1} dy.$$

Theorem 2.3. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$ and $\beta > 1/p$. Then the operator $R_{\alpha,\beta}$ is bounded from $L^p(R^2_+)$ to $L^q_v(R^2_+)$ if and only if*

$$A \equiv \sup_{\substack{0<h<a \\ j\in Z}} \left(\int_a^{a+h} \int_{2^j}^{2^{j+1}} v(x,y)dxdy \right)^{1/q} h^{\alpha-1/p} 2^{j(\beta-1/p)} < \infty. \tag{2.3}$$

Moreover, $\|R_{\alpha,\beta}\| \approx A$.

Theorem 2.3 follows from the integral representation

$$\begin{aligned} R_{\alpha,\beta} f(x,y) &= \int_0^x \int_0^{y/2} (\cdots) dt d\tau + \int_0^x \int_{y/2}^y (\cdots) dt d\tau \\ &\equiv \int_{x/2}^x \int_0^{y/2} (\cdots) dt d\tau + \int_{x/2}^x \int_{y/2}^y (\cdots) dt d\tau, \end{aligned}$$

inequality (2.1) and Theorem D. Theorem 2.2 can be obtained in a similar manner. In this case we need to use Theorem G instead of Theorem D (see [KM5] for details).

Now we present the conditions which yield the boundedness of truncated and ball potentials with multiple kernels. Let

$$T_{\alpha,\beta} f(x,y) = \int_{|t|\le 2|x|} \int_{|\tau|\le 2|y|} f(t,\tau)|x-t|^{\alpha-n}|y-\tau|^{\beta-n} dt d\tau, \quad \alpha,\beta > 0.$$

Theorem 2.4. *Let $1 < p \le q < \infty$. Suppose that $\alpha, \beta > n/p$. Then the operator $T_{\alpha,\beta}$ is bounded $L^p(R^{2n})$ to $L^q_v(R^{2n})$ if and only if*

$$\sup_{a,b>0} \left(\int_{|x|>a} \int_{|y|>b} v(x,y)|x|^{(\alpha-n)q}|y|^{(\beta-n)q} dxdy \right)^{1/q} (ab)^{n/p'} < \infty.$$

For the multiple ball potential

$$B_{\alpha,\beta}f(x,y)=\int_{|t|<|x|}\int_{|\tau|<|y|}f(t,\tau)\frac{(|x|^2-|t|^2)^{\alpha}}{|x-t|^n}\frac{(|y|^2-|\tau|^2)^{\beta}}{|y-\tau|^n}dtd\tau,\ \ \alpha,\beta>0,$$

we have

Theorem 2.5. *Let $1<p\le q<\infty$. Suppose that $\alpha,\beta>n/p$. Then the operator $B_{\alpha,\beta}$ is bounded from $L^p(R^{2n})$ to $L^q_v(R^{2n})$ if and only if*

$$\sup_{a,b>0}\left(\int_{|x|>a}\int_{|y|>b}v(x,y)|x|^{(2\alpha-n)q}|y|^{(2\beta-n)q}dxdy\right)^{1/q}(ab)^{n/p'}<\infty.$$

Now we consider the "mixed potentials" with multiple kernels:

$$(I_{\alpha}R_{\beta})f(x,y)=\int_0^{\infty}\int_0^{y}|x-t|^{\alpha-1}(y-\tau)^{\beta-1}f(t,\tau)dtd\tau,\ \ x,y>0,$$

$$(W_{\alpha}R_{\beta})f(x,y)=\int_x^{\infty}\int_0^{y}(t-x)^{\alpha-1}(y-\tau)^{\beta-1}f(t,\tau)dtd\tau,\ \ x,y>0.$$

Theorem 2.6. *Let $1<p\le q<\infty$. Suppose that $0<\alpha<1/p$, $\beta>1/p$. Then the operator $I_{\alpha}R_{\beta}$ is bounded from $L^p(R^2_+)$ to $L^q_v(R^2_+)$ if and only if*

$$\sup_{\substack{h,a,j\\0<h<a,j\in Z}}\left(\int_a^{a+h}\int_{2^j}^{2^{j+1}}v(x,y)dxdy\right)^{1/q}h^{\alpha-1/p}2^{j(\beta-1/p)}<\infty.$$

Theorem 2.7. *Let $1<p<\infty$. Suppose that $0<\alpha<1/p$ and $\beta>1/p$. Then the operator $W_{\alpha}R_{\beta}$ is bounded from $L^p(R^2_+)$ to $L^p_v(R^2_+)$ if and only if $R_{\alpha}V_j\in L^{p'}_{loc}(R_+)$ for all $j\in Z$ and*

$$R_{\alpha}[R_{\alpha}V_j]^{p'}(x)\le cR_{\alpha}[V_j](x),\ x>0,\ j\in Z,\tag{2.4}$$

where V_j are defined as follows

$$V_j(x)\equiv\int_{2^j}^{2^{j+1}}v(x,y)y^{\beta p-1}dy.$$

Theorem 2.8. *Let $1<p<\infty$. Suppose that $0<\alpha<1/p$ and $\beta>1/p$. Then for the boundedness of $I_{\alpha}R_{\beta}$ from $L^p(R^2_+)$ to $L^p_v(R^2_+)$, it is necessary and sufficient that $R_{\alpha}V_j,W_{\alpha}V_j\in L^{p'}_{loc}(R_+)$ for all $j\in Z$ and conditions* (2.2), (2.4) *hold.*

To end this section we discuss integral transforms with general multiple kernels.

Let

$$Kf(x,y)=\int_0^{x}\int_0^{y}f(t,\tau)k_1(x,t)k_2(y,\tau)dtd\tau,\ \ x,y>0,$$

where k_i $(i=1,2)$ are positive kernels.

We say that a kernel $k:\{(x,y):0<y<x<\infty\}\to(0,\infty)$ belongs to V $(k\in V)$ if there exists a positive constant c_1 such that for all x,t,z with $0<t<z<x<\infty$ the inequality

$$k(x,t)\le c_1k(x,z)$$

holds, and k belongs to V_λ ($k \in V_\lambda$) ($1 < \lambda < \infty$) if there exists a positive constant c_2 such that for all $x, x > 0$, the inequality

$$\int_{x/2}^{x} k^{\lambda'}(x,y)dy \le c_2 x k^{\lambda'}(x,x/2)$$

is fulfilled, where $\lambda' = \lambda/(\lambda-1)$.

For example, if $k(x,y) = (x-y)^{\alpha-1}\chi_{\{(x,y):0<y<x\}}$ with $1/p < \alpha \le 1$, then $k \in V \cap V_p$. For other examples of kernels k satisfying the conditions mentioned above see [M2], [EKM], Section 2.7.

THEOREM 2.9. *Let $1 < p \le q < \infty$. Suppose that the kernels k_1 and k_2 belong to $V \cap V_p$. Then the following statements are equivalent:*

(i) the operator K is bounded from $L^p(R^2_+)$ to $L^q_v(R^2_+)$;

(ii)

$$B \equiv \sup_{a,b>0} \left(\int_a^\infty \int_b^\infty v(x,y) k_1^q(x,x/2) k_2^q(y,y/2) dxdy \right)^{1/q} (ab)^{1/p'} < \infty;$$

(iii)

$$\tilde{B} \equiv \sup_{k,j\in Z} \left(\int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} v(x,y)(xy)^{q/p'} k_1^q(x,x/2) k_2^q(y,y/2) dxdy \right)^{1/q} < \infty.$$

Moreover, $\|K\| \approx B \approx \tilde{B}$.

3. Two-weighted inequalities for two-dimensional Hardy operators. Let us consider the operators

$$H_2 f(x,y) = \int_0^x \int_0^y f(t,\tau)dtd\tau,$$

$$H_2' f(x,y) = \int_x^\infty \int_y^\infty f(t,\tau)dtd\tau, \quad x,y>0.$$

The solution of the two-weight problem for two-dimensional Hardy operator has been given by E. Sawyer [Sa2]:

THEOREM I. *Let $1 < p \le q < \infty$. Then for the boundedness of the operator H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ it is necessary and sufficient that the following three conditions are satisfied:*

(i)

$$\sup_{a,b>0} (H_2' v(a,b))^{1/q} (H_2\sigma(a,b))^{1/p'} \equiv A < \infty, \quad \sigma = w^{1-p'}; \tag{3.1}$$

(ii)

$$\int_0^a \int_0^b (H_2\sigma)^q v \le A^q [H_2\sigma(a,b)]^{q/p}$$

for all $a, b > 0$;

(iii)

$$\int_a^\infty \int_b^\infty (H_2' v)^{p'} \sigma \le A^{p'} [H_2' v(a,b)]^{p'/q'}$$

for all $a, b > 0$.

Earlier in [Sy], [Se] were derived some sufficient conditions for the validity of the corresponding multiple Hardy inequality. When the weight function is a product of two one-dimensional weights, i.e. $w(x,y) = w_1(x)w_2(y)$, the boundedness criteria from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ for the operator H_2 have been found in [W]. Here we present some new results concerning the two-weight inequalities for the operator H_2, not necessarily for the weights which are product of two functions of separate variables on the right side.

Definition 3.1. A nonnegative function $\rho : R^2_+ \to R^1$ is said to be a weight function with doubling condition uniformly with respect to $y \in R_+$ if there exists a positive constant c such that for arbitrary $t > 0$ and almost all $x > 0$ the condition

$$\int_0^{2t} \rho(x,y)dy \le c \int_0^t \rho(x,y)dy$$

holds. In this case we write that $\rho \in DC(y)$. Analogously we define the class $DC(x)$.

Let us formulate the results for more general two-dimensional Hardy-type transform

$$R_{\alpha,\beta} f(x,y) = \int_0^x \int_0^y \frac{f(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} dt d\tau, \quad \alpha,\beta \ge 1.$$

Theorem 3.1. *Let $1 < p \le q < \infty$ and let $\alpha,\beta \ge 1$. Suppose that $w^{1-p'} \in DC(y)$. Then the operator $R_{\alpha,\beta}$ is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if the following two conditions hold:*

$$A_1 \equiv \sup_{a,b>0} \left(\int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)p'}} dx dy \right)^{1/p'} \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dx dy \right)^{1/q} < \infty;$$

$$A_2 \equiv \sup_{a,b>0} \left(\int_0^a \int_0^b w^{1-p'}(x,y) dx dy \right)^{1/p'} \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q} y^{(1-\beta)q}} dx dy \right)^{1/q} < \infty.$$

Moreover, $\|R_{\alpha,\beta}\| \approx \max\{A_1, A_2\}$.

Theorem 3.2. *Let $1 < p \le q < \infty$ and let $\alpha,\beta \ge 1$. Suppose that $w^{1-p'} \in DC(x)$. Then the operator $R_{\alpha,\beta}$ is bounded from $L^p_w(R^2_+)$ to $L^q_v(R_+)$ if and only if the following two conditions hold:*

$$B_1 \equiv \sup_{a,b>0} \left(\int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(b-y)^{(1-\beta)p'}} dx dy \right)^{1/p'} \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{x^{(1-\alpha)q}} dx dy \right)^{1/q} < \infty;$$

$$B_2 \equiv \sup_{a,b>0} \left(\int_0^a \int_0^b w^{1-p'}(x,y) dx dy \right)^{1/p'} \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(y-b)^{(1-\beta)q} x^{(1-\alpha)q}} dx dy \right)^{1/q} < \infty.$$

Moreover, $\|R_{\alpha,\beta}\| \approx \max\{B_1, B_2\}$.

Proofs of Theorems 3.1-3.2 are based on Theorem B and the following statement (see, e.g., [M3]):

Proposition 3.1. *Let $1 < p \le q < \infty$ and let $\{a_n\}$, $\{b_n\}$ be positive sequences. The inequality*

$$\Big(\sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{n} g_k \Big|^q a_n^q \Big)^{1/q} \le c \Big(\sum_{n=-\infty}^{\infty} |g_n|^p b_n^p \Big)^{1/p} \tag{3.2}$$

holds with the positive constant c independent of $\{g_k\}$ $(g_k \in l^p_{b^p_n}(Z))$ if and only if

$$B =: \sup_{n\in Z}\Big(\sum_{k=n}^{\infty} a_k^q\Big)^{1/q}\Big(\sum_{k=-\infty}^{n} b_k^{-p'}\Big)^{1/p'} < \infty.$$

Moreover, if c is the best constant in (3.2)*, then*

$$B \le c \le Bq^{\frac{1}{q}}\Big(\frac{q}{q-1}\Big)^{(p-1)/p}.$$

Corollary 3.1. *Let $1 < p \le q < \infty$ and let $w^{1-p'} \in DC(x)$ or $w^{1-p'} \in DC(y)$. Then the operator H_2 is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if condition* (3.1) *holds.*

A more general form of this result is the following statement, which we formulate for H_2:

Theorem 3.3. *Let $1 < p \le q < \infty$. Assume that the weight function w satisfies the condition*

$$\sup_{x>0,k\in Z}\Big(\sum_{j=k}^{\infty}\Big(\int_0^{2^j} w^{1-p'}(x,y)dy\Big)^{1-p}\Big)\Big(\int_0^{2^{k+1}} w^{1-p'}(x,y)dx\Big)^{p-1} < \infty.$$

Then the boundedness of H_2 from $L^p(R^2_+)$ to $L^q_v(R^2_+)$ is equivalent to the condition (3.1)*.*

The following theorem states that when the weight w on the right-hand side has the form $w(x,y) = w_1(x)w_2(y)$ then the first condition in Sawyer's theorem is equivalent to the boundedness of the operator H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$.

Theorem 2.4. *Let $1 < p \le q < \infty$ and $w(x,y) = w_1(x)w_2(y)$. Then the operator H_2 is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if*

$$A_1 \equiv \sup_{a,b>0}\Big(\int_0^a\int_0^b w^{1-p'}(x,y)dxdy\Big)^{1/p'}\Big(\int_a^{\infty}\int_b^{\infty} v(x,y)dxdy\Big)^{1/q} < \infty.$$

The results of this section were published in [M3] and announced in [M4].

4. Multiple Riesz potentials. In this section we deal with the following operators:

$$(I_{\alpha,\beta}f)(x,y) = \int_R\int_R f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1}dtd\tau;$$

$$(J_{\alpha,\beta}f)(x,y) = \int_R\int_{|\tau|<2|y|} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1}dtd\tau,$$

where $0 < \alpha, \beta < 1$.

Let us first formulate the results for $J_{\alpha,\beta}$.

Theorem 4.1. *Let $1 < p < q < \infty$, $1/p - 1/q < \alpha < 1$, $1/p < \beta < 1$. Then the operator $J_{\alpha,\beta}$ is bounded from $L^p(R^2)$ to $L^q_v(R^2)$ if and only if*

$$\sup_{\substack{a\in R\\ r>0\\ k\in Z}}\Big(\int_{a-r}^{a+r}\int_{2^k<|y|<2^{k+1}} v(x,y)dxdy\Big)^{1/q} r^{\alpha-1/p}2^{k(\beta-1/p)} < \infty.$$

In the diagonal case $p = q$ we have

THEOREM 4.2. *Let $1 < p < \infty$, $0 < \alpha < 1$, $1/p < \beta < 1$. Then the operator $J_{\alpha,\beta}$ is bounded from $L^p(R^2)$ to $L^p_v(R^2)$ if and only if there exists a positive constant c such that for a.a. $x \in R$ and $k \in Z$ the inequality*

$$K_\alpha[K_\alpha V_j]^{p'}(x) \le K_\alpha[V_j](x)$$

holds, where K_α is the one-dimensional potential

$$K_\alpha f(x) = \int_R \frac{f(y)}{|x-y|^{1-\alpha}} dy$$

and

$$V_j(x) \equiv \int_{2^j<|y|<2^{j+1}} v(x,y)|y|^{\beta p-1} dy.$$

Further, the following statement holds.

THEOREM 4.3. *Let $1 < p < q < \infty$. Suppose that $1/p - 1/q < \alpha < 1$, $1/p < \beta < 1$. Then the two-weight inequality*

$$\left(\int_R \int_R |(J_{\alpha,\beta} f)(x,y)|^q v(x,y) dx dy\right)^{1/q} \le c \left(\int_R \int_R |f(x,y)|^p u(x) dx dy\right)^{1/p}$$

holds if and only if

(*i*)

$$\sup_{\substack{a\in R\\ r>0\\ k\in Z}} \left(\int_{|x-a|>r} \int_{2^k<|y|<2^{k+1}} \frac{v(x,y)}{|x-a|^{(1-\alpha)q}} dx dy\right)^{1/q} \times \left(\int_{|x-a|<r} u^{1-p'}(x) dx\right)^{1/p'} 2^{k(\beta-1/p)} < \infty;$$

(*ii*)

$$\sup_{\substack{a\in R\\ r>0\\ k\in Z}} \left(\int_{|x-a|<r} \int_{2^k<|y|<2^{k+1}} v(x,y) dx dy\right)^{1/q} \times \left(\int_{|x-a|>r} \frac{u^{1-p'}(x)}{|x-a|^{(\alpha-1)p'}} dx\right)^{1/p'} 2^{k(\beta-1/p)} < \infty.$$

DEFINITION 4.1. We say that a two-dimensional weight function ρ defined on R^2 satisfies the reverse doubling condition at 0 uniformly with respect to x if there exist constants $\eta_1 > 1$ and $\eta_2 > 1$ such that for all $x \in R$ and $r > 0$ the inequality

$$\int_{B(0,\eta_1 r)} \rho(x,y) dy \ge \eta_2 \int_{B(0,r)} \rho(x,y) dy$$

holds. In this case we write $\rho \in RD_0(y,\eta_1,\eta_2)$ (or $\rho \in RD_0(y)$).

DEFINITION 4.2. The weight $\rho : R^2 \to R$ belongs to the class $RD_\infty(y,\lambda_1,\lambda_2)$ (or $RD_\infty(y)$), $\lambda_1, \lambda_2 > 1$, if the inequality

$$\int_{R^2\setminus B(0,r/\lambda_1)} \rho(x,y) dy \ge \lambda_2 \int_{R^2\setminus B(0,r)} \rho(x,y) dy$$

holds for all $r > 0$ and $x \in R$.

Analogously we can define the classes $RD_0(x)$, $RD_\infty(x)$.

Theorem 4.4. *Let $1<p<q<\infty$, $0<\alpha,\beta<1$. Suppose that $w^{1-p'}\in RD_0(y,\eta_1,\eta_2)$ and $\frac{w^{1-p'}(x,y)}{|y|^{(1-\beta)p'}}\in RD_\infty(y,\mu_1,\mu_2)$. Assume also that w satisfies the condition: there exists a positive constant c such that for all $y\in R$ and $t\in R$ the inequality*

$$\int_{\frac{|y|}{\eta_1}<|\tau|<\mu_1|y|}\frac{w^{1-p'}(t,\tau)}{|y-\tau|^{(1-\beta)p'}}d\tau\le\frac{c}{|y|^{(1-\beta)p'}}\int_{\frac{|y|}{\eta_1\mu_1}<|\tau|<\frac{|y|}{\eta_1}}w^{1-p'}(t,\tau)d\tau \tag{4.1}$$

holds. Then $I_{\alpha,\beta}$ is bounded from $L^p_w(R^2)$ to $L^q_v(R^2)$ if and only if the following two conditions are satisfied:

(*i*)

$$\sup_{\substack{a\in R\\ r>0\\ s>0}}\bigg(\int_{|x-a|<r}\int_{|y|>s/\mu_1^2}\frac{w^{1-p'}(x,y)}{|y|^{(1-\beta)p'}}dxdy\bigg)^{1/p'}\times\bigg(\int_{|x-a|>r}\int_{|y|<s}\frac{v(x,y)}{|x-a|^{(1-\alpha)q}}dxdy\bigg)^{1/q}<\infty;$$

(*ii*)

$$\sup_{\substack{a\in R\\ r>0\\ s>0}}\bigg(\int_{|x-a|>r}\int_{|y|>s/\mu_1^2}\frac{w^{1-p'}(x,y)}{|x-a|^{(1-\alpha)p'}|y|^{(1-\beta)p'}}dxdy\bigg)^{1/p'}\times\bigg(\int_{|x-a|<r}\int_{|y|<s}v(x,y)dxdy\bigg)^{1/q}<\infty.$$

Remark 4.1. For example, the weight function $w(x,y)$ which is increasing with respect to y for a.a. $x\in R$ satisfies the condition (4.1).

Corollary 4.1. *Let $1<p<q<\infty$, $0<\alpha<1$, $1/p<\beta<1$. Then the two-weight inequality*

$$\bigg(\int_R\int_R|(I_{\alpha,\beta}f)(x,y)|^q v(x,y)dxdy\bigg)^{1/q}\le c\bigg(\int_R\int_R|f(x,y)|^p|y|^\gamma dxdy\bigg)^{1/p}$$

holds if and only if

(*i*)

$$\beta p-1<\gamma<p-1;$$

(*ii*)

$$\sup_{\substack{a\in R\\ r>0\\ t>0}}\bigg(\int_{a-r}^{a+r}\int_{|y|>t}\frac{v(x,y)}{|y|^{(1-\alpha)q}}dxdy\bigg)^{1/q}r^{\alpha-1/p}2^{k(-\gamma/p+1/p')}<\infty.$$

To formulate the next results we shall need some definitions.

Definition 4.3. We say that a weight u defined on R satisfies the reverse Hölder's inequality if there exist $r>1$ and $c>0$ such that

$$\bigg(\frac{1}{t}\int_{|x|<t}u^r(x)dx\bigg)^{1/r}\le\frac{c}{t}\int_{|x|<t}u(x)dx \tag{4.2}$$

holds for all $t>0$. In this case we write $u\in RH_0(r)$.

Note that if the weight function u satisfies condition (4.2), then it also satisfies

$$\left(\frac{1}{t}\int_{t<|x|<\lambda t} u^r(x)dx\right)^{1/r} \le \frac{c}{t}\int_{|x|<\lambda t} u(x)dx,$$

for all t and $\lambda > 1$, where the constant c is independent of t. In the latter case we write $u \in RH_{an}(r)$.

REMARK 4.2. For example, the power-type weight $u(x) = |x|^\gamma$ belongs to $RH_{an}(r)$ for all $\gamma \in R$.

DEFINITION 4.4. A two-dimensional weight $\rho : R^2 \to R$ satisfies the reverse Hölder's inequality uniformly with respect to x ($u \in RH_0(r, y)$), if there exist $r > 1$ and $c > 0$ such that for all $t > 0$ and $x \in R$ the inequality

$$\left(\frac{1}{t}\int_{|y|<t} \rho^r(x,y)dy\right)^{1/r} \le \frac{c}{t}\int_{|y|<t} \rho(x,y)dy$$

holds. We say that $\rho \in RH_{an}(r, y)$ if

$$\left(\frac{1}{t}\int_{t<|y|<\lambda t} \rho^r(x,y)dy\right)^{1/r} \le \frac{c}{t}\int_{|y|<\lambda t} \rho(x,y)dy$$

holds, for all $t > 0$, $\lambda > 1$ and $x \in R$, and the constant c is independent of t and x.

PROPOSITION 4.1. *The following statements are equivalent:*

(i) The weight $\rho \in RH_0(r, y)$;

(ii) $\rho^r \in A^0_\infty(y)$, i.e. there exist positive constants δ and c such that for all $a > 0$ and any measurable subset $E \subset [-a, a]$ the inequality

$$\frac{\rho^r_x(E)}{\rho^r_x([-a,a])} \le c\left(\frac{|E|}{a}\right)^\delta$$

holds, where $\rho^r_x(S) \equiv \int_S \rho^r(x,y)dy$;

(iii) $\rho^r \in A^0_p(y)$ for some $p \ge 1$. By the definition $\rho^r \in A^0_p(y)$ if

$$\sup_{a,x}\left(\frac{1}{a}\int_{-a}^{a} \rho^r(x,y)dy\right)\left(\frac{1}{a}\int_{-a}^{a} \rho^{r/(1-p)}(x,y)dy\right)^{p-1} < \infty \;\; for \;\; p > 1,$$

and

$$\sup_{a,x}\left(\frac{1}{a}\int_{-a}^{a} \rho^r(x,y)dy\right) \operatorname*{ess\,sup}_{y\in[-a,a]} \frac{1}{\rho^r(x,y)} < \infty \;\; for \;\; p = 1.$$

The condition (iii) of the latter proposition is the Muckenhoupt's condition written with respect to the intervals of the form $[-a, a]$.

Proposition 4.1 can be obtained in the standard way (see, for example, [Mu2], [GR], Ch. 4, [Fe], [CF], [SW]), where similar problems are discussed for the weight classes RH, A_∞ and A_p which are defined on all cubes of R^n.

Now we are able to formulate some results concerning the two-weight inequalities for $I_{\alpha,\beta}$.

Let us begin with the case $w \equiv 1$.

THEOREM 4.5. *Let $1 < p < \infty$, $0 < \alpha, \beta < 1/p$. Suppose that $v \in RH_{an}(\frac{1}{\alpha p}, x)$, $RH_{an}(\frac{1}{\beta p}, y)$. Then $I_{\alpha,\beta}$ is bounded from $L^p(R^2)$ to $L^p_v(R^2)$ if and only if*

$$\sup_{\substack{t>0\\ \tau>0}} \left(\int_{|x|<t} \int_{|y|<\tau} v(x,y)dxdy \right)^{1/p} t^{\alpha-1/p}\tau^{\beta-1/p} < \infty. \tag{4.3}$$

In the case of two weights we have

THEOREM 4.6. *Let $1 < p < \infty$, $0 < \alpha, \beta < 1$. Suppose that the weights v and $w^{1-p'}$ belong to the classes $RH_0(\frac{1}{\alpha}, x)$; $RH_0(\frac{1}{\beta}, y)$. Then $I_{\alpha,\beta}$ is bounded from $L^p_w(R^2)$ to $L^p_v(R^2)$ if and only if*

$$B \equiv \sup_{\substack{t>0\\ \tau>0}} t^{\alpha-1}\tau^{\beta-1} \left(\int_{|x|<t} \int_{|y|<\tau} v(x,y)dxdy \right)^{1/p} \times \left(\int_{|x|<t} \int_{|y|<\tau} w^{1-p'}(x,y)dxdy \right)^{1/p'} < \infty. \tag{4.4}$$

REMARK 4.3. According to Proposition 4.1, Theorem 4.6 remains true if we replace the assumption $v, w^{1-p'} \in RH_0(\frac{1}{\alpha}, x), RH_0(\frac{1}{\beta}, y)$ by the condition $v^{1/\alpha}, w^{\frac{1-p'}{\alpha}} \in A^0_\infty(x)$; $v^{1/\beta}, w^{\frac{1-p'}{\beta}} \in A^0_\infty(y)$.

THEOREM 4.7. *Let $1 < p < \infty$, $0 < \alpha, \beta < 1$. Suppose that v and $w^{1-p'}$ belong to the classes $RD_0(x)$, $RD_0(y)$, $RH_{an}(1/\alpha, x)$, $RH_{an}(1/\beta, y)$. Then $I_{\alpha,\beta}$ is bounded from $L^p_w(R^2)$ to $L^p_v(R^2)$ if and only if (4.4) holds.*

Now we give sketches of the proofs of the main results of this section.

Theorems 4.1, 4.3 and 4.4 follow from the following lemmas:

LEMMA 4.1. *Let $1 < p < q < \infty$, $0 < \alpha, \beta < 1$. Suppose that $w^{1-p'} \in RD_0(y, \eta_1, \eta_2)$. Then the operator*

$$I^{(1)}_{\alpha,\beta} f(x,y) = \int_R \int_{|\tau|<|y|/\eta_1} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1}dtd\tau$$

is bounded from $L^p_w(R^2)$ to $L^q_v(R^2)$ if and only if

(a)

$$\sup_{\substack{a\in R\\ r>0\\ s>0}} \left(\int_{|x-a|<r} \int_{|y|<s} w^{1-p'}(x,y)dxdy \right)^{1/p'} \times \left(\int_{|x-a|>r} \int_{|y|>s} \frac{v(x,y)}{|x-a|^{(1-\alpha)q}|y|^{(1-\beta)q}} dxdy \right)^{1/q} < \infty;$$

(b)

$$\sup_{\substack{a\in R\\ r>0\\ s>0}} \left(\int_{|x-a|>r} \int_{|y|<s} \frac{w^{1-p'}(x,y)}{|x-a|^{(1-\alpha)p'}} dxdy \right)^{1/p'} \times \left(\int_{|x-a|<r} \int_{|y|>s} \frac{v(x,y)}{|y|^{(1-\beta)q}} dxdy \right)^{1/q} < \infty.$$

LEMMA 4.2. *Let $1 < p < q < \infty$, $0 < \alpha, \beta < 1$. Suppose that $w^{1-p'} \in RD_\infty(y, \mu_1, \mu_2)$. Then the operator*

$$I^{(2)}_{\alpha,\beta} f(x,y) = \int_R \int_{|\tau|>\mu_1|y|} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1} dt d\tau$$

is bounded from $L^p_w(R^2)$ to $L^p_v(R^2)$ if and only if (i) and (ii) of Theorem 4.4 hold.

These lemmas can be proved using Theorem A of Section 1 and two-weight criteria for the discrete Hardy operator (see Proposition 3.1). Further, due to the condition $w^{1-p'} \in RD_0(y)$ we have that (i) and (ii) of Theorem 4.4 guarantees (a) and (b) of Lemma 4.1.

The next lemma follows from Theorem A, Hölder's inequalities and Proposition 3.1.

LEMMA 4.3. *Let $1 < p < q < \infty$, $0 < \alpha, \beta < 1$. Suppose that $w^{1-p'}$ satisfies condition (4.1). Then conditions (i) and (ii) of Theorem 4.4 guarantees the boundedness of the operator*

$$I^{(3)}_{\alpha,\beta} f(x,y) = \int_R \int_{|y|/\eta_1<|\tau|<\mu_1|y|} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1} dt d\tau$$

from $L^p_w(R^2)$ to $L^p_v(R^2)$.

Now representing $I_{\alpha,\beta}$ as

$$I_{\alpha,\beta} f = I^{(1)}_{\alpha,\beta} f + I^{(2)}_{\alpha,\beta} f + I^{(3)}_{\alpha,\beta}$$

and using Lemmas 4.1–4.3 we get sufficiency of Theorem 4.4. Necessity follows easily applying the two-weight inequality to the appropriate class of functions. Theorem 4.1 and 4.3 can be derived from Theorem 4.4. In this case $\mu_1, \mu_2, \eta_1, \eta_2 = 2$ and the conditions $w^{1-p'} \in RD_0(y)$ $w^{1-p'}(x,y)/|y|^{(1-\beta)p'} \in RD_\infty(y)$ are automatically satisfied.

The proof of Theorem 4.2 is based on the appropriate result concerning the one-dimensional potential K_α.

To prove Theorems 4.5 and 4.7 we need the following statements:

PROPOSITION 4.2. *Let $1 < p < \infty$, $0 < \alpha < 1/p$. Suppose that a one-dimensional weight u belong to $RH_{an}(\frac{1}{\alpha p})$. Then K_α is bounded from $L^p(R)$ to $L^p_v(R)$ if and only if*

$$\sup_{t>0} t^{\alpha p-1} \int_{|x|<t} v(x)dx < \infty.$$

PROPOSITION 4.3. *Let $1 < p < \infty$, $0 < \alpha < 1$. Suppose that one-dimensional weights $u_1^{1-p'}$ and u_2 belong to $RH_{an}(\frac{1}{\alpha})$. Then K_α is bounded from $L^p_{u_1}(R)$ to $L^p_{u_2}(R)$ if and only if the following three simple conditions are satisfied:*

(*i*)

$$\sup_{t>0} t^{\alpha-1} \left(\int_{-t}^{t} u_2(x)dx\right)^{1/p} \left(\int_{-t}^{t} u_1^{1-p'}(x)dx\right)^{1/p'} < \infty; \tag{4.5}$$

(*ii*)

$$\sup_{t>0} \left(\int_{|x|>t} u_2(x)|x|^{(\alpha-1)p} dx\right)^{1/p} \left(\int_{|x|<t} u_1^{1-p'}(x)dx\right)^{1/p'} < \infty;$$

(iii)

$$\sup_{t>0}\Big(\int_{|x|<t} u_2(x)dx\Big)^{1/p}\Big(\int_{|x|>t} u_1^{1-p'}(x)|x|^{(\alpha-1)p'}dx\Big)^{1/p'}<\infty.$$

It should be noted that when the weights $u_1^{1-p'}$ and u_2 belong to $RH_0(1/\alpha)$, then they satisfy doubling condition for all balls of the type $B(0,r)$, and consequently we have $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)$. So we can formulate

Proposition 4.4. *Let $1<p<\infty$, $0<\alpha<1$. Suppose that one-dimensional weights $u_1^{1-p'}$ and u_2 belong to $RH_0(\frac{1}{\alpha})$. Then K_α is bounded from $L^p_{u_1}(R)$ to $L^p_{u_2}(R)$ if and only if* (4.5) *holds.*

Remark 4.4. This statement can be also formulated for the Riesz potential I_α defined on R^n. Namely if the weights $u_1^{1-p'}$ and u_2 defined on R^n satisfy the reverse Hölder's inequality for all balls centered at the origin and for the parameter n/α (or if $u_1^{n/\alpha(1-p)}$, $u_2^{n/\alpha}$ belongs to the class A_∞ defined not over all balls but on the balls centered at the origin), then I_α is bounded from $L^p_{u_1}$ to $L^p_{u_2}$ if and only if

$$\sup_{t>0} t^{\alpha/n-1}\Big(\int_{B(0,t)} u_2(x)dx\Big)^{1/p}\Big(\int_{B(0,t)} u_1^{1-p'}(x)dx\Big)^{1/p'}<\infty.$$

On the other hand, in the paper [P] it is shown that for the boundedness of I_α from $L^p_{u_1}(R^n)$ to $L^p_{u_2}(R^n)$ it is necessary and sufficient that

$$\sup_{Q}|Q|^{\alpha/n-1}\Big(\int_Q u_2(x)dx\Big)^{1/p}\Big(\int_Q u_1^{1-p'}(x)dx\Big)^{1/p'}<\infty,$$

where the supremum is taken over all dyadic cubes, provided that $u_1^{1-p'}$ and u_2 belong to the class A_∞ defined with respect to all dyadic cubes in R^n.

Propositions 4.3 and 4.4 can be obtained using two-weight criteria for the Hardy-type operators

$$Hf(x)=\int_{|y|<|x|} f(y)dy;\quad H'f(x)=\int_{|y|>|x|} f(y)dy$$

and the Hardy-Littlewood-Sobolev theorem for the operator K_α.

From Proposition 4.2 it follows

Lemma 4.4. *Let $1<p<\infty$, $0<\alpha,\beta<1/p$. Assume that $v\in RH_{an}(\frac{1}{\alpha p},x)$. Then the following are equivalent:*

(*i*) *The operator*

$$J^{(1)}_{\alpha,\beta}f(x,y)=\int_R\int_{|\tau|<|y|/2} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1}dtd\tau$$

is bounded from $L^p(R^2)$ to $L^p_v(R^2)$;

(*ii*) *The operator*

$$J^{(2)}_{\alpha,\beta}f(x,y)=\int_R\int_{|\tau|>2|y|} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1}dtd\tau$$

is bounded from $L^p(R^2)$ to $L^p_v(R^2)$;

(*iii*) *Condition* (4.3) *holds.*

Proof of Theorem 4.5. Let us represent $I_{\alpha,\beta}$ as follows:

$$I_{\alpha,\beta}f = \sum_{i=1}^{3} J^{(i)}_{\alpha,\beta}f,$$

where $J^{(1)}$ and $J^{(2)}$ are defined in the latter lemma and

$$J^{(3)}_{\alpha,\beta}f(x,y) \equiv \int_R \int_{|y|/2<|\tau|<2|y|} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1}dtd\tau.$$

Lemma 4.4 guarantees the boundedness of $J^{(1)}_{\alpha,\beta}$ and $J^{(2)}_{\alpha,\beta}$. Further, the following estimate is obvious:

$$\begin{aligned} J^{(3)}_{\alpha,\beta}f &\le \int_{|t|<|x|/2}\int_R(\cdots) + \int_{|t|>|x|/2}\int_R(\cdots) + \int_{|x|/2<|t|<2|x|}\int_{|y|/2<|\tau|<2|y|}(\cdots) \\ &\equiv \sum_{j=1}^{3} J^{(3,j)}_{\alpha,\beta}f. \end{aligned}$$

The operators $J^{(3,1)}_{\alpha,\beta}$ and $J^{(3,2)}_{\alpha,\beta}$ are similar to $J^{(1)}_{\alpha,\beta}$ and $J^{(2)}_{\alpha,\beta}$ respectively. If we replace the assumption $v \in RH_{an}(\frac{1}{\alpha p}, x)$ by $v \in RH_{an}(\frac{1}{\beta p}, y)$, then the statement analogous to Lemma 4.4 holds for $J^{(3,1)}_{\alpha,\beta}$ and $J^{(3,2)}_{\alpha,\beta}$. Further, condition (4.3) guarantees also the boundedness of $J^{(3,3)}_{\alpha,\beta}$ under the restriction $v \in RH_{an}(\frac{1}{\alpha p}, x)$, $v \in RH_{an}(\frac{1}{\beta p}, y)$. ∎

The proof of Theorem 4.7 is similar to that of Theorem 4.5. In this case we use Proposition 4.3 and the reverse doubling condition. The Hardy-Littlewood-Sobolev and generalized Minkowski's inequalities yield the norm estimate for $J^{(3,3)}_{\alpha,\beta}$.

To get Theorem 4.6 we need

LEMMA 4.5. *Let a weight $\rho(x,y)$ belongs to $RH_0(r,y)$ for some $r>1$. Then $\rho \in DC_0(y)$, i.e. there exists a positive constant b such that for all $t>0$ and $x \in R$ the inequality*

$$\int_{|x|<2t} \rho(x,y)dy \le b\int_{|x|<t} \rho(x,y)dy$$

holds.

Further, it is known (see, e.g., [ST]) that $DC_0(y) \Rightarrow RD_0(y)$. Now Theorem 4.7 completes the proof of Theorem 4.6.

5. Weighted inequalities for fractional integrals on nonhomogeneous spaces.

Throughout this section we assume that $(X, \mathbf{d}, \mu)$ is a topological space X, endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and there exists a non-negative real-valued function (quasimetric) $\mathbf{d} : X \times X \to R^1$ satisfying the conditions:

(i) $\mathbf{d}(x,x) = 0$ for arbitrary $x \in X$;

(ii) $\mathbf{d}(x,y) > 0$ for arbitrary $x, y \in X$, $x \neq y$;

(iii) there exists a positive constant a_0 such that for all $x, y \in X$ the following inequality holds:

$$\mathbf{d}(x,y) \le a_0 \mathbf{d}(y,x);$$

(iv) there exists a positive constant a_1 such that the inequality

$$\mathbf{d}(x,y) \le a_1(\mathbf{d}(x,z) + \mathbf{d}(z,y))$$

holds for arbitrary $x, y, z \in X$;

(v) for every neighbourhood V of the point $x \in X$ there exists a positive number r such that the ball

$$B(x,r) = \{y \in X : \mathbf{d}(x,y) < r\}$$

with center in x and radius r is contained in V;

(vi) the balls $B(x,r)$ are measurable for all $x \in X$, $r > 0$ and, in addition, $0 < \mu B(x,r) < \infty$.

The spaces $(X, \mathbf{d}, \mu)$ with the above mentioned properties are called *nonhomogeneous spaces.*

In the sequel we shall assume that $\mu(X) = \infty$, $\mu\{a\} = 0$ for all $a \in X$; and $B(x,r_2) \setminus B(x,r_1) \ne \emptyset$ for all x, r_1 and r_2 ($x \in X$, $0 < r_1 < r_2 < \infty$).

We consider the integral operator of the form:

$$T_\alpha f(x) = \int_X \frac{f(y)}{\mathbf{d}(x,y)^{s-\alpha}} d\mu(y), \quad 0 < \alpha < s.$$

Theorem 5.1 ([KM3]). *Let $1 < p < q < \infty$. The operator T_α is bounded from $L^p(X)$ into $L^q(X)$ if and only if there exists a positive constant $c > 0$ such that*

$$\mu B(x,r) \le c r^\beta, \quad \beta = \frac{pq(s-\alpha)}{pq+p-q},$$

for arbitrary balls $B(x,r)$.

The latter result in the case of Euclidean spaces has been obtained by the first author in [K1]–[K2].

In particular, from the latter statement it follows

Corollary 5.1. *Let $0 < \alpha < s$, $1 < p < s/\alpha$ and $1/q = 1/p - \alpha/s$. Then T_α acts boundedly from $L^p(X)$ into $L^q(X)$ if and only if*

$$\mu B(x,r) \le c r^s.$$

Sufficiency of this corollary in the case when $\mathbf{d}$ is a metric has been independently considered in [GG] (for the sufficient condition in the case of $X = R^n$ see also [GM]).

Now we concentrate on weighted inequalities for T_α, in particular we have

Theorem 5.2 ([KM6]). *Let s be a positive number. Suppose that $1 < p \le q < \infty$, $s/p - s/q \le \alpha < s$, $\alpha \ne s/p$, $\alpha p - s < \beta < s(p-1)$ and $\lambda = q(s/p + \beta/p - \alpha) - s$. Then the inequality*

$$\left(\int_X |T_\alpha f(x)|^q \mathbf{d}(x_0,x)^\lambda d\mu(x)\right)^{1/q} \le c\left(\int_X |f(x)|^p \mathbf{d}(x_0,x)^\beta d\mu(x)\right)^{1/p} \tag{5.1}$$

for the operator T_α *with the positive constant* c *independent of* f *and* x_0 *holds if and only if*

$$\sup_{a\in X,\, r>0} \frac{\mu B(a,r)}{r^s} < \infty. \tag{5.2}$$

The *proof* of this statement is based on Corollary 5.1 and the following theorems concerning the Hardy-type operators defined on measure spaces:

$$H_{x_0} f(x) = \int_{\{y:\mathbf{d}(x_0,y)\le \mathbf{d}(x_0,x)\}} f(y)d\mu(y),$$

$$H'_{x_0} f(x) = \int_{\{y:\mathbf{d}(x_0,y)\ge \mathbf{d}(x_0,x)\}} f(y)d\mu(y),$$

where x_0 is a fixed point of X.

THEOREM C ([EKM], Ch. 1). *Let* $1 < p \le q < \infty$. *Suppose that* v *and* w *are* μ- *a.e. positive functions on* X. *Then*

(*a*) *The operator* H_{x_0} *is bounded from* $L^p_w(X)$ *to* $L^q_v(X)$ *if and only if*

$$A_1 \equiv \sup_{t\ge 0}\bigg(\int_{\{y:\mathbf{d}(x_0,y)\ge t\}} v(y)d\mu(y)\bigg)^{1/q}\bigg(\int_{\{y:\mathbf{d}(x_0,y)\le t\}} w^{1-p'}(y)d\mu(y)\bigg)^{1/p'} < \infty,$$

$p' = p/(p-1)$;

(*b*) *The operator* H'_{x_0} *is bounded from* $L^p_w(X)$ *to* $L^q_v(X)$ *if and only if*

$$A_2 \equiv \sup_{t\ge 0}\bigg(\int_{\{y:\mathbf{d}(x_0,y)\le t\}} v(y)d\mu(y)\bigg)^{1/q}\bigg(\int_{\{y:\mathbf{d}(x_0,y)\ge t\}} w^{1-p'}(y)d\mu(y)\bigg)^{1/p'} < \infty.$$

Moreover, there exist positive constants c_j, $j = 1,\dots,4$, *depending only on* p *and* q *such that*

$$c_1A_1 \le \|H_{x_0}\| \le c_2A_1, \ \ c_3A_2 \le \|H'_{x_0}\| \le c_4A_2.$$

REMARK 5.1. It follows immediately from (5.2) that $\mu\{a\} = 0$ for all $a \in X$. Therefore in sufficiency of Theorem 5.2 we do not require that the measure μ has any atoms.

From Theorem 5.2 it is easy to obtain the next corollary for the operator

$$I^{x_0}_\alpha f(x) = \mathbf{d}(x_0,x)^{-\alpha s}\int_X \frac{f(y)}{\mathbf{d}(x,y)^{s-\alpha}}d\mu(y).$$

COROLLARY 5.2. *Let* $1 < p < \infty$, $0 < \alpha < s/p$. *Then the inequality*

$$\bigg(\int_X |I^{x_0}_\alpha f(x)|^p d\mu(x)\bigg)^{1/p} \le c\bigg(\int_X |f(x)|^p d\mu(x)\bigg)^{1/p}, \tag{5.3}$$

where the positive constant c *does not depend on* x_0 *and* f, *holds if and only if the measure* μ *satisfies the condition* (5.2).

It should be stressed that Theorem 5.2 contains the extension of the well-known theorem by E. M. Stein and G. Weiss [StW] concerning two-weight inequality for the Riesz potentials.

A non-negative Borel measure m on $\mathbb{C}$ is called a Radon measure if m is finite on compact sets and

$$m(A) = \sup m(K) = \inf m(U)$$

for every Borel set A, where the supremum is taken over all compact sets $K \subset A$ and the infimum is over all open sets U containing A. We say that a Borel measure m on $\mathbb{C}$ is a Carleson measure if m is a Radon measure and there exists a constant $C := C(m) \geq 0$ such that

$$m(D(z, \varepsilon)) \leq C\varepsilon$$

for all disks $D(z, \varepsilon) := \{\tau \in \mathbb{C} : |\tau - z| < \varepsilon\}$.

Proposition 5.1. *Let m be a Radon measure on $\mathbb{C}$. Suppose that $1 < p \leq q < \infty$, $1/p - 1/q \leq \alpha < 1$, $\alpha \neq 1/p$, $\alpha p - 1 < \beta < p - 1$ and $\lambda = q(1/p + \beta/p - \alpha) - 1$. Then the two-weight inequality*

$$\bigg(\int_{\mathbb{C}} |K_\alpha f(z)|^q |z - z_0|^\lambda dm(z)\bigg)^{1/q} \leq c\bigg(\int_{\mathbb{C}} |f(z)|^p |z - z_0|^\beta dm(z)\bigg)^{1/p}$$

for the operator

$$K_\alpha f(z) = \int_{\mathbb{C}} \frac{f(\zeta)}{|\zeta - z|^{1-\alpha}} dm(\zeta),$$

with the positive constant c independent of f and z_0, $z_0 \in \mathbb{C}$, holds if and only if m is a Carleson measure.

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