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MOMENT AND TAIL ESTIMATES FOR MULTIDIMENSIONAL CHAOSES GENERATED BY SYMMETRIC RANDOM VARIABLES WITH LOGARITHMICALLY CONCAVE TAILS

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Abstract. Two kinds of estimates are presented for tails and moments of random multidimensional chaoses $S = \sum a_{i_1,\dots,i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)}$ generated by symmetric random variables $X_{i_1}^{(1)}, \dots, X_{i_d}^{(d)}$ with logarithmically concave tails. The estimates of the first kind are generalizations of bounds obtained by Arcones and Giné for Gaussian chaoses. They are exact up to constants depending only on the order d. Unfortunately, suprema of empirical processes are involved. The second kind estimates are based on comparison between moments of S and moments of some related Rademacher chaoses. The estimates for pth moment are exact up to a factor $(\max(1, \ln p))^{d^2}$.

1. Introduction. Let $S = \sum a_{i_1,\dots,i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)}$ be a random chaos of order d generated by independent random variables $X_i^{(r)}$. In this paper we deal with tail and moment estimates of S in the case when $X_i^{(r)}$ are symmetric random variables with logarithmically concave tails. (This is also the case of Gaussian chaoses.) Exact estimates (up to universal constants) for tails and moments of S are not known for $d \geq 3$. (For $d = 1, 2$ see [La1], [La2].)

Recently R. Latała [La] proved exact estimates (involving only deterministic quantities) for moments and tails of Gaussian chaoses of any order.

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Arcones and Giné (see [AG]) obtained exact bounds for Gaussian chaoses but involving expected values of suprema of empirical processes. In this paper we extend their results giving analogous estimates for chaoses generated by symmetric random variables with logarithmically concave tails. We use similar methods as authors of [AG], i.e. a concentration of measure phenomenon proved by Talagrand (see [Ta]). We adapt also some techniques from [La2].

In the last section of the paper we give estimates of another type. They involve moments of chaoses generated only by Rademacher variables and for pth moment they are exact up to a factor $(\max(1, \ln p))^{d^2}$. However, it seems that it is much more complex to investigate Rademacher than Gaussian chaoses. In the proofs we use some techniques from [LaLo].

2. Generalization of the results of Arcones and Giné.

Let us start with

2.1. *Notation and formulation of the results.* Let d and n be positive integers and $(a_{i_1,\dots,i_d})_{1\le i_1,\dots,i_d\le n}$ be a multidimensional matrix of reals. For $h = (h_1,\dots,h_n) \in R^n$ let $|h|$ denote $\sqrt{h_1^2+\cdots+h_n^2}$.

In [AG] (see page 120) the following estimate was proven for tails of Gaussian chaos $S = \sum a_{i_1,\dots,i_d} g_{i_1}^{(1)}\cdots g_{i_d}^{(d)}$, generated by $N(0,1)$ variables $g_i^{(r)}, 1\le i\le n, 1\le r\le d$. If numbers $\sigma(S)$ and $M_r, 1\le r\le d$, satisfy the conditions

$$\sigma(S) = \sup_{|h^{(k)}|\le 1, 1\le k\le d} \sum a_{i_1,\dots,i_d} h_{i_1}^{(1)}\cdots h_{i_d}^{(d)},$$

$$P\Big(\sup_{|h^{(k)}|\le 1, r+1\le k\le d} \sum a_{i_1,\dots,i_d} \prod_{j=1}^{r} g_{i_j}^{(j)} \prod_{k=r+1}^{d} h_{i_k}^{(k)} \ge M_r\Big) \le \frac{1}{2d},$$

then for all $t>0$

$$P\bigg(|S| \ge \sigma(S)t^d + \sum_{r=1}^{d}\binom{d}{r} t^{d-r} M_r\bigg) \le \frac{1}{2} e^{-t^2/2}.$$

We will prove an analogous estimate for more general chaoses. Henceforth $X_i^{(r)}$, $1\le i\le n, 1\le r\le d$, will denote independent, symmetric random variables with logarithmically concave tails so that

$$P(|X_i^{(r)}|\ge t) = e^{-N_i^{(r)}(t)}, \text{ for } 1\le i\le n, 1\le r\le d, t\ge 0,$$

where $N_i^{(r)} : [0;+\infty) \to [0;+\infty)$ is a convex, strictly increasing function normalized in such a way that

$$N_i^{(r)}(1) = 1.$$

For $1\le i\le n$ and $1\le r\le d$ let us define

$$\tilde N_i^{(r)}(t) = \begin{cases} t^2 & \text{if } |t|\le 1,\\ N_i^{(r)}(|t|) & \text{if } |t|>1.\end{cases}$$

Further, let

$$\tilde B_p^{(r)} = \Big\{ b^{(r)} \in R^n : \sum \tilde N_i^{(r)}(b_i^{(r)}) \le p \Big\}.$$

For $p \geq 1$ and $D \subset \{1, 2, \ldots, d\}$ let

$$M_{D,p} = E \sup_{b^{(k)} \in \tilde{B}_p^{(k)}, k \in D} \sum a_{i_1,\ldots,i_d} \prod_{j \in D'} X_{i_j}^{(j)} \prod_{k \in D} b_{i_k}^{(k)}, \tag{1}$$

where $D' = \{1, 2, \ldots, d\} \backslash D$.

Remark. For $D = \emptyset$ we apply the convention

$$\sup_{b^{(k)} \in \tilde{B}_p^{(k)}, k \in D} \sum a_{i_1,\ldots,i_d} \prod_{j \in D'} X_{i_j}^{(j)} \prod_{k \in D} b_{i_k}^{(k)} = \Big| \sum a_{i_1,\ldots,i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)} \Big|.$$

We use letters c and C to denote universal positive constants that may change from line to line and $c(d)$ and $C(d)$ to denote positive constants, depeding only on d ($c(d)$ and $C(d)$ may also differ at each occurrence).

Relation $A \sim_d B$ means that $c(d)A \leq B \leq C(d)A$.

The symbol $\#D$ denotes the cardinality of the set D.

We may now state the results of this section.

Theorem 1. *Let us consider chaos S of order d, generated by the variables $X_i^{(r)}$,*

$$S = \sum a_{i_1,\ldots,i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)}.$$

For $p \geq 1$ the following estimates hold:

$$P\Big(|S| \geq C(d) \sum_{D \subset \{1,2,\ldots,d\}} M_{D,p}\Big) \leq e^{-p}, \tag{2}$$

$$P\Big(|S| \geq c(d) \sum_{D \subset \{1,2,\ldots,d\}} M_{D,p}\Big) \geq \min(c_1(d), e^{-p}), \tag{3}$$

moreover

$$\|S\|_p \sim_d \sum_{D \subset \{1,2,\ldots,d\}} M_{D,p}. \tag{4}$$

Corollary 1 (Semihypercontractivity of $X_i^{(r)}$). *For $p \geq 1$ and $\lambda \geq 1$*

$$\|S\|_{\lambda p} \leq \lambda^d C_1(d) \|S\|_p. \tag{5}$$

Proof. For $\lambda \geq 1, 1 \leq k \leq d$ we have $\tilde{B}_{\lambda p}^{(k)} \subset \lambda \tilde{B}_p^{(k)}$. From this it follows that

$$M_{D,\lambda p} \leq \lambda^{\#D} M_{D,p}.$$

Now from the above and (4) for $C_1(d) = C(d)/c(d)$ it follows

$$\|S\|_{\lambda p} \leq C(d) \sum_{D \subset \{1,2,\ldots,d\}} M_{D,\lambda p} \leq C(d) \sum_{D \subset \{1,2,\ldots,d\}} \lambda^{\#D} M_{D,p}$$
$$\leq C(d)\lambda^d \sum_{D \subset \{1,2,\ldots,d\}} M_{D,p} \leq C(d)\lambda^d \frac{1}{c(d)} \|S\|_p \leq \lambda^d C_1(d) \|S\|_p. \ \blacksquare$$

We also have the following well known result, which will be used in the sequel:

Corollary 2 (Bonami type inequality). *For $p \geq 1$*

$$\|S\|_p \leq C_2(d) p^d \sqrt{\sum a_{i_1,\ldots,i_d}^2}. \tag{6}$$

Proof. Notice that $\|S\|_2 \sim_d \sqrt{\sum a^2_{i_1,\ldots,i_d}}$ so for $p \in [1;2)$, (6) follows from the monotonicity of the moments, and for $p \geq 2$ from Corollary 1 we have

$$\|S\|_p \leq (p/2)^d C_1(d)\|S\|_2 \leq C_2(d)p^d\sqrt{\sum a^2_{i_1,\ldots,i_d}}. \ \blacksquare$$

REMARK. Theorem 1 is also valid (possibly with worse constants) for undecoupled chaoses of order d, that is, for random variables of the form

$$S_{undec} = \sum_{1\leq i_1<\cdots<i_d\leq n} a_{i_1,\ldots,i_d} X_{i_1}\cdots X_{i_d},$$

where symmetric r.v.'s X_i are independent with log-concave tails. It is an immediate consequence of the result of de la Peña and Montgomery-Smith ([dlPM]) that moments and tails of $\tilde{S}$ are comparable (with constants depending only on d) with moments and tails of the decoupled chaos

$$S_{dec} = \sum_{\pi}\sum_{1\leq i_1<\cdots<i_d\leq n} a_{i_1,\ldots,i_d} X^{(1)}_{i_{\pi(1)}}\cdots X^{(d)}_{i_{\pi(d)}},$$

where the first sum is taken over all permutations π of the set $\{1,2,\ldots,d\}$ and r.v.'s $X^{(r)}_i$ are independent copies of X_i.

REMARK. Let us mention that there is also an analogue of Theorem 1 for more general variables than S, namely for variables of the form

$$S_T = \sup_{\mathbf{a}\in T}\sum a_{i_1,\ldots,i_d} X^{(1)}_{i_1}\cdots X^{(d)}_{i_d}$$

(T is a nonempty family of multidimensional real matrices), with numbers $M_{D,p}$ substituted by numbers $M_{T,D,p}$, defined as

$$M_{T,D,p} = E\sup_{\mathbf{a}\in T}\sup_{b^{(k)}\in\tilde{B}^{(k)}_p,k\in D}\sum a_{i_1,\ldots,i_d}\prod_{j\in D'} X^{(j)}_{i_j}\prod_{k\in D} b^{(k)}_{i_k}.$$

The method of proof is similar to the method of proof of Theorem 1. Other methods for investigating the behaviour of S_T for Rademacher chaoses, based on entropy and tensorization, were developed in [BBLM].

2.2. *Proof of Theorem 1.* We will use the concentration properties of the measure μ with the density $\frac{1}{2}e^{-|x|}$ with respect to the Lebesgue measure, proved by Talagrand.

For any positive integer N and product measure $\mu^{\otimes N}$ on R^N the following concentration phenomenon holds:

$$\mu^{\otimes N}(A+V_s) \geq 1 - \{\mu^{\otimes N}(A)\}^{-1}e^{-s}, \tag{7}$$

where $V_s = \{x \in R^N : \sum\min(|x_i|, x_i^2) \leq 36s\}$. Proof of (7) may be found in [Ta], an alternative, simpler proof was presented in [Ma].

To prove estimates for tails and moments of S from above let us first notice that $X^{(r)}_i = \hat{X}^{(r)}_i + \hat{\hat{X}}^{(r)}_i$ for some independent symmetric random variables $\hat{X}^{(r)}_i, \hat{\hat{X}}^{(r)}_i$ such that

$$P(|\hat{X}^{(r)}_i| \geq t) = e^{-\hat{N}^{(r)}_i(t)}, \text{ where } \hat{N}^{(r)}_i(t) = \begin{cases} t & \text{if } 0\leq t\leq 1,\\ N^{(r)}_i(t) & \text{if } t>1\end{cases}$$

and $|\hat{\hat{X}}_i^{(r)}| \le 1$ a.e. By the contraction principle (cf. [KW]), since $E|\hat{X}_i^{(r)}| \ge \int_0^1 e^{-t}dt > \frac{1}{2}$, we have

$$\begin{aligned}\|S\|_p &= \Big\|\sum a_{i_1,\ldots,i_d}\prod_{j=1}^{d}(\hat{X}_{i_j}^{(j)} + \hat{\hat{X}}_{i_j}^{(j)})\Big\|_p \\ &\le \sum_{D\subset\{1,2,\ldots,d\}}\Big\|\sum a_{i_1,\ldots,i_d}\prod_{j\in D'}\hat{X}_{i_j}^{(j)}\prod_{k\in D}\hat{\hat{X}}_{i_k}^{(k)}\Big\|_p \\ &\le \sum_{D\subset\{1,2,\ldots,d\}} 2^{\#D}\Big\|\sum a_{i_1,\ldots,i_d}\prod_{j\in D'}\hat{X}_{i_j}^{(j)}\prod_{k\in D}\hat{X}_{i_k}^{(k)}\Big\|_p \\ &= 3^d\Big\|\sum a_{i_1,\ldots,i_d}\hat{X}_{i_1}^{(1)}\cdots\hat{X}_{i_d}^{(d)}\Big\|_p.\end{aligned}$$

So, in order to prove estimate in (4) from above, i.e.

$$\|S\|_p \le C(d)\sum_{D\subset\{1,2,\ldots,d\}} M_{D,p},$$

it is enough to prove that

$$\|\hat{S}\|_p \le C(d)\sum_{D\subset\{1,2,\ldots,d\}} M_{D,p}, \tag{8}$$

where $\hat{S} = \sum a_{i_1,\ldots,i_d}\hat{X}_{i_1}^{(1)}\cdots\hat{X}_{i_d}^{(d)}$. Below we prove that

$$\hat{M}_{D,p} \le 3^{\#D'} M_{D,p}, \tag{9}$$

where

$$\hat{M}_{D,p} = E\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D}\sum a_{i_1,\ldots,i_d}\prod_{j\in D'}\hat{X}_{i_j}^{(j)}\prod_{k\in D} b_{i_k}^{(k)}.$$

Further, we will prove

$$\|\hat{S}\|_p \le C(d)\sum_{D\subset\{1,2,\ldots,d\}} \hat{M}_{D,p}. \tag{10}$$

This together with (9) will give (8).

Having an estimate of $\|S\|_p$ from above we immediately get, by a standard application of Chebyshev's inequality, the estimate of tails of S from above (2).

In order to prove (9) we will use the following

LEMMA 1. *For $p \ge 0, D = \{k_1,\ldots,k_l\} \subset \{1,2,\ldots,d\}, j \in D'$ and real numbers $a_{\mathbf{i}}$, where $\mathbf{i} = (i_j, i_{k_1},\ldots,i_{k_l})$, we have*

$$E\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D}\sum a_{\mathbf{i}}\hat{X}_{i_j}^{(j)}\prod_{k\in D} b_{i_k}^{(k)} \le 3E\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D}\sum a_{\mathbf{i}}X_{i_j}^{(j)}\prod_{k\in D} b_{i_k}^{(k)}.$$

Proof. We have

$$\begin{aligned}E\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D}\sum a_{\mathbf{i}}\hat{X}_{i_j}^{(j)}\prod_{k\in D} b_{i_k}^{(k)} \le{}& E\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D}\sum a_{\mathbf{i}}\hat{X}_{i_j}^{(j)} I_{\{|\hat{X}_i^{(r)}|\le 1\}}\prod_{k\in D} b_{i_k}^{(k)} \\ &+ E\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D}\sum a_{\mathbf{i}}\hat{X}_{i_j}^{(j)} I_{\{|\hat{X}_i^{(r)}|>1\}}\prod_{k\in D} b_{i_k}^{(k)}.\end{aligned}$$

Now, since $E|X_i^{(r)}| > \frac{1}{2}$, by the contraction principle,

$$E \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{\mathbf{i}}\hat{X}_{i_j}^{(j)} I_{\{|\hat{X}_i^{(r)}|\le 1\}} \prod_{k\in D} b_{i_k}^{(k)} \le 2E \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{\mathbf{i}} X_{i_j}^{(j)} \prod_{k\in D} b_{i_k}^{(k)}.$$

Further

$$\begin{aligned} E &\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{\mathbf{i}}\hat{X}_{i_j}^{(j)} I_{\{|\hat{X}_i^{(r)}|>1\}} \prod_{k\in D} b_{i_k}^{(k)} \\ &= E \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{\mathbf{i}} X_{i_j}^{(j)} I_{\{|X_i^{(r)}|>1\}} \prod_{k\in D} b_{i_k}^{(k)} \\ &\le E \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{\mathbf{i}} X_{i_j}^{(j)} \prod_{k\in D} b_{i_k}^{(k)}. \end{aligned}$$

From the above inequalities we get the assertion. ∎

Applying Lemma 1 for consecutive j's belonging to D' we get (9):

$$\begin{aligned} \hat{M}_{D,p} &= E \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{i_1,\ldots,i_d} \prod_{j\in D'} \hat{X}_{i_j}^{(j)} \prod_{k\in D} b_{i_k}^{(k)} \\ &\le 3^{\#D'} E \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{i_1,\ldots,i_d} \prod_{j\in D'} X_{i_j}^{(j)} \prod_{k\in D} b_{i_k}^{(k)} \\ &= 3^{\#D'} M_{D,p}. \end{aligned}$$

Let now $\hat{M}_i^{(r)}$ be inverse of $\hat{N}_i^{(r)}$ and for $x \in (-\infty;0)$ define $\hat{M}_i^{(r)}(x) = -\hat{M}_i^{(r)}(-x)$, then for any Borel set $B \subset R^n$

$$\begin{aligned} &P(\hat{S} \in B) \\ &= \mu^{\otimes d\cdot n}\Big(\Big\{(x^{(1)},\ldots,x^{(d)}) \in R^{dn} : \sum a_{i_1,\ldots,i_d} \hat{M}_{i_1}^{(1)}(x_{i_1}^{(1)}) \cdots \hat{M}_{i_d}^{(d)}(x_{i_d}^{(d)}) \in B\Big\}\Big). \end{aligned}$$

For $D \subset \{1,2,\ldots,d\}, p \ge 1$, define

$$A_{D,p} = \Big\{x \in R^{dn} : \sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{i_1,\ldots,i_d} \prod_{j\in D'} \hat{M}_{i_j}^{(j)}(x_{i_j}^{(j)}) \prod_{k\in D} b_{i_k}^{(k)} \le 2^{d+1}\hat{M}_{D,p}\Big\}$$

and

$$A_p = \bigcap_{D\subset\{1,2,\ldots,d\}} A_{D,p}.$$

By Chebyshev's inequality we get

$$\begin{aligned} \mu^{\otimes dn}(A_{D,p}) &= P\Big(\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k\in D} \sum a_{i_1,\ldots,i_d} \prod_{j\in D'} \hat{X}_{i_j}^{(j)} \prod_{k\in D} b_{i_k}^{(k)} \le 2^{d+1}\hat{M}_{D,p}\Big) \\ &\ge 1 - \frac{1}{2^{d+1}}. \end{aligned}$$

Hence

$$\mu^{\otimes dn}(A_p) \ge 1 - \Big(\sum_{D\subset\{1,2,\ldots,d\}} A'_{D,p}\Big) \ge 1 - 2^d \frac{1}{2^{d+1}} = \frac{1}{2}. \tag{11}$$

Let now $R^{dn} \ni x = (x^{(1)}, \dots, x^{(d)}) = (y^{(1)}, \dots, y^{(d)}) + (z^{(1)}, \dots, z^{(d)}) = y + z$, where $y \in A_p$, $z \in V_s = \{z \in R^{dn} : \sum \min(|z_i|, z_i^2) \le 36s\}$. We have the estimate

$$\tag{12} \begin{aligned} &\Big|\sum a_{i_1,\dots,i_d} \hat{M}_{i_1}^{(1)}(x_{i_1}^{(1)}) \cdots \hat{M}_{i_d}^{(d)}(x_{i_d}^{(d)})\Big| \\ &= \Big|\sum a_{i_1,\dots,i_d} \prod_{j=1}^{d} \{\hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) + \hat{M}_{i_j}^{(j)}(x_{i_j}^{(j)}) - \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)})\}\Big| \\ &\le \sum_{D \subset \{1,2,\dots,d\}} \Big|\sum a_{i_1,\dots,i_d} \prod_{j \in D'} \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) \prod_{k \in D} \{\hat{M}_{i_k}^{(k)}(x_{i_k}^{(k)}) - \hat{M}_{i_k}^{(k)}(y_{i_k}^{(k)})\}\Big|. \end{aligned}$$

By concavity of $\hat{M}_i^{(r)}$ on $[0; +\infty)$

$$|\hat{M}_i^{(r)}(x_i^{(r)}) - \hat{M}_i^{(r)}(y_i^{(r)})| \le 2\hat{M}_i^{(r)}(|x_i^{(r)} - y_i^{(r)}|) = 2\hat{M}_i^{(r)}(|z_i^{(r)}|).$$

Since $z \in V_s$, then

$$\sum_{i,r} \tilde{N}_i^{(r)}(|\hat{M}_i^{(r)}(z_i^{(r)})|) = \sum_{i,r} \min((z_i^{(r)})^2, |z_i^{(r)}|) \le 36s.$$

So $(\hat{M}_i^{(k)}(z_i^{(k)})) \in \tilde{B}_{36s}^{(k)}$ for $1 \le k \le r$ and

$$\tag{13} \begin{aligned} &\Big|\sum a_{i_1,\dots,i_d} \prod_{j \in D'} \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) \prod_{k \in D} \{\hat{M}_{i_k}^{(k)}(x_{i_k}^{(k)}) - \hat{M}_{i_k}^{(k)}(y_{i_k}^{(k)})\}\Big| \\ &\le \sup_{z \in V_s} \sum a_{i_1,\dots,i_d} \prod_{j \in D'} \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) \prod_{k \in D} \{2\hat{M}_{i_k}^{(k)}(z_{i_k}^{(k)})\} \\ &\le 2^{\#D} \sup_{b^{(k)} \in \tilde{B}_{36s}^{(k)}, k \in D} \sum a_{i_1,\dots,i_d} \prod_{j \in D'} \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) \prod_{k \in D} b_{i_k}^{(k)}. \end{aligned}$$

For $\lambda \ge 1$ we have $\tilde{B}_{\lambda p}^{(k)} \subset \lambda \tilde{B}_p^{(k)}$. From this for $s \ge p/36$ we get

$$\tag{14} \begin{aligned} &\sup_{b^{(k)} \in \tilde{B}_{36s}^{(k)}, k \in D} \sum a_{i_1,\dots,i_d} \prod_{j \in D'} \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) \prod_{k \in D} b_{i_k}^{(k)} \\ &\le \left(\frac{36s}{p}\right)^{\#D} \sup_{y \in A_p, b^{(k)} \in \tilde{B}_p^{(k)}, k \in D} \sum a_{i_1,\dots,i_d} \prod_{j \in D'} \hat{M}_{i_j}^{(j)}(y_{i_j}^{(j)}) \prod_{k \in D} b_{i_k}^{(k)} \\ &\le \left(\frac{36s}{p}\right)^{\#D} \hat{M}_{D,p}. \end{aligned}$$

The last inequality follows from the definition of A_p. From (12), (13) and (14) we get for $x \in A_p + V_s$, $s \ge p/36$ the estimate

$$\tag{15} \Big|\sum a_{i_1,\dots,i_d} \hat{M}_{i_1}^{(1)}(x_{i_1}^{(1)}) \cdots \hat{M}_{i_d}^{(d)}(x_{i_d}^{(d)})\Big| \le \sum_{D \subset \{1,2,\dots,d\}} 2^{\#D} \left(\frac{36s}{p}\right)^{\#D} \hat{M}_{D,p}.$$

From concentration inequality (7) and (11)

$$\tag{16} \mu^{\otimes dn}(A_p + V_s) \ge 1 - 2e^{-s}.$$

From (15) and (16), taking $s = mp$, $m = 1, 2, \dots$, we get the estimate for $\|\hat{S}\|_p$ from above (10):

$$\begin{aligned}\|\hat{S}\|_p^p &\le \Big(\sum_{D\subset\{1,2,\ldots,d\}} (72)^{\#D}\hat{M}_{D,p}\Big)^p \mu^{\otimes dn}(A_p+V_p)\\ &\quad+\sum_{m=2}^{\infty}\Big(\sum_{D\subset\{1,2,\ldots,d\}}(72m)^{\#D}\hat{M}_{D,p}\Big)^p\mu^{\otimes dn}((A_p+V_{mp})\backslash(A_p+V_{(m-1)p}))\\ &\le 2\sum_{m=1}^{\infty}(72m)^{dp}\Big(\sum_{D\subset\{1,2,\ldots,d\}}\hat{M}_{D,p}\Big)^p e^{-(m-1)p}\\ &\le \Big(C(d)\sum_{D\subset\{1,2,\ldots,d\}}\hat{M}_{D,p}\Big)^p.\end{aligned}$$

Now we will prove the estimates for tails and moments of S from below:

$$\|S\|_p \ge c(d)\sum_{D\in\{1,2,\ldots,d\}} M_{D,p},$$

$$P(|S|\ge c(d)\sum_{D\in\{1,2,\ldots,d\}} M_{D,p}) \ge \min(c_1(d), e^{-p}).$$

We will proceed by induction in d. For $d=1$ the estimate is proven in [La1]. Let us assume that it is true for chaoses of orders $1,\ldots,d$ and let now

$$S=\sum a_{i_1\ldots i_d i_{d+1}} X_{i_1}^{(1)}\cdots X_{i_d}^{(d)}\cdot X_{i_{d+1}}^{(d+1)}.$$

For $D=\emptyset$ we have $M_{D,p}=E|S|$ and the inequality

$$\|S\|_p \ge c(d+1)M_{D,p} \tag{17}$$

is satisfied for $p\ge 1$ with $c(d+1)=1$. For $D=\{1,2,\ldots,d+1\}$ we have

$$M_{D,p}=\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k=1,\ldots,d,d+1}\sum a_{i_1\ldots i_d i_{d+1}} b_{i_1}^{(1)}\cdots b_{i_d}^{(d)}\cdot b_{i_{d+1}}^{(d+1)}.$$

By the induction hypothesis

$$\begin{aligned}\|S\|_p &= \Big(E_{X^{(1)},\ldots,X^{(d)}}\Big(\Big(E_{X^{(d+1)}}\Big|\sum a_{i_1\ldots i_d i_{d+1}}X_{i_1}^{(1)}\cdots X_{i_d}^{(d)}X_{i_{d+1}}^{(d+1)}\Big|^p\Big)^{1/p}\Big)^p\Big)^{1/p}\\ &\ge \Big(E_{X^{(1)},\ldots,X^{(d)}}\Big(c(1)\sup_{b^{(d+1)}\in\tilde{B}_p^{(d+1)}}\sum a_{i_1\ldots i_d i_{d+1}}X_{i_1}^{(1)}\cdots X_{i_d}^{(d)}b_{i_{d+1}}^{(d+1)}\Big)^p\Big)^{1/p}\\ &\ge c(1)\sup_{b^{(d+1)}\in\tilde{B}_p^{(d+1)}}\Big(E_{X^{(1)},\ldots,X^{(d)}}\Big|\sum a_{i_1\ldots i_d i_{d+1}}X_{i_1}^{(1)}\cdots X_{i_d}^{(d)}b_{i_{d+1}}^{(d+1)}\Big|^p\Big)^{1/p}\\ &\ge c(1)\cdot c(d)\sup_{b^{(d+1)}\in\tilde{B}_p^{(d+1)}}\sup_{b^{(k)}\in\tilde{B}_p^{(k)},k=1,\ldots,d}\sum a_{i_1\ldots i_d i_{d+1}}b_{i_1}^{(1)}\cdots b_{i_d}^{(d)}b_{i_{d+1}}^{(d+1)}\\ &= c(d+1)M_{D,p}.\end{aligned} \tag{18}$$

For $D\ne\emptyset$ and $D\ne\{1,2,\ldots,d+1\}$ we have even easier estimates. Again, by the induction hypothesis

$$(19)\quad \|S\|_p = \Big(E_{X^{(j)},j\in D'}\Big(\Big(E_{X^{(k)},k\in D}\Big|\sum a_{i_1\dots i_d i_{d+1}}X^{(1)}_{i_1}\cdots X^{(d)}_{i_d}X^{(d+1)}_{i_{d+1}}\Big|^p\Big)^{1/p}\Big)^p\Big)^{1/p}$$
$$\geq \Big(E_{X^{(j)},j\in D'}\Big(c(\#D)\sup_{b^{(k)}\in\tilde{B}^{(k)}_p k\in D}\sum a_{i_1,\dots,i_d,i_{d+1}}\prod_{j\in D'}X^{(j)}_{i_j}\prod_{k\in D}b^{(k)}_{i_k}\Big)^p\Big)^{1/p}$$
$$\geq c(\#D)E_{X^{(j)},j\in D'}\sup_{b^{(k)}\in\tilde{B}^{(k)}_p,k\in D}\sum a_{i_1,\dots,i_d,i_{d+1}}\prod_{j\in D'}X^{(j)}_{i_j}\prod_{k\in D}b^{(k)}_{i_k}$$
$$= c(d+1)M_{D,p}.$$

From (17), (18) and (19) we obtain the estimate for $\|S\|_p$ from below:

$$\|S\|_p \geq c(d+1)\sum_{D\subset\{1,2,\dots,d+1\}}M_{D,p},$$

which together with the estimate from above gives (4).

In order to prove (3) we will use (5) and the Paley-Zygmund inequality (cf. [KW]) (notice that the inequality (5) is already proven). The Paley-Zygmund inequality states that for any nonnegative, nondegenerate r. v. Y and for $t\in(0;1)$, $P(Y\geq tEY)\geq (1-t)^2\frac{(EY)^2}{EY^2}$. So for $q\geq 1$ we have

$$P\Big(|S|\geq\frac{1}{2}\|S\|_q\Big) = P\Big(|S|^q\geq\Big(\frac{1}{2}\Big)^q E|S|^q\Big)\geq\Big(1-\Big(\frac{1}{2}\Big)^q\Big)^2\frac{(E|S|^q)^2}{E|S|^{2q}}$$
$$\geq\frac{1}{4}\frac{(E|S|^q)^2}{C_1(d)^{2q}2^{2dq}(E|S|^q)^2} = (C_1(d)^2 2^{2d+2})^{-q} = e^{-C_2(d)q}.$$

From this for $p\geq\max(1,C_2(d))$, taking $q=p/C_2(d)$, $c(d)=\frac{1}{2}C_1^{-1}(d)C_2^{-d}(d)$, we get $c(d)\|S\|_p\leq\frac{1}{2}\|S\|_q$ and

$$P(|S|\geq c(d)\|S\|_p)\geq P\Big(|S|\geq\frac{1}{2}\|S\|_q\Big)\geq e^{-C_2(d)q}=e^{-p}.$$

Now, for any $p\geq 1$, taking $c_1(d)=e^{-\max(1,C_2(d))}$, we obtain

$$P(|S|\geq c(d)\|S\|_p)\geq\min(c_1(d),e^{-p}).$$

3. Estimates by moments of Rademacher chaoses. In this section we will apply the previously introduced notation. Some additional notation is presented below.

Let $r^{(k)}_i, 1\leq i\leq n, 1\leq k\leq d$, denote independent Rademacher variables (independent also from the variables $X^{(k)}_i, 1\leq i\leq n, 1\leq k\leq d$). For $p\geq 1, 1\leq k\leq d$ define

$$B^{(k)}_p = \Big\{x^{(k)}\in R^n : \sum N^{(k)}_i(|x^{(k)}_i|)\leq p \;\&\; (x^{(k)}_i=0 \text{ or } |x^{(k)}_i|\geq 1 \text{ for } i=1,\dots,n)\Big\}.$$

We will prove the following

THEOREM 2. *Let us consider a chaos S of order d, generated by the variables $X^{(r)}_i$,*

$$S=\sum a_{i_1,\dots,i_d}X^{(1)}_{i_1}\cdots X^{(d)}_{i_d}.$$

For $p \geq 1$ the following estimates hold:

$$\|S\|_p \leq C(d) \sum_{I \subset \{1,2,\ldots,d\}} \max(1, \ln p)^{d\#I} \sup_{x^{(j)} \in B_p^{(j)}, j \in I} \Big\| \sum a_{i_1 \ldots i_d} \prod_{j \in I} x_{i_j}^{(j)} \prod_{k \in I'} r_{i_k}^{(k)} \Big\|_p \tag{20}$$

and

$$\|S\|_p \geq c(d) \sum_{I \subset \{1,2,\ldots,d\}} \sup_{x^{(j)} \in B_p^{(j)}, j \in I} \Big\| \sum a_{i_1 \ldots i_d} \prod_{j \in I} x_{i_j}^{(j)} \prod_{k \in I'} r_{i_k}^{(k)} \Big\|_p. \tag{21}$$

Remark. Recently R. Latała [La] proved exact estimates for moments and tails of Gaussian chaoses of any order. Below we present moment estimates for Gaussian chaoses of order 3:

$$\begin{aligned}
& c(\sqrt{p}\|A\|_{HS} + p(\|A\|_{\{1\}\{2,3\}} + \|A\|_{\{2\}\{1,3\}} + \|A\|_{\{3\}\{1,2\}}) + p\sqrt{p}|||A|||) \\
& \leq \Big\| \sum a_{ijk} g_i^{(1)} g_j^{(2)} g_k^{(3)} \Big\|_p \\
& \leq \ C(\sqrt{p}\|A\|_{HS} + p(\|A\|_{\{1\}\{2,3\}} + \|A\|_{\{2\}\{1,3\}} + \|A\|_{\{3\}\{1,2\}}) + p\sqrt{p}|||A|||).
\end{aligned}$$

In the above inequalities A denotes the matrix (a_{ijk}) and

$$\|A\|_{HS} = \|A\|_{\{1,2,3\}} := \sup\Big\{ \sum_{ijk} a_{ijk} x_{ijk} : \sum_{ijk} x_{ijk}^2 \leq 1 \Big\} = \sqrt{\sum_{ijk} a_{ijk}^2},$$

$$\|A\|_{\{1\}\{2,3\}} := \sup\Big\{ \sum_{ijk} a_{ijk} x_i y_{jk} : \sum_i x_i^2 \leq 1, \sum_{jk} y_{jk}^2 \leq 1 \Big\}.$$

Similarly define $\|A\|_{\{2\}\{1,3\}}, \|A\|_{\{3\}\{1,2\}}$ and

$$|||A||| = \|A\|_{\{1\},\{2\},\{3\}} := \sup\Big\{ \sum a_{ijk} x_i y_j z_k : \sum_i x_i^2 \leq 1, \sum_j y_j^2 \leq 1, \sum_k z_k^2 \leq 1 \Big\}.$$

Proof of Theorem 2. The estimate from above immediately follows by the iteration argument from the following

Lemma 2. *With the same assumptions as in Theorem 2*

$$\begin{aligned}
\|S\|_p \leq {} & C(d) \Big\| \sum a_{i_1 \ldots i_d} r_{i_1}^{(1)} \cdot X_{i_2}^{(2)} \cdots X_{i_d}^{(d)} \Big\|_p \\
& + C(d) \max(1, \ln p)^d \sup_{x^{(1)} \in B_p^{(1)}} \Big\| \sum a_{i_1 \ldots i_n} x_{i_1}^{(1)} \cdot X_{i_2}^{(2)} \cdots X_{i_d}^{(d)} \Big\|_p.
\end{aligned} \tag{22}$$

We postpone the proof of Lemma 2 to subsection 3.1 and now we will prove (21), i.e. the estimate from below. From contractive properties of Rademacher variables and from

(4) we get for any $I \subset \{1, 2, \dots, d\}$

$$\begin{aligned}\|S\|_p &= \Big(E_{X^{(j)}, j\in I} E_{X^{(k)}, k\in I'} \Big|\sum a_{i_1\dots i_d} \prod_{j\in I} X^{(j)}_{i_j} \prod_{k\in I'} X^{(k)}_{i_k}\Big|^p\Big)^{1/p} \\
&= \Big(E_{X^{(j)}, j\in I} E_{X^{(k)}, r^{(k)}, k\in I'} \Big|\sum a_{i_1\dots i_d} \prod_{j\in I} X^{(j)}_{i_j} \prod_{k\in I'} (|X^{(k)}_{i_k}| r^{(k)}_{i_k})\Big|^p\Big)^{1/p} \\
&\ge \Big(E_{X^{(j)}, j\in I} E_{r^{(k)}, k\in I'} \Big|\sum a_{i_1\dots i_d} \prod_{j\in I} X^{(j)}_{i_j} \prod_{k\in I'} (E_{X^{(k)}} |X^{(k)}_{i_k}| r^{(k)}_{i_k})\Big|^p\Big)^{1/p} \\
&\ge c(d) \Big(E_{r^{(k)}, k\in I'} E_{X^{(j)}, j\in I} \Big|\sum a_{i_1\dots i_d} \prod_{j\in I} X^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big|^p\Big)^{1/p} \\
&\ge c(d) \Big(E_{r^{(k)}, k\in I'} \sup_{x^{(j)} \in \tilde{B}^{(j)}_p, j\in I} \Big|\sum a_{i_1\dots i_d} \prod_{j\in I} x^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big|^p\Big)^{1/p} \\
&\ge c(d) \sup_{x^{(j)} \in \tilde{B}^{(j)}_p, j\in I} \Big\|\sum a_{i_1\dots i_d} \prod_{j\in I} x^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big\|_p.\end{aligned}$$

Since $B^{(k)}_p \subset \tilde{B}^{(k)}_p$, we get

$$\begin{aligned}\|S\|_p &\ge c(d) \sup_{x^{(j)} \in \tilde{B}^{(j)}_p, j\in I} \Big\|\sum a_{i_1\dots i_d} \prod_{j\in I} x^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big\|_p \\
&\ge c(d) \sup_{x^{(j)} \in B^{(j)}_p, j\in I} \Big\|\sum a_{i_1\dots i_d} \prod_{j\in I} x^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big\|_p.\end{aligned}$$

Now

$$\begin{aligned}\|S\|_p &\ge \frac{1}{2^d} \sum_{I\subset\{1,2,\dots,d\}} c(d) \sup_{x^{(j)} \in B^{(j)}_p, j\in I} \Big\|\sum a_{i_1\dots i_d} \prod_{j\in I} x^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big\|_p \\
&\ge c(d) \sum_{I\subset\{1,2,\dots,d\}} \sup_{x^{(j)} \in B^{(j)}_p, j\in I} \Big\|\sum a_{i_1\dots i_d} \prod_{j\in I} x^{(j)}_{i_j} \prod_{k\in I'} r^{(k)}_{i_k}\Big\|_p\end{aligned}$$

and (21) follows. ■

3.1. *Proof of Lemma 2.* We will proceed by induction. For $d = 1$ and any $(a_i) \in R^n$ we have

$$\frac{1}{C(1)}\Big\|\sum a_i X^{(1)}_i\Big\|_p \le \Big\|\sum a_i r^{(1)}_i\Big\|_p + \sup\Big\{\sum a_i x^{(1)}_i : (x^{(1)}_i) \in B^{(1)}_p\Big\}. \tag{23}$$

This follows (cf. [La1]) from the moment estimates for linear combinations of $X^{(1)}_i$ and $r^{(1)}_i$, and it is an even better estimate than (20). Let us assume that $d \ge 2$ and (20) is already proven for chaoses of order $1, 2, \dots, d-1$.

First we will prove that the sum $S = \sum a_{i_1\dots i_d} X^{(1)}_{i_1} \cdots X^{(d)}_{i_d}$ may be split into two sums $S_1 + S_2$ such that S_1 contains summands corresponding to multiindices $(i_1, i_2, \dots, i_d)$ with $1 \le i_1 \le p^{C(d)}$ and $\|S_2\|_p$ is comparable with $C(d)\|\sum a_{i_1\dots i_d} r^{(1)}_{i_1} \cdot X^{(2)}_{i_2} \cdots X^{(d)}_{i_d}\|_p$.

First notice that multiindices $(i_1, \dots, i_d)$ may be rearranged in such a way that

$$\text{if } j_1 \le j_2 \text{ then } \sum_{i_2,\dots,i_d} a^2_{i_{j_1} i_2\dots i_d} \ge \sum_{i_2,\dots,i_d} a^2_{i_{j_2} i_2\dots i_d}. \tag{24}$$

From (23) we get

$$\frac{1}{C(1)}\Big\|\sum a_i X_i^{(1)}\Big\|_p \le \Big\|\sum a_i r_i^{(1)}\Big\|_p + \sup\Big\{\sum a_i x_i^{(1)} : (x_i^{(1)}) \in B_p^{(1)}\Big\} \le \Big\|\sum a_i r_i^{(1)}\Big\|_p + p\max|a_i|. \tag{25}$$

Now, from (25) we get

$$\begin{aligned}\Big\|\sum_{i_1>p^{C(d)},i_2,\dots,i_d} a_{i_1\dots i_d} X_{i_1}^{(1)}\cdots X_{i_d}^{(d)}\Big\|_p &\le C(1)\Big\|\sum a_{i_1\dots i_d} r_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p \\ &\quad + C(1)p\Big\|\max_{i_1>p^{C(d)}}\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_d} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p.\end{aligned} \tag{26}$$

Using (24), we will estimate

$$\Big\|\max_{i_1>p^{C(d)}}\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_d} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p.$$

We have

$$\begin{aligned}&\Big\|\max_{i_1>p^{C(d)}}\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_n} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p \\ &\qquad\le \sum_{k=0}^{\infty}\Big\|\max_{2^k p^{C(d)}<i_1\le 2^{k+1}p^{C(d)}}\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_d} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p.\end{aligned}$$

Now, using a Bonami type inequality (i.e. Corollary 2 from the previous section)

$$\Big\|\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_d} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p \le C(d)p^d\sqrt{\sum_{i_2,\dots,i_d} a^2_{i_1 i_2\dots i_d}}.$$

Hence, by Chebyshev's inequality, for $t\ge 1$

$$P\Big(\Big|\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_d} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big| \ge C(d)t\sqrt{\sum a^2_{i_1 i_2\dots i_d}}\Big) \le e^{-t^{1/d}}.$$

From the inequality above, integrating by parts, for $t_0\ge 1$ we obtain

$$\begin{aligned}&\Big\|\Big(\Big|\sum_{i_2,\dots,i_d} a_{i_1 i_2\dots i_d} X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big| - C(d)t_0\sqrt{\sum_{i_2,\dots,i_d} a^2_{i_1 i_2\dots i_d}}\Big)_+\Big\|_p \\ &\qquad\le C(d)\big(\sup_{t_0\le t\le\infty} te^{-t^{1/d}/(2p)}\big)\sqrt{\sum_{i_2,\dots,i_d} a^2_{i_1 i_2\dots i_d}}.\end{aligned} \tag{27}$$

Let $t_0(k)$ satisfy

$$\sup_{t_0(k)\le t\le\infty} te^{-t^{1/d}/(2p)} \le \frac{1}{2^k p^{C(d)}},$$

for example

$$t_0(k) = C(d)\left[p\ln p + pk\right]^d. \tag{28}$$

Denote by I_k the set of integers belonging to the interval $(2^k p^{C(d)}; 2^{k+1}p^{C(d)}]$.

From (27), taking $t_0(k)$ defined as in (28), we get

$$\Big\|\max_{i_1\in I_k}\sum_{i_2,\ldots,i_d} a_{i_1i_2\ldots i_d}X^{(2)}_{i_2}\cdots X^{(d)}_{i_d}\Big\|_p \le C(d)t_0(k)\max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}$$
$$+\sum_{i_1=2^kp^{C(d)}+1}^{2^{k+1}p^{C(d)}}\Big\|\Big(\sum_{i_2,\ldots,i_d} a_{i_1i_2\ldots i_d}X^{(2)}_{i_2}\cdots X^{(d)}_{i_d} - C(d)t_0\max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}\Big)_+\Big\|_p$$
$$\le C(d)\left[p\ln p + pk\right]^d\max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}$$
$$+ C(d)\sum_{i_1=2^kp^{C(d)}+1}^{2^{k+1}p^{C(d)}}\big(\sup_{t_0\le t\le\infty} te^{-t^{1/d}/(2p)}\big)\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}$$
$$\le C(d)\left[2p\ln p + pk\right]^d\max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}} + \max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}$$
$$\le C(d)\left[2p\ln p + pk\right]^d\max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}.$$

Now, using the estimate

$$\max_{i_1\in I_k}\sqrt{\sum_{i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}} \le \sqrt{\frac{1}{2^kp^{C(d)}}\sum_{i_1,i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}},$$

which follows from (24), we get

$$C(d)p\Big\|\max_{i_1>p^{C(d)}}\sum_{i_2,\ldots,i_d} a_{i_1i_2\ldots i_n}X^{(2)}_{i_2}\cdots X^{(d)}_{i_d}\Big\|_p$$
$$\le C(d)p\sum_{k=0}^{\infty}\Big\|\max_{2^kp^{C(d)}<i_1\le 2^{k+1}p^{C(d)}}\sum_{i_2,\ldots,i_d} a_{i_1i_2\ldots i_d}X^{(2)}_{i_2}\cdots X^{(d)}_{i_d}\Big\|_p$$
$$\le\sum_{k=0}^{\infty} C(d)p\left[2p\ln p + pk\right]^d\sqrt{\frac{1}{2^kp^{C(d)}}\sum_{i_1,i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}.$$

Notice that for $C(d)$ large enough, the last sum may be bounded by

$$C(d)\sqrt{\sum_{i_1,i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}}$$

and therefore

(29)
$$C(d)p\Big\|\max_{i_1>p^{C(d)}}\sum_{i_2,\ldots,i_d} a_{i_1i_2\ldots i_n}X^{(2)}_{i_2}\cdots X^{(d)}_{i_d}\Big\|_p$$
$$\le C(d)\sqrt{\sum_{i_1,i_2,\ldots,i_d} a^2_{i_1i_2\ldots i_d}} \le C(d)\Big\|\sum a_{i_1\ldots i_d}r^{(1)}_{i_1}\cdot X^{(2)}_{i_2}\cdots X^{(d)}_{i_d}\Big\|_2.$$

By (26) and (29) we have

$$\Big\| \sum_{i_1 > p^{C(d)}, i_2, \ldots, i_d} a_{i_1 \ldots i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)} \Big\|_p \le C(d) \Big\| \sum a_{i_1 \ldots i_d} r_{i_1}^{(1)} \cdot X_{i_2}^{(2)} \cdots X_{i_d}^{(d)} \Big\|_p.$$

So let us consider

$$S_1 = \sum_{i_1 \le p^{C(d)}, i_2, \ldots, i_d} a_{i_1 i_2 \ldots i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)}.$$

By (23) we have

$$\frac{1}{C(1)} \|S_1\|_p \le \Big\| \sum_{i_1 \le p^{C(d)}, i_2, \ldots, i_d} a_{i_1 \ldots i_d} r_{i_1}^{(1)} \cdot X_{i_2}^{(2)} \cdots X_{i_d}^{(d)} \Big\|_p + \Big\| \sup_{x^{(1)} \in B_p^{(1)}} \sum_{i_1 \le p^{C(d)}, i_2, \ldots, i_d} a_{i_1 \ldots i_d} x_{i_1}^{(1)} \cdot X_{i_2}^{(2)} \cdots X_{i_d}^{(d)} \Big\|_p. \tag{30}$$

Recall the definition of $B_p^{(1)}$,

$$B_p^{(1)} = \Big\{ x^{(1)} \in R^n : \sum N_i^{(1)}(|x_i^{(1)}|) \le p \text{ and } (x_i^{(1)} = 0 \text{ or } |x_i^{(1)}| \ge 1 \text{ for } i = 1, \ldots, n) \Big\}.$$

Since $i_1 \le p^{C(d)}$ we may assume that $B_p^{(1)} \subset R^{p^{C(d)}}$. Let us define

$$\hat{B}_p^{(1)} = \{ (\hat{x}_i) \in R^{p^{C(d)}} : \exists (x_i) \in B_p^{(1)}, \hat{x}_i = \operatorname{sign}(x_i) \cdot \lfloor |x_i| \rfloor \}.$$

We will prove that $\hat{B}_p^{(1)}$ has cardinality no greater than $p^{C(d)p}$ and that for any $(a_i) \in R^{p^{C(d)}}$

$$\sup_{x \in B_p^{(1)}} \sum a_i x_i \ge \max_{x \in \hat{B}_p^{(1)}} \sum a_i x_i \ge \frac{1}{3} \sup_{x \in B_p^{(1)}} \sum a_i x_i. \tag{31}$$

Indeed, by the definition of $B_p^{(1)}$, for any $(x_i) \in B_p^{(1)}$ we have

$$\#\{ i : x_i \ne 0 \} \le p, \#\{ i : |x_i| > p \} = 0$$

and since all vectors from $\hat{B}_p^{(1)}$ have integer coordinates, then

$$\#\hat{B}_p^{(1)} \le \binom{p^{C(d)}}{p} (2p+1)^p \le p^{C(d)p}.$$

In order to prove (31) notice that $\hat{B}_p^{(1)} \subset B_p^{(1)}$, so the first inequality is obvious. To show the second, take $(x_i^0) \in B_p^{(1)}$ such that

$$\sum a_i x_i^0 \ge \frac{2}{3} \sup_{x \in B_p^{(1)}} \sum a_i x_i.$$

We may additionally assume that $a_i x_i^0 \ge 0$ for all i's. Take $\hat{x}_i^0 = \operatorname{sign}(x_i^0) \cdot \lfloor |x_i^0| \rfloor$. Since for all i's $x_i^0 = 0$ or $|x_{\ss}^0| \ge 1$, so

$$a_{\ss} x_i^0 \ge a_i \hat{x}_i^0 \ge \frac{1}{2} a_i x_i^0 \text{ for all } i\text{'s},$$

hence

$$\sup_{x\in \hat{B}_p^{(1)}} \sum a_i x_i \ge \frac{1}{2}\sum a_i x_i^0 \ge \frac{1}{3}\sup_{x\in B_p^{(1)}} \sum a_i x_i.$$

Using properties of $\hat{B}_p^{(1)}$, for any $q \ge p$ we have

$$\begin{aligned}
(32)\qquad & \Big\| \sup_{x^{(1)}\in B_p^{(1)}} \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p \\
&\le 3\Big\| \max_{x^{(1)}\in \hat{B}_p^{(1)}} \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p \\
&\le 3\Big(\sum_{x^{(1)}\in \hat{B}_p^{(1)}} E\Big| \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big|^q\Big)^{1/q} \\
&\le 3(\#\hat{B}_p^{(1)})^{1/q} \max_{x^{(1)}\in \hat{B}_p^{(1)}} \Big\| \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_q.
\end{aligned}$$

Taking $q = p\max(1, \ln p)$ and using Corollary 2 we get

$$\begin{aligned}
&(\#\hat{B}_p^{(1)})^{1/q} \max_{x^{(1)}\in \hat{B}_p^{(1)}} \Big\| \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_q \\
&\le (\#\hat{B}_p^{(1)})^{1/(p\max(1,\ln p))} \Big(\frac{p\max(1,\ln p)}{p}\Big)^{d-1} \times \\
&\times \max_{x^{(1)}\in \hat{B}_p^{(1)}} \Big\| \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p \\
&\le C(d)\max(1,\ln p)^{d-1} \sup_{x^{(1)}\in B_p^{(1)}} \Big\| \sum_{i_1\le p^{C(d)}, i_2,\ldots,i_d} a_{i_1\ldots i_d} x_{i_1}^{(1)}\cdot X_{i_2}^{(2)}\cdots X_{i_d}^{(d)}\Big\|_p.
\end{aligned}$$

And now, from the above inequality, from (32) and (30) the inequality (22) follows.

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