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MIXED NORM CONDITION NUMBERS FOR THE UNIVARIATE BERNSTEIN BASIS

TOM LYCHE

Institutt for Informatikk and CMA, University of Oslo
P.O. Box 1053 Blindern, 0316 Oslo, Norway
E-mail: tom@ifi.uio.no

KARL SCHERER

Institut für Angewandte Mathematik, Universität Bonn
Wegelstr. 6, 53115 Bonn, Germany
E-mail: unm11cr@ibm.rhrz.uni-bonn.de

Abstract. We study mixed norm condition numbers for the univariate Bernstein basis for polynomials of degree n, that is, we measure the stability of the coefficients of the basis in the l_q-sequence norm whereas the polynomials to be represented are measured in the L_p-function norm. The resulting condition numbers differ from earlier results obtained for $p = q$.

1. Introduction. In order to approximate a function from a finite dimensional vector space one should choose a basis for the space which is well conditioned in the sense that small relative errors in the coefficients of the approximation do not lead to large relative changes in its function values. In this respect a basis of orthogonal polynomials is a good choice when the approximating space consists of univariate polynomials. In computer aided geometric design the Bernstein basis

$$B_{j,n}(x) := \binom{n}{j}\left(\frac{1+x}{2}\right)^j\left(\frac{1-x}{2}\right)^{n-j}, \quad j = 0,\dots,n, \tag{1}$$

is a popular choice. Although it is not as well conditioned as orthogonal polynomials it is better conditioned than the power basis and it has the additional properties of nonnegativity on the interval in question (which will be $[-1, 1]$ in this paper) and form a

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partition of unity so that

$$B_j^n(x) \ge 0,\ x \in [-1,1],\quad \sum_{j=0}^n B_j^n(x) = 1,\ x \in \mathbb{R},\ \text{and}\ \int_{-1}^1 B_j^n(x)dx = \frac{2}{n+1}. \tag{2}$$

The p-norm condition numbers of linearly independent functions $\phi^T = (\phi_0, \dots, \phi_n)$ can be defined by

$$\kappa_p(\phi) = S_1(\phi) * S_2(\phi)$$

$$\text{where}\quad S_1(\phi) = \sup_{c\neq 0} \frac{\|\phi^T c\|_p}{\|c\|_p},\quad S_2(\phi) = \sup_{c\neq 0} \frac{\|c\|_p}{\|\phi^T c\|_p},\quad 1 \le p \le \infty.$$

Here

$$\|c\|_p = \begin{cases} (\sum_{j=0}^n |c_j|^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{0\le j\le n} |c_j| & \text{if } p = \infty, \end{cases}$$

is the usual l_p sequence norm of the vector $c = (c_0, \dots, c_n)^T$ and

$$\|f\|_p = \begin{cases} (\int_\Omega |f(x)|^p dx)^{1/p} & \text{if } 1 \le p < \infty, \\ \sup_{x\in\Omega} |f(x)| & \text{if } p = \infty, \end{cases}$$

denotes the standard L_p function norm of f on the domain of definition Ω which can be a domain in $\mathbb{R}^s$ for some positive integer s. The condition number measures in the p-norm setting how much the relative size of the function can differ from the relative size of its coefficients. We are interested both in asymptotic estimates in n and s and in characterizing the extremals, i.e. the functions for which the supremums in $S_1(\phi)$ and $S_2(\phi)$ are obtained. In addition to the examples already mentioned much work has been carried out on the condition numbers for the univariate B-spline basis [1, 3, 4, 10] and the Bernstein basis [3, 5, 6, 7, 9] for polynomials of degree $\le n$ defined on a simplex in $\mathbb{R}^s$.

It is striking that the number 2^n comes up both for B-splines and for the Bernstein basis in one and several variables.

- For B-splines of degree n de Boor showed in 1973 that the condition number is independent of the knots, see [1]. For $p = \infty$ the lower bound $2^{n-\frac{1}{2}}\left(1 - \frac{1}{n}\right)$ for the condition number was shown in 1978 in [4]. In fact the lower bound was obtained by representing the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$ in terms of the Bernstein basis on $[-1, 1]$. The upper bound $(n+1)2^{n+1}$ was given in [10] and this is valid for all values of p.
- For the univariate Bernstein basis a 2^n upper bound for all p's was given by Ciesielski at a conference in 1983 with the proceedings published in 1987, [3]. The exact asymptotic estimate for the Bernstein basis was shown in [7] and [9] to be of order $n^{-1/(2p)}2^n$. This was achieved by interpolating the $p = \infty$ estimate in [4] with $p = 1$ estimates derived in [7][1].
- For s-variate triangular Bernstein basis the condition number for $p = 2$ was determined exactly and for general p an upper bound of order $n^{s+1/4}2^n$ was shown in [6]. In [5] an upper bound was given for $p = \infty$ which does not depend on the space dimension s.

[1]This result is based on the conjecture that the extremal polynomial for the supremum S_2 has n distinct roots in $(-1, 1)$. However the proof of the corresponding Theorem 4.2 in [7] contains a gap as was pointed out to us by J. Domsta. A note with a correct proof of this theorem will appear in [8].

In this paper we will consider the univariate Bernstein basis and mixed norms. The mixed (p,q)-norm condition numbers of linearly independent functions $\phi^T = (\phi_1, \dots, \phi_n)$ can be defined by

$$\kappa_{p,q}(\phi) = S_1(\phi) * S_2(\phi)$$

$$\text{where } S_1(\phi) = \sup_{c \neq 0} \frac{\|\phi^T c\|_p}{\|c\|_q}, \quad S_2(\phi) = \sup_{c \neq 0} \frac{\|c\|_q}{\|\phi^T c\|_p}, \quad 1 \leq p, q \leq \infty.$$

We denote the mixed (p,q)-norm condition numbers of the Bernstein basis of degree n by $\kappa_{n,p,q}$ and the supremums by S_1 and S_2. Using the interval $[-1,1]$ to define the Bernstein basis polynomials is no restriction since the condition number is independent of the interval we use to define the Bernstein basis polynomials (as long as we use the whole interval).

Figure 1 shows what is previously known for this condition number. We display $1/p$ along the horizontal axis and $1/q$ along the vertical axis. We know the asymptotic values along the diagonal $1/p = 1/q$ for $0 \leq 1/q \leq 1$. On the diagonal the extremal polynomial for S_1 is simply the constant polynomial $f = 1$ for any p, while for S_2 the extremal polynomials are known for $p = \infty$ (the Chebyshev polynomial of degree n) and $p = 2$ (the Legendre polynomial of degree n). In this paper we do accurate asymptotic estimates for both S_1, S_2 along the vertical and horizontal axis in Figure 1. With these values and the values along the diagonal one can obtain values for the whole square by interpolation.

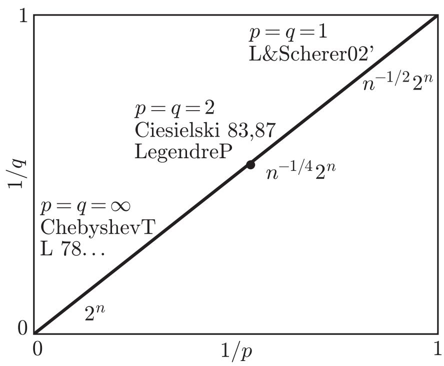

Fig. 1. Asymptotic behavior of the condition number for the univariate Bernstein basis

The content of this paper is as follows. In Section 2 we determine the first supremum S_1 in the whole square. The value is exact below the diagonal and almost exact in the upper part of the square. In Section 3 we compute S_2 along the vertical line $1/p = 0$ in Figure 1 and show that the Chebyshev polynomial of degree n is extremal on that line. In Section 4 we attack S_2 on the horizontal line $q = \infty$. For this we need to estimate the size of dual linear functionals. A precise estimate is quite difficult to obtain and for the estimates we derive a refinement of the recurrence relation for the dual polynomials proved by Ciesielski in 1983, see [3].

Throughout the paper we will make repeated use of Wallis inequality and a version of Stirling's inequality:

$$\frac{2^{2n}}{\sqrt{\pi(n+\frac{1}{2})}} \le \binom{2n}{n} \le \frac{2^{2n}}{\sqrt{\pi n}}, \quad \text{(Wallis' inequality)}, \tag{3}$$

$$\left(\frac{z}{e}\right)^z \sqrt{2\pi z} \le \Gamma(z+1) \le \left(\frac{z}{e}\right)^z \sqrt{2\pi z}\left(1+\frac{1}{4z}\right), \quad z \ge 1, \quad \text{(Stirling's inequality)}, \tag{4}$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the gamma function.

2. The first supremum. For the first supremum we can give reasonably sharp lower and upper bounds and exact values on the bottom horizontal line, the left vertical line and the diagonal line in Figure 1.

LEMMA 1. *For the first supremum we have*

$$\sup_{c\neq 0} \frac{\|\sum_{j=0}^n c_j B_{j,n}\|_p}{\|c\|_q} = \frac{2^{1/p}}{(n+1)^{1/q}}, \quad 1 \le p \le q \le \infty, \tag{5}$$

with equality for $c = c^*$ and $c_j^* = 1$ all j, while if $1 \le q \le p \le \infty$ then

$$(1/e)^{1/e}\frac{2^{1/p}}{(n+1)^{1/p}} \le (1/p)^{1/p}\frac{2^{1/p}}{(n+1)^{1/p}} \le \sup_{c\neq 0}\frac{\|\sum_{j=0}^n c_j B_{j,n}\|_p}{\|c\|_q} \le \frac{2^{1/p}}{(n+1)^{1/p}}. \tag{6}$$

In particular, if $p = \infty$ then

$$\sup_{c\neq 0}\frac{\|\sum_{j=0}^n c_j B_{j,n}\|_\infty}{\|c\|_q} = 1, \quad 1 \le q \le \infty, \tag{7}$$

with equality for $c = c^*$ with $c_j^* = \delta_{j,n}$ or $c_j^* = \delta_{j,0}$.

Proof. From Hölder's inequality and (2) we have the well known inequality ([7])

$$\Big\|\sum_{j=0}^n c_j B_{j,n}\Big\|_p = \Big\|\sum_{j=0}^n c_j B_{j,n}^{1/p} B_{j,n}^{1-1/p}\Big\|_p \le \frac{2^{1/p}}{(n+1)^{1/p}}\|c\|_p, \quad 1 \le p \le \infty, \tag{8}$$

with equality if $c_j = 1$ for all j. For $1 \le p \le q \le \infty$ the classical inequality $\|c\|_p \le (n+1)^{1/p-1/q}\|c\|_q$ holds, again with equality for $c_j = 1$ for all j. But then (5) follows. Consider next the case $1 \le q \le p \le \infty$. The upper bound follows by combining (8) with the classical inequality $\|c\|_p \le \|c\|_q$. For the lower bound we choose one of the polynomials $f = B_{0,n}$ or $f = B_{n,n}$. For such f we have $\|c\|_q = 1$ and compute

$$\|B_{n,n}\|_p = \left(\int_{-1}^1 \left(\frac{1+x}{2}\right)^{np} dx\right)^{1/p} = \left(\frac{1}{p}\right)^{1/p}\frac{2^{1/p}}{(n+1/p)^{1/p}} \ge \left(\frac{1}{p}\right)^{1/p}\frac{2^{1/p}}{(n+1)^{1/p}}.$$

For $p = \infty$ the upper and lower bound in (6) are equal and we have equality in (7) for $c = c^* = (\delta_{j,0})_{j=0}^n$ or $c^* = (\delta_{j,n})_{j=0}^n$. ■

3. The second supremum when $p = \infty$ and $1 \le q \le \infty$. In this case the value of the second supremum can be determined exactly.

Lemma 2. *For $1 \le q \le \infty$ we have*

$$\sup_{c\neq 0} \frac{\|c\|_q}{\|\sum_{j=0}^n c_j B_{j,n}\|_\infty} = \Big(\sum_{j=0}^n \gamma_{j,n}^q\Big)^{1/q},$$

where $\gamma_{0,n} = 1$ and

$$\gamma_{i,n} := \binom{2n-1}{2i-1} \Big/ \binom{n-1}{i-1}, \quad i = 1, \dots, n.$$

Proof. It is well known (see i.e. [4]) that the Bernstein-Bézier form of the Chebyshev polynomial of degree n is given by

$$T_n(x) = \sum_{j=0}^n (-1)^{n-j} \gamma_{j,n} B_{j,n}(x).$$

Using this we have shown in [5] that for any sequence $c = \{c_j\}$,

$$|c_i| \ge \gamma_{i,n} \Big\| \sum_{j=0}^n c_j B_{j,n} \Big\|_\infty, \quad i = 0, 1, \cdots, n.$$

Taking the q-th power and summing gives

$$\|c\|_q \ge \Big(\sum_{j=0}^n \gamma_{j,n}^q\Big)^{1/q} \Big\| \sum_{j=0}^n c_j B_{j,n} \Big\|_\infty.$$

Equality follows with the choice $\sum_{j=0}^n c_j B_{j,n} = f^*$ where $f^*(x) = T_n(x)$. ∎

4. The second supremum for $q = \infty$ and $1 \le p \le \infty$. For $q = \infty$ it follows from (5) that the value of the first supremum is equal to 2. Thus

$$\kappa_{n,p,\infty} = 2^{1/p} S_2.$$

We will use a dual functional representation to estimate the second supremum S_2.

From a corollary on page 127 in [2] we immediately obtain that the BB-coefficients of a polynomial $f = \sum_{j=0}^n d_j B_{j,n}$ can be written in the form

$$d_j = \langle g_j^{(n+1)}, f \rangle := \int_{-1}^1 g_j^{(n+1)}(t) f(t) dt, \quad j = 0, \dots, n, \tag{9}$$

where $g_j \in \Pi_{2n+1}$ is the unique solution of the interpolation problem

$$D^r g_j(-1) = 0, \quad D^r g_j(1) = D^r N_{n-j}(1), \quad r = 0, 1, \dots, n \tag{10}$$

with

$$N_j(x) = (x+1)^j (x-1)^{n-j}/n!, \quad j = 0, 1, \dots, n. \tag{11}$$

We claim that the solution of this problem is given by

$$g_j(x) = N_{n-j}(x) \int_{-1}^x N_j(t) dt \; / \beta_j, \tag{12}$$

where

$$N_j(x) = (x+1)^j (x-1)^{n-j}/n!, \quad \beta_j := \int_{-1}^1 N_j(t) dt, \quad j = 0, 1, \dots, n.$$

Indeed, we have by r-fold differentiation of (12) for any $j, r = 0, 1, \dots, n$

$$\beta_j g_j^{(r)}(x) = N_{n-j}^{(r)}(x) \int_{-1}^{x} N_j(t)dt + \sum_{l=0}^{r-1} \binom{r}{l} N_{n-j}^{(l)}(x) N_j^{(r-l-1)}(x). \tag{13}$$

Since $N_j^{(k)}(-1) = 0$ for $k < j$ it can be shown that $g_j^{(r)}(-1) = 0$ for $j, r = 0, 1, \dots, n$. Similarly for $x = 1$ we have $N_j^{(k)}(1) = 0$ for $k < n - j$ and the sum in (13) vanishes also for $x = 1$ and $j, r = 0, 1, \dots, n$. Thus

$$\beta_j g_j^{(r)}(1) = N_{n-j}^{(r)}(1) \int_{-1}^{1} N_j(t)dt = N_{n-j}^{(r)}(1)\beta_j, \quad j, r = 0, 1, \dots, n$$

and (10) holds. By the preceding considerations we get now for $1 \le p \le \infty$

Lemma 3. *We have*

$$\kappa_{n,p,\infty} = 2^{1/p} \max_{0 \le j \le n} \sup_{f \in \Pi_n} \frac{|\langle g_j^{(n+1)}, f \rangle|}{\|f\|_p}, \tag{14}$$

Proof. By definition of f and (9)

$$2^{-1/p}\kappa_{n,p,\infty} = \sup_{(d_j) \ne 0} \frac{\max_{0 \le j \le n} |d_j|}{\|\sum_{j=0}^{n} d_j B_{j,n}\|_p} = \sup_{f \in \Pi_n} \max_{0 \le j \le n} \frac{|\langle g_j^{(n+1)}, f \rangle|}{\|f\|_p}. \blacksquare$$

In [7] we estimated the condition number by splitting a polynomial into a sum of a Legendre polynomial and a remainder term of lower degree. The asymptotic behavior was then determined by the Legendre term. We could try the same idea here and split the kernel of the functional into a sum of two polynomials, a Legendre polynomial P_n of degree n and a remainder polynomial S_n. The contribution to $\kappa_{n,1}$ from P_n can then be computed exactly. In this respect note that by (13)

$$\begin{aligned} \beta_j g_j^{(n+1)}(x) &= \frac{d^n}{dx^n} \left[\frac{(x+1)^n (x-1)^n}{(n!)^2} + N'_{n-j}(x) \int_{-1}^{x} N_j(t)dt \right] \\ &= \frac{2^n}{n!} P_n(x) + \frac{d^n}{dx^n} \left[N'_{n-j}(x) \int_{-1}^{x} N_j(t)dt \right]. \end{aligned}$$

However, the remainder with respect to the Legendre polynomial here is not of lower degree which makes it difficult to estimate it.

Therefore we use a different approach from [3]. A dual polynomial basis $B_{j,n}^*$ for the Bernstein polynomials $B_{i,n}$ i.e., polynomials $B_{j,n}^*$ of degree n satisfying

$$\int_{-1}^{1} B_{j,n}^*(t) B_{i,n}(t)\, dt = \delta_{i,j}, \quad 0 \le j, i \le n$$

is considered. Comparison of this identity with (9) shows that

$$g_j^{(n+1)}(x) = B_{j,n}^*(x). \tag{15}$$

In addition to Lemma 3 one can prove

Lemma 4. *For $1/p + 1/p' = 1$,*

$$2^{-1/p}\kappa_{n,p,\infty} = \max_{0\le j\le n}\left\{\operatorname{dist}(1;\operatorname{span}\{B_{i,n}\}_{i\ne j})_p\right\}^{-1}$$
$$= \max_{0\le j\le n}\sup_{f\in\Pi_n}\frac{|\langle B^*_{j,n}, f\rangle|}{\|f\|_p} \le \max_{0\le j\le n}\|B^*_{j,n}\|_{p'}.$$

Proof. The inequality follows immediatly from Hölder's inequality. By definition

$$\operatorname{dist}(1;\operatorname{span}\{B_{i,n}\}_{i\ne j})_p = \inf_{\substack{(a_i)\\ a_j=1}}\Big\|\sum_{i=0}^{n} a_i B_{i,n}(x)\Big\|_p = \inf_{\substack{f\in\Pi_n\\ \langle B^*_{j,n}, f\rangle = 1}}\|f\|_p$$
$$= \left\{\sup_{f\in\Pi_n}\frac{|\langle B^*_{j,n}, f\rangle|}{\|f\|_p}\right\}^{-1}.$$

Using (15) and maximizing over j the equality parts of the lemma follows from Lemma 3. ∎

In [3] bounds for $\|B^*_{j,n}\|_p$ were established for the cases $p = 1, 2$ and ∞ using a recurrence relation for the dual polynomials. For convenience we prove this recurrence relation here.

Lemma 5. *Suppose $\int_{-1}^{1} B^*_{i,n}(t)B^*_{j,n}(t)dt = \delta_{ij}$ for $0 \le i, j \le n$. Then $B^*_{0,0} = 1$ and for $n \ge 1$*

$$B^*_{j,n} = \frac{j}{n}B^*_{j-1,n-1} + \frac{n-j}{n}B^*_{j,n-1} + \left(n+\frac{1}{2}\right)(-1)^{n-j}\binom{n}{j}P_n, \quad j = 0, 1, \ldots, n, \tag{16}$$

where P_n is the Legendre polynomial of degree n normalized so that $P_n(1) = 1$.

Proof. Suppose $f \in \Pi_{n-1}$. By duality and degree raising

$$f = \sum_{j=0}^{n-1}\langle f, B^*_{j,n-1}\rangle B_{j,n-1} = \sum_{j=0}^{n-1}\langle f, B^*_{j,n-1}\rangle\left(\frac{j+1}{n}B_{j+1,n} + \frac{n-j}{n}B_{j,n}\right)$$
$$= \sum_{j=0}^{n}\langle f, \frac{j}{n}B^*_{j-1,n-1} + \frac{n-j}{n}B^*_{j,n-1}\rangle B_{j,n} = \sum_{j=0}^{n}\langle f, B^*_{j,n}\rangle B_{j,n}$$

and since the Bernstein basis is a basis the polynomial $g := B^*_{j,n} - \frac{j}{n}B^*_{j-1,n-1} - \frac{n-j}{n}B^*_{j,n-1}$ must be orthogonal to Π_{n-1}. Thus $g = cP_n$ for some constant c which can be determined from the Bernstein basis representation of the Legendre polynomial

$$P_n = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{j,n}.$$

Again by duality

$$(-1)^{n-j}\binom{n}{j} = \langle B^*_{j,n}, P_n\rangle = \langle g, P_n\rangle = \langle cP_n, P_n\rangle = \frac{c}{n+\frac{1}{2}}.$$

This determines c and the proof is complete. ∎

From this one obtains for the quantity $F_n := \max_{0\le j\le n}\|B^*_{j,n}\|_{p'}$ the recursive inequality

$$F_n \le F_{n-1} + \left(n+\frac{1}{2}\right)\binom{n}{[n/2]}\|P_n\|_{p'}. \tag{17}$$

For the $L^p[-1,1]$-norm of the Legendre polynomial we have

$$\|P_n\|_p \le \begin{cases} 2^{1/p}(2n+1)^{-1/2} & \text{if } 1\le p\le 2,\\ (n+\frac{1}{2})^{-1/p} & \text{if } 2\le p\le\infty. \end{cases}$$

Indeed, for $1\le p\le 2$ we have

$$\int_{-1}^{1}|P_n(x)|^p dx \le \left(\int_{-1}^{1} 1dx\right)^{1-p/2}\left(\int_{-1}^{1}|P_n(x)|^2 dx\right)^{p/2} = 2^{1-p/2}\left(n+\frac{1}{2}\right)^{-p/2}$$

and for $p\ge 2$ we note that $\int_{-1}^{1}|P_n(x)|^p dx \le \int_{-1}^{1}|P_n(x)|^2 dx = (n+\frac{1}{2})^{-1}$ since $|P_n(x)|\le 1$ for $x\in[-1,1]$. Combining this with the Wallis inequality (3) the crucial term in (17) has the asymptotic behavior

$$\left(n+\frac{1}{2}\right)\binom{n}{[n/2]}\|P_n\|_{p'} \le \begin{cases} 2^n(n+\frac{1}{2})^{1/p}/\sqrt{\pi n} & \text{if } 1\le p\le 2,\\ 2^{n+\frac{1}{2}-\frac{1}{p}}(n+\frac{1}{2})^{1/2}/\sqrt{\pi n} & \text{if } 2\le p\le\infty. \end{cases} \tag{18}$$

By applying this recursively in (17) the resulting bound for F_n and hence by Lemma 4 the upper one for $\kappa_{n,p,\infty}$ would be of the same order. The problem is as we will see that this bound is not asymptotically sharp in n.

5. Refined estimates for $q=\infty$.

We define for $0\le j\le n$, $n\ge 0$, and $1\le p\le\infty$ the quantities

$$\begin{aligned}
S^*_{j,n} &:= \frac{n-j}{n}B^*_{j,n-1} + \frac{j}{n}B^*_{j-1,n-1}, \quad n\ge 1,\\
A_{j,n,p} &:= \sup_{f\in\Pi_n}\frac{|\langle B^*_{j,n},f\rangle|}{\|f\|_p}, \quad A_{n,p} = \sup_{0\le j\le n} A_{j,n,p},\\
R_{j,n,p} &:= \sup_{f\in\Pi_n}\frac{|\langle S^*_{j,n},f\rangle|}{\|f\|_p}, \quad R_{n,p} = \sup_{0\le j\le n} R_{j,n,p}, \quad n\ge 1,\\
C_{n,p} &:= \sup_{f\in\Pi_n}\frac{|\langle P_n,f\rangle|}{\|f\|_p},\\
D_{j,n,p} &:= (n+1/2)\binom{n}{j}C_{n,p}, \quad D_{n,p} = \sup_{0\le j\le n} D_{j,n,p},\\
E_{j,n-1,p} &:= (n-1/2)\binom{n}{j}\frac{n-2j}{n}C_{n-1,p}, \quad E_{n-1,p} = \sup_{0\le j\le n-1} E_{j,n-1,p}, \quad n\ge 1.
\end{aligned} \tag{19}$$

Taking inner products with $f\in\Pi_n$ in the recurrence (16) we obtain

$$A_{j,n,p} = R_{j,n,p} + (-1)^{n-j}D_{j,n,p}, \tag{20}$$

or by Lemma 4

$$D_{n,p} - R_{n,p} \le 2^{-1/p}\kappa_{n,p,\infty} = A_{n,p} \le D_{n,p} + R_{n,p}. \tag{21}$$

Clearly $R_{j,n,p} \le \sup_{f\in\Pi_n} |\langle B^*_{j,n-1}, f\rangle|/\|f\|_p$, but the estimate we obtain from (17), (18) and (21) is not good enough for a sharp asymptotic estimate. We will show that

$$\lim_{n\to\infty} \frac{\kappa_{n,p,\infty}}{D_{n,p}} = 2^{1/p}. \tag{22}$$

For this we derive and use a recurrence relation for the quantities $S^*_{j,n}$ in (19).

LEMMA 6. *For $j = 0,1,\dots,n$, $n \ge 2$, and $1 \le p \le \infty$ the following recurrence relations hold:*

$$S^*_{j,n} = \frac{j}{n} S^*_{j-1,n-1} + \frac{n-j}{n} S^*_{j,n-1} + (-1)^{n-j+1}\left(n-\frac{1}{2}\right)\binom{n}{j}\frac{n-2j}{n}P_{n-1}, \tag{23}$$

$$R_{j,n,p} = \frac{j}{n} R_{j-1,n-1,p} + \frac{n-j}{n} R_{j,n-1,p} + (-1)^{n-j+1} E_{j,n-1,p}. \tag{24}$$

Moreover,

$$S^*_{0,1} = S^*_{1,1} = 1, \quad R_{0,1,p} = R_{1,1,p} = 2^{1-1/p}. \tag{25}$$

Proof. Substituting in (16) we obtain

$$\begin{aligned} S^*_{j,n} = {}& \frac{j}{n}\left[S^*_{j-1,n-1} + (-1)^{n-j}\left(n-\frac{1}{2}\right)\binom{n-1}{j-1}P_{n-1}\right] \\ &+ \frac{n-j}{n}\left[S^*_{j,n-1} + (-1)^{n+1-j}\left(n-\frac{1}{2}\right)\binom{n-1}{j}P_{n-1}\right]. \end{aligned}$$

For simplification note that

$$\begin{aligned} \frac{n-j}{n}\binom{n-1}{j} - \frac{j}{n}\binom{n-1}{j-1} &= \binom{n-1}{j} - \frac{j}{n}\left[\binom{n-1}{j} + \binom{n-1}{j-1}\right] \\ &= \binom{n-1}{j} - \binom{n-1}{j-1} = \binom{n}{j}\frac{n-2j}{n} \end{aligned}$$

so that (23) follows.

Since $S^*_{j,n} \in \Pi_{n-1}$ we have $\langle S^*_{j,n}, P_n\rangle = 0$ and we can restrict the supremum to Π_{n-1}:

$$\sup_{f\in\Pi_n} \frac{|\langle S^*_{j,n-1}, f\rangle|}{\|f\|_p} = \sup_{f\in\Pi_{n-1}} \frac{|\langle S^*_{j,n-1}, f\rangle|}{\|f\|_p}, \quad \sup_{f\in\Pi_n} \frac{|\langle P_{n-1}, f\rangle|}{\|f\|_p} = \sup_{f\in\Pi_{n-1}} \frac{|\langle P_{n-1}, f\rangle|}{\|f\|_p}.$$

But then (24) follows. That $S^*_{0,1} = 1$ and $S^*_{1,1} = 1$ follows immediately from the definition in (19). Now $R_{0,1,p} = R_{1,1,p} = \sup_{f\in\Pi_1}\|f\|_1/\|f\|_p \le 2^{1-1/p}$ with equality for $f = 1$. ∎

LEMMA 7. *For $n \ge 1$ and $1 \le p \le \infty$ we have*

$$D_{n,p} - R_{1,p} - \sum_{k=1}^{n-1} E_{k,p} \le 2^{-1/p}\kappa_{n,p,\infty} \le D_{n,p} + R_{1,p} + \sum_{k=1}^{n-1} E_{k,p}. \tag{26}$$

Proof. By (21) it is enough to show that $R_{n,p} \le R_{n-1,p} + E_{n-1,p}$. But this follows immediately from (24). ∎

In order to show (22) we need to show that $D_{n,p}$ dominates the sum $R_{1,p} + \sum_{k=1}^{n-1} E_{k,p}$. We start by estimating $C_{n,p}$ and $D_{n,p}$.

LEMMA 8. *For $n \geq 1$ and $1 \leq p \leq \infty$*

$$k_n^{-1} 2^{-1/p} \sqrt{\pi}\left(n+\frac{1}{2}\right)^{-1/2} \leq C_{n,p} \leq 2^{-1/p}\sqrt{\pi}\left(n+\frac{1}{2}\right)^{-1/2}, \tag{27}$$

and

$$k_n^{-1} 2^{n-1-1/p} \leq D_{n,p} \leq k_n 2^{n-1/p}, \tag{28}$$

where $k_n = (1+\frac{1}{2n})^{1/2}$.

Proof. By the same argument as in [7] we get

$$C_{n,p} = \frac{|\langle x^n, P_n\rangle|}{\inf_{g\in\Pi_{n-1}} \|x^n - g\|_p} = \frac{1}{\|x^n - g_p\|_p} \frac{2^{n+1}}{(2n+1)\binom{2n}{n}},$$

where g_p is defined by $\|x^n - g_p\| := \inf_{g\in\Pi_{n-1}} \|x^n - g\|_p$. The exact value of $\|x^n - g_p\|_p$ can be determined in cases $p = 1, 2, \infty$. Thus we find

$$C_{n,2} = \|P_n\|_2 = (n+1/2)^{-1/2}, \qquad C_{n,1} = C_{n,\infty} = \frac{4^n}{(2n+1)\binom{2n}{n}}. \tag{29}$$

For arbitrary $p \in [1,\infty]$ one could think of applying the Riesz-Thorin theorem to the linear operator $T(f) := \langle P_n, f\rangle$ in the numerator of $C_{n,p}$ but this operator is only defined for $f \in \Pi_n$ and hence the theorem is not applicable.

Since

$$\begin{aligned}
\|x^n - g_p\|_p &\leq \|x^n - g_\infty\|_p \leq 2^{1/p}\|x^n - g_\infty\|_\infty, \\
\|x^n - g_1\|_1 &\leq \|x^n - g_p\|_1 \leq 2^{1-1/p}\|x^n - g_p\|_p,
\end{aligned}$$

we obtain

$$2^{-1/p} C_{n,\infty} \leq C_{n,p} \leq 2^{1-1/p} C_{n,1}, \ 1 \leq p \leq \infty. \tag{30}$$

Wallis inequality gives

$$C_{n,1} = C_{n,\infty} \leq \frac{4^n\sqrt{\pi}(n+\frac{1}{2})^{1/2}}{2(n+\frac{1}{2})4^n} = \frac{\sqrt{\pi}}{2}\left(n+\frac{1}{2}\right)^{-1/2}$$

and

$$C_{n,1} = C_{n,\infty} \geq \frac{4^n\sqrt{\pi}(n)^{1/2}}{2(n+\frac{1}{2})4^n} = \frac{\sqrt{\pi}}{2k_n}\left(n+\frac{1}{2}\right)^{-1/2}.$$

Combining this with (30) proves (27).

Using Wallis inequality and what we already proved

$$D_{n,1} = D_{n,\infty} \leq (n+\frac{1}{2})\binom{n}{[n/2]}\frac{\sqrt{\pi}}{2}\left(n+\frac{1}{2}\right)^{-1/2} \leq \left(n+\frac{1}{2}\right)^{1/2}\frac{2^n}{\sqrt{\pi n}}\frac{\sqrt{\pi}}{2} = 2^{n-1}k_n,$$

$$D_{n,1} = D_{n,\infty} \geq (n+\frac{1}{2})\frac{2^n}{\sqrt{\pi(n+\frac{1}{2})}}\frac{\sqrt{\pi}}{2k_n}\left(n+\frac{1}{2}\right)^{-1/2} = \frac{2^{n-1}}{k_n}.$$

By (30)

$$\frac{2^{n-1-1/p}}{k_n} \leq 2^{-1/p} D_{n,\infty} \leq D_{n,p} \leq 2^{1-1/p} D_{n,1} \leq k_n 2^{n-1/p}. \ \blacksquare$$

Note that $\|B^*_{j,n}\| = \|B^*_{n-j,n}\|_{p'}$ in view of $B_{j,n}(-x) = B^n_{n-j}(x)$. Thus one needs to consider in the following only $0 \le j \le n/2$.

In order to give a sharp estimate for the terms $E_{k,p}$ in Lemma 7 we consider

$$j \le j^* := n/2 - \ \gamma \sqrt{n}, \qquad \text{for some } 0 < \gamma < \sqrt{n}/2.$$

Using (4) gives

$$\begin{aligned}\binom{n}{j} \le \binom{n}{j^*} &= \frac{\Gamma(n+1)}{\Gamma(j^*+1)\Gamma(n-j^*+1)} \\ &\le \frac{\sqrt{n/2\pi}}{\sqrt{(\frac{n}{2}-\gamma\sqrt{n})(\frac{n}{2}+\gamma\sqrt{n})}} \ \frac{n^n(1+\mathcal{O}(1/n))}{(\frac{n}{2}-\gamma\sqrt{n})^{\frac{n}{2}-\gamma\sqrt{n}}(\frac{n}{2}+\gamma\sqrt{n})^{\frac{n}{2}+\gamma\sqrt{n}}} \\ &= \frac{1}{\sqrt{2\pi(n/4-\gamma^2)}} \ \frac{n^n(1+\mathcal{O}(1/n))}{(\frac{n^2}{4}-\gamma^2 n)^{\frac{n}{2}}} \left(\frac{\frac{n}{2}-\gamma\sqrt{n}}{\frac{n}{2}+\gamma\sqrt{n}}\right)^{\gamma\sqrt{n}} \\ &= \frac{1}{\sqrt{2\pi(n/4-\gamma^2)}} \ \frac{2^n(1+\mathcal{O}(1/n))}{(1-\frac{2\gamma^2}{n/2})^{\frac{n}{2}}} \left(1-\frac{4\gamma}{\sqrt{n}+2\gamma}\right)^{\gamma\sqrt{n}}\end{aligned}$$

Then we obtain the asymptotic formula

$$\binom{n}{j} \le \frac{e^{-2\gamma^2}\ 2^n}{\sqrt{\pi n/2}}(1+\mathcal{O}(1/\gamma\ \sqrt{n})), \qquad j \le j^* \le n/2 - \gamma\ \sqrt{n}.$$

We use this for k instead of n and $\gamma = k^{1/4}$ and observe that then $|\frac{k-2j}{k}| \le 2k^{-1/4}$. Considering the two cases $j \le j_k := k/2 - \gamma\ \sqrt{k}$ and $j_k \le j \le k/2$ we obtain

$$\left(k-\frac{1}{2}\right)\binom{k}{j}\left|\frac{k-2j}{k}\right| \le \sqrt{\frac{2k}{\pi}}\ 2^k \ \begin{cases} e^{-2\sqrt{k}}(1+\mathcal{O}(k^{-1/4})), & \text{if } \ j \le j_k, \\ 2\ k^{-1/4}(1+\mathcal{O}(k^{-1})), & \text{if } \ j \ge j_k. \end{cases}$$

and consequently by (19)

$$E_{k,p} \le 2^{1/2-1/p}\ 2^k \ \begin{cases} e^{-2\sqrt{k}}(1+\mathcal{O}(k^{-1/4})), & \text{if } \ j \le j_k, \\ 2\ k^{-1/4}(1+\mathcal{O}(k^{-1})), & \text{if } \ j \ge j_k. \end{cases}$$

For k or n sufficiently large that the second case dominates. From this we conclude that for $1 \le p \le \infty$

$$E_{k,p} \le 2\ k^{-1/4}\ D_{k,p}(1+\mathcal{O}(k^{-1/4})), \qquad k \text{ large enough.}$$

Inserting this estimate for $E_{k,p}$ in (26) we see that the term $D_{n,p}$ dominates the rest. Indeed sums of this form can be bounded by

$$\sum_{k=1}^{n} \frac{2^k}{k^\alpha} \le \sum_{k=1}^{[(n+1)/2]} 2^k \ + \ \frac{1}{[(n+1)/2]^\alpha} \sum_{k=[(n+1)/2]}^{n} 2^k \le 2^{1+(n+1)/2} + \frac{2^{n+1}}{(n/2)^\alpha}\ .$$

Thus we can conclude in view of Lemma 8 from (26)

Theorem 9. *For $1 \le p \le \infty$ and n large enough,*

$$\kappa_{n,p,\infty} = 2^{1/p} D_{n,p}(1+\mathcal{O}(n^{-1})).$$

In particular the value of $\kappa_{n,p,\infty}$ is asymptotically up to factor $1/2$ equal to 2^n.

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