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APPROXIMATION FROM SPARSE GRIDS AND FUNCTION SPACES OF DOMINATING MIXED SMOOTHNESS

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Abstract. We investigate the convergence and the rate of convergence in $\| \cdot |L_p\|$, $1 < p < \infty$, of a bivariate interpolating (with respect to a sparse grid) trigonometric polynomial in the framework of Sobolev spaces of dominating mixed smoothness.

1. Introduction. The present article is a continuation of the investigations of the approximation properties of trigonometric interpolation with respect to uniform grids, see [5, 4, 19, 21, 14]; we now study the bivariate situation with respect to a sparse grid. More precisely, we investigate the rate of convergence of the Smolyak algorithm (applied to trigonometric interpolation on uniform grids) for functions belonging to a Sobolev space of dominating mixed smoothness. This continues earlier work of Smolyak [17], Temlyakov [19], Wasilkowski, Woźniakowski [23] and the author [15]. At the end of this article we add a comment on consequences of our estimates for the problem of optimal recovery.

To prove our main assertion we make use of the Fourier series of the interpolatory trigonometric polynomial, a special decomposition of the function in the Fourier image (related to the function spaces) and a Fourier multiplier theorem due to Lizorkin.

2. Interpolation on sparse grids. As usual, $\mathbb{N}$ stands for the natural numbers, by $\mathbb{N}_0$ we denote the natural numbers including 0 and by $\mathbb{Z}^d$ the d-tuples of integers. Let $\mathbb{T} = [0, 2\pi)$. Further, let

$$\mathcal{D}_m(t) := \sum_{|k| \le m} e^{ikt}, \quad t \in \mathbb{T},\ m \in \mathbb{N}_0,$$

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be the Dirichlet kernel and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\, \mathcal{D}_m(t - t_\ell)\,, \qquad t_\ell = \frac{2\pi\ell}{2m+1}\,.$$

Then I_m is the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. As usual, let

$$c_k(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(t)\, e^{-ikt}\, dt\,, \qquad k \in \mathbb{Z}^d\,,$$

be the Fourier coefficient of $f \in L_1(\mathbb{T}^d)$. The Fourier series $S[I_m f]$ of $I_m f$ is then given by

$$S[I_m f](t) = \sum_{k=-m}^{m} \Big(\sum_{\ell=-\infty}^{\infty} c_{k+\ell(2m+1)}(f) \Big)\, e^{ikt}\,.$$

Let

$$Q_{m,\ell} := \{ n \in \mathbb{Z} :\ \ell\,(2m+1) - m \le n \le \ell\,(2m+1) + m \}\,, \qquad m \in \mathbb{N}, \quad \ell \in \mathbb{Z}\,.$$

Hence,

$$Q_{m,\ell} \cap Q_{m,\ell'} = \emptyset \qquad \text{if} \quad \ell \neq \ell' \qquad \text{and} \qquad \bigcup_{\ell=-\infty}^{\infty} Q_{m,\ell} = \mathbb{Z}\,.$$

The Fourier series of $I_m f$ can be rewritten as

$$S[I_m f](t) = \sum_{\ell=-\infty}^{\infty} e^{-i\ell(2m+1)t} \sum_{k \in Q_{m,\ell}} c_k(f)\, e^{ikt}\,, \tag{1}$$

at least if f belongs to the Wiener algebra.

We do not need the complete sequence of interpolatory polynomials of a given function. We concentrate on a dyadic subsequence. To have a convenient notation we put $L_j := I_{2^j}$, $j = 0, 1, \ldots$. By $L_{j,k} := L_j \otimes L_k$ we denote the tensor product of L_j and L_k. The sampling operators B_m we are going to study are defined as

$$B_m := \sum_{j=0}^{m} L_{j,m-j} - \sum_{j=0}^{m-1} L_{j,m-j-1}\,, \quad m = 1, 2, \ldots\,.$$

This is Smolyak's construction (sometimes called Smolyak algorithm or blending operators) with respect to the L_j, cf. e.g. [3, 16, 17, 21, 23]. We collect a few elementary properties of B_m. Let

$$\mathcal{T}_m := \left\{ \left(\frac{2\pi\ell_1}{2^{j+1}+1}, \frac{2\pi\ell_2}{2^{m-j+1}+1} \right) :\ 0 \le \ell_1 \le 2^{j+1},\ 0 \le \ell_2 \le 2^{m-j+1}, j = 0, \ldots, m \right\}.$$

Then we have the following.

Lemma 1.

(i) B_m uses samples of f from the sparse grid $\mathcal{T}_m \cup \mathcal{T}_{m-1}$.

(ii) $c_k(B_m f) = 0$ if

$$k \notin H_m := \{ (\ell_1, \ell_2) :\ \exists r \in (\mathbb{N}_0 \cap [0, m]) \ \ s.t.\ |\ell_1| \le 2^r \ \ \text{and} \ \ |\ell_2| \le 2^{m-r} \}\,.$$

(iii) Suppose that f is a trigonometric polynomial with harmonics from H_m. Then $B_m f = f$.

Proof. The proof of these statements is elementary, but see also [20]. ∎

3. Function spaces of dominating mixed smoothness

3.1. *Sobolev spaces.* If r is a natural number and $1 < p < \infty$, then the Sobolev space $S_p^r W(\mathbb{T}^2)$ of dominating mixed smoothness of order r is defined as the collection of all $f \in L_p(\mathbb{T}^2)$ such that

$$\frac{\partial^{2r} f}{\partial x_1^r\, \partial x_2^r}\,,\ \frac{\partial^r f}{\partial x_1^r}\,,\ \frac{\partial^r f}{\partial x_2^r} \in L_p(\mathbb{T}^2)\,.$$

For general $r > 0$ one may use

$$\sum_{k\in\mathbb{Z}^2} c_k(f)(1+|k_1|^2)^{r/2}\,(1+|k_2|^2)^{r/2}\,e^{ikx} \in L_p(\mathbb{T}^2)\,.$$

We endow these classes with the norm

$$\|\, f\,|S_p^r W(\mathbb{T}^2)\| := \Big\| \sum_{k\in\mathbb{Z}^2} c_k(f)(1+|k_1|^2)^{r/2}\,(1+|k_2|^2)^{r/2}\,e^{ikx}\, \Big| L_p(\mathbb{T}^2)\Big\|\,.$$

3.2. *Lizorkin-Triebel and Besov spaces.* For us it is convenient to introduce Triebel-Lizorkin and Besov spaces by making use of a Littlewood-Paley decomposition, cf. [9, 13]. Let

$$\begin{aligned} P_0 &= (-1,1)\,, \qquad P_j = \{x\,:\, 2^{j-1} \le |x| < 2^j\,\}\,, \qquad j\in\mathbb{N}\,,\\ P_{j,k} &= P_j \times P_k\,, \qquad j,\,k\in\mathbb{N}_0\,. \end{aligned}$$

As an abbreviation we shall use

$$f_{j,k}(x) = \sum_{\ell\in P_{j,k}} c_\ell(f)\,e^{i\ell x}\,, \qquad x\in\mathbb{T}^2, \quad j,k\in\mathbb{N}_0\,,$$

which results in

$$f = \sum_{j=0}^\infty\sum_{k=0}^\infty f_{j,k}\,,$$

at least in the sense of periodic distributions.

Let $1<p<\infty$, $1<q<\infty$, and $r>0$. Then the Lizorkin-Triebel space $S^r_{p,q}F(\mathbb{T}^2)$ of dominating mixed smoothness is the collection of all functions $f\in L_p(\mathbb{T}^2)$ such that

$$\|\, f\,|S^r_{p,q}T(\mathbb{T}^2)\| := \Big\|\Big(\sum_{j=0}^\infty\sum_{k=0}^\infty 2^{r(j+k)q}\,|f_{j,k}|^q\Big)^{1/q}\,\Big|L_p(\mathbb{T}^2)\Big\| < \infty\,. \tag{2}$$

These classes generalize the Sobolev scale. More precisely,

$$S^r_{p,2}F(\mathbb{T}^2) = S^r_p W(\mathbb{T}^2) \quad \text{(equivalent norms)}\,, \tag{3}$$

cf. e.g. [13, 2.3.1] for the non-periodic case.

Let $1<p<\infty$, $1\le q\le\infty$, and $r>0$. Then the Besov space $S^r_{p,q}B(\mathbb{T}^2)$ of dominating mixed smoothness is the collection of all functions $f\in L_p(\mathbb{T}^2)$ such that

$$\|\, f\,|S^r_{p,q}B(\mathbb{T}^2)\| := \Big(\sum_{j=0}^\infty\sum_{k=0}^\infty 2^{r(j+k)q}\,\|f_{j,k}\,|L_p(\mathbb{T}^2)\|^q\Big)^{1/q} < \infty\,. \tag{4}$$

Obviously, from the definitions it follows $S^r_{p,p}B(\mathbb{T}^2) = S^r_{p,p}F(\mathbb{T}^2)$. For $r > 1/p$ and all q it is known that

$$(S^r_pW(\mathbb{T}^2) \cup S^r_{p,q}F(\mathbb{T}^2) \cup S^r_{p,q}B(\mathbb{T}^2)) \hookrightarrow C(\mathbb{T}^2)$$

holds, cf. [13, 2.4.1]. So, for $r > 1/p$ interpolation of functions f belonging to one of these classes makes sense.

Important for us will also be the following interpolation formula. Here $[\,\cdot\,,\,\cdot\,]_\Theta$ denotes the complex interpolation functor. Let $0 < \Theta < 1$ and $1 < p_0, p_1, q_0, q_1 < \infty$. Then

$$[S^{r_0}_{p_0,q_0}F(\mathbb{T}^2),\, S^{r_1}_{p_1,q_1}F(\mathbb{T}^2)]_\Theta = S^r_{p,q}F(\mathbb{T}^2) \quad \text{(equivalent norms)}\,, \tag{5}$$

where

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,, \qquad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\,, \qquad \text{and} \qquad r = (1-\Theta)\, r_0 + \Theta\, r_1\,,$$

cf. [12] for the nonperiodic case.

4. The approximation power of B_m

4.1. *The approximation power of B_m for functions belonging to the Triebel–Lizorkin classes of dominating mixed smoothness.* Let I be the identity operator (we do not indicate the space where I is considered, hoping this will be clear from the context). We write $a \sim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \le b \le c\, a\,.$$

Our main result in [15] has been the following.

PROPOSITION 1. *Suppose $1 < p < \infty$, $1 \le q \le \infty$, and $r > 1/p$. Then*

$$\|\, I - B_m \,:\, S^r_{p,q}B(\mathbb{T}^2) \to L_p(\mathbb{T}^2)\| \sim m^{1-1/q}\, 2^{-mr}\,. \tag{6}$$

Now we are going to prove a counterpart for the Lizorkin-Triebel classes.

PROPOSITION 2. *Suppose $1 < p, q < \infty$ and $r > 1$. Then*

$$\|\, I - B_m \,:\, S^r_{p,q}F(\mathbb{T}^2) \to L_p(\mathbb{T}^2)\| \sim m^{1-1/q}\, 2^{-mr}\,. \tag{7}$$

Proof. Step 1. Preparations. Because of the density of the trigonometric polynomials in $S^r_{p,q}F(\mathbb{R}^2)$ (under the restrictions of the proposition) we assume that f is a trigonometric polynomial. We shall employ the same decomposition of the error $f - B_m f$ as in [15], where we investigated the same problem for Besov spaces instead of Lizorkin-Triebel spaces. For given m we shall use the splitting $f = f_1 + f_2 + f_3 + f_4 + f_5$, where

$$f_1 = \sum_{u+v\le m} f_{u,v}\,, \quad f_2 = \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} f_{u,v}\,, \quad f_3 = \sum_{u=0}^{m} \sum_{v=m+1}^{\infty} f_{u,v}\,,$$

$$f_4 = \sum_{u=m+1}^{\infty} \sum_{v=0}^{m} f_{u,v} \quad \text{and} \quad f_5 = \sum_{u=m+1}^{\infty} \sum_{v=m+1}^{\infty} f_{u,v}\,.$$

Moreover, in [15] we proved

$$\|\, f_i - B_m f_i \,|L_p(\mathbb{T}^2)\| \le c\, 2^{-mr}\, \|\, f_i \,|S^r_{p,\infty}B(\mathbb{T}^2)\|\,, \qquad i = 3, 4, 5\,.$$

Since $S^r_{p,q}F(\mathbb{T}^2) \hookrightarrow S^r_{p,\infty}B(\mathbb{T}^2)$ this is enough to guarantee the desired estimate for these parts of the error. Furthermore, Lemma 1 implies $f_1 = B_m f_1$. So it remains to consider $\| f_2 - B_m f_2 \,|L_p(\mathbb{T}^2)\|$.

Step 2. Estimate of $\|f_2 - B_m f_2 \,|L_p(\mathbb{T}^2)\|$. Using the projection property of L_j we derive

$$((I - L_j) \otimes (L_{m-j} - L_{m-j-1}))\, f_{u,v} = 0 \tag{8}$$

if either $j \ge u$ or if $m - j - 1 \ge v$. Furthermore, we recall the identity

$$\begin{aligned} I \otimes I - B_m \,&=\, (I - L_m) \otimes L_0 + I \otimes (I - L_m) \\ &\quad + \sum_{j=0}^{m-1} (I - L_j) \otimes (L_{m-j} - L_{m-j-1})\,, \end{aligned} \tag{9}$$

valid for each $m \in \mathbb{N}$, cf. [3, Prop. 1.4/2] or [23]. Altogether this implies $f_2 - B_m f_2 = T_1 + T_2$, where

$$T_1 \,=\, \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} ((I - L_j) \otimes (L_{m-j} - L_{m-j-1}))\, f_{u,v}\,, \tag{10}$$

$$T_2 \,=\, \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} ((I - L_m) \otimes L_0)\, f_{u,v} + (I \otimes (I - L_m)) f_{u,v}\,. \tag{11}$$

Substep 2.1. Estimate of T_1. We rewrite T_1 by making use of the Fourier series of the terms on the right-hand side. To avoid double indices we put:

$$I^j_\ell = Q_{2^j,\ell} \qquad \text{and} \qquad I^{j,k}_{\ell_1,\ell_2} = I^j_{\ell_1} \times I^k_{\ell_2}\,,$$

$j \in \mathbb{N}_0$, $\ell, \ell_1, \ell_2 \in \mathbb{Z}$. In view of (1) we find the identities

$$\begin{aligned} &((I - L_j) \otimes (L_{m-j} - L_{m-j-1}))\, f_{u,v} \\ &\quad = \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i(2^{m-j+1}+1)\ell_2 x_2} \sum_{k \in I^{j,m-j}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\ &\quad - \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i(2^{m-j}+1)\ell_2 x_2} \sum_{k \in I^{j,m-j-1}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\ &\quad - \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i((2^{j+1}+1)\ell_1 x_1 + (2^{m-j+1}+1)\ell_2 x_2)} \sum_{k \in I^{j,m-j}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\ &\quad + \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i((2^{j+1}+1)\ell_1 x_1 + (2^{m-j}+1)\ell_2 x_2)} \sum_{k \in I^{j,m-j-1}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)}\,. \end{aligned}$$

Observe that on the right-hand side the terms with $\ell_1 = \ell_2 = 0$ sum up to zero. So we shall use this identity with $|\ell_1| + |\ell_2| > 0$. Furthermore, comparing $I^{j,m-j}_{\ell_1,\ell_2}$ and $P_{u,v}$ and $I^{j,m-j-1}_{\ell_1,\ell_2}$ and $P_{u,v}$, respectively, we see that all sums (with respect to ℓ_1, ℓ_2) are finite. Let

$$
\begin{aligned}
h_{u,v,j,\ell_1,\ell_2} := \; & e^{-i(2^{m-j+1}+1)\ell_2 x_2} \sum_{k\in I^{j,m-j}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
& - e^{-i(2^{m-j}+1)\ell_2 x_2} \sum_{k\in I^{j,m-j-1}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
& - e^{-i((2^{j+1}+1)\ell_1 x_1+(2^{m-j+1}+1)\ell_2 x_2)} \sum_{k\in I^{j,m-j}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
& + e^{-i((2^{j+1}+1)\ell_1 x_1+(2^{m-j}+1)\ell_2 x_2)} \sum_{k\in I^{j,m-j-1}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \, .
\end{aligned}
$$

For the absolute value of these functions one has the obvious estimate

$$
|h_{u,v,j,\ell_1,\ell_2}| \le \underbrace{2\Big|\sum_{k\in I^{j,m-j}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)}\Big| + 2\Big|\sum_{k\in I^{j,m-j-1}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)}\Big|}_{:=\widetilde{h}_{u,v,j,\ell_1,\ell_2}} \, .
$$

Defining

$$
\begin{aligned}
g_{1,u,v,j} &= \sum_{|\ell_1|>0} \sum_{|\ell_2|>0} h_{u,v,j,\ell_1,\ell_2} \, , \\
g_{2,u,v,j} &= \sum_{|\ell_1|>0} h_{u,v,j,\ell_1,0} \, , \\
g_{3,u,v,j} &= \sum_{|\ell_2|>0} h_{u,v,j,0,\ell_2} \, .
\end{aligned}
$$

we see that the identity (10) can be rewritten now in the form

$$
T_1 = \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{i=1}^{3} g_{i,u,v,j} \, .
$$

Substep 2.1.1. Estimate of $\sum_{u,v,j} g_{1,u,v,j}$. We compare the coverings induced by $I^{j,m-j}_{\ell_1,\ell_2}$ and $P_{u,v}$, respectively. Suppose $|\ell_1|, |\ell_2| \ge 1$. Elementary calculations yield that

$$
I^{j,m-j}_{\ell_1,\ell_2} \cap P_{u,v} \neq \emptyset
$$

implies

$$
\max(1, 2^{u-j-4}) \le |\ell_1| < 2^{u-j} \qquad \text{and} \qquad \max(1, 2^{v-m+j-4}) \le |\ell_2| < 2^{v-m+j} \, .
$$

We put $J_k := [\max(1, 2^{k-4}), 2^k)$, $k \in \mathbb{N}$. Our decomposition of the approximation error will be applied together with a vector-valued Fourier multiplier theorem of Lizorkin, cf. [7], which has been transferred to the periodic case in [11], see also [13, Th. 3.4.3/3]. It says that a sequence of rectangles with sides parallel to the axes is a Fourier multiplier for the space $L_p(\ell_q)$ $(1 < p, q < \infty)$. Here the norm of the corresponding operator neither depends on the centres of these rectangles nor on their side-length. Hence, using Hölder's inequality, $r > 1$, and the quoted Fourier multiplier assertion we obtain

$$\begin{aligned}
&\Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} g_{1,u,v,j} \, \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le \Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} \sum_{|\ell_2| \in J_{v-m+j} \cup J_{v-m+j+1}} |\widetilde{h}_{u,v,j,\ell_1,\ell_2}| \, \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le c_1 \Big\| \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} \sum_{|\ell_2| \in J_{v-m+j} \cup J_{v-m+j+1}} \\
&\qquad (u+v-m)^{-1} \, 2^{-(u+v-m)} \, 2^{(u+v)rq} \, |\widetilde{h}_{u,v,j,\ell_1,\ell_2}|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^2) \Big\| \\
&\quad \times \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} \sum_{|\ell_2| \in J_{v-m+j} \cup J_{v-m+j+1}} (u+v-m)^{q'/q} \\
&\qquad \times 2^{(u+v-m)q'/q} \, 2^{-(u+v)rq'} \Big)^{1/q'} \\
&\le c_2 \, m^{1/q'} \, 2^{-mr} \Big\| \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} \sum_{|\ell_2| \in J_{v-m+j} \cup J_{v-m+j+1}} \\
&\qquad (u+v-m)^{-1} \, 2^{-(u+v-m)} \, 2^{(u+v)rq} \, |f_{u,v}|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le c_3 \, m^{1/q'} \, 2^{-mr} \Big\| \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} 2^{(u+v)rq} \, |f_{u,v}|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le c_3 \, m^{1/q'} \, 2^{-mr} \, \| \, f \, | S^r_{p,q} F(\mathbb{T}^2) \| \, . \qquad (12)
\end{aligned}$$

Here c_3 does not depend on m and f.

Substep 2.1.2. Estimate of $\sum_{u,v,j} g_{i,u,v,j}$, $i = 2, 3$. Analogously to the previous step we conclude

$$\begin{aligned}
&\Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} g_{2,u,v,j} \, \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le \Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} |\widetilde{h}_{u,v,j,\ell_1,0}| \, \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le c_1 \Big\| \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} 2^{-(u+v)+m} \, 2^{(u+v)rq} \, |f_{u,v}|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^2) \Big\| \\
&\qquad \times \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{j=m-v}^{u-1} \sum_{|\ell_1| \in J_{u-j}} 2^{(u+v-m)q'/q} \, 2^{-(u+v)rq'} \Big)^{1/q'} \\
&\le c_2 \, m^{1/q'} \, 2^{-mr} \Big\| \Big(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} 2^{(u+v)rq} \, |f_{u,v}|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^2) \Big\| \\
&\le c_2 \, m^{1/q'} \, 2^{-mr} \, \| \, f \, | S^r_{p,q} F(\mathbb{T}^2) \| \, , \qquad (13)
\end{aligned}$$

where c_2 does not depend on m and f. The estimate of $\sum_{u,v,j} g_{3,u,v,j}$ can be done similarly. Now, putting (12) and (13) together we obtain the desired estimate of T_1 from above.

Substep 2.2. Estimate of T_2. Similarly as in Substep 2.1 we conclude

$$\begin{aligned}
&((I - L_m) \otimes L_0)\, f_{u,v} + (I \otimes (I - L_m))\, f_{u,v} \\
&\quad = \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i(2^1+1)\ell_2 x_2} \sum_{k \in I^{m,0}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
&\quad - \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i(2^{m+1}+1)\ell_1 x_1 + (2^1+1)\ell_2 x_2} \sum_{k \in I^{m,0}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
&\quad - \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} e^{-i(2^{m+1}+1)\ell_2 x_2)} \sum_{k \in I^{0,m}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
&\quad + \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} \sum_{k \in I^{0,m}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)}\,.
\end{aligned}$$

As before, the terms on the right-hand side with $\ell_1 = \ell_2 = 0$ sum up to zero. So we shall use this identity with $|\ell_1| + |\ell_2| > 0$. Furthermore, let

$$\begin{aligned}
h_{u,v,\ell_1,\ell_2} := {} & e^{-i(2^1+1)\ell_2 x_2} \sum_{k \in I^{m,0}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
& - e^{-i(2^{m+1}+1)\ell_1 x_1 + (2^1+1)\ell_2 x_2} \sum_{k \in I^{m,0}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
& - e^{-i(2^{m+1}+1)\ell_2 x_2} \sum_{k \in I^{0,m}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)} \\
& + \sum_{k \in I^{0,m}_{\ell_1,\ell_2}} c_k(f_{u,v})\, e^{i(k_1x_1+k_2x_2)}\,,
\end{aligned}$$

and

$$\begin{aligned}
g_{1,u,v} &= \sum_{|\ell_1|>0} \sum_{|\ell_2|>0} h_{u,v,\ell_1,\ell_2}\,, \\
g_{2,u,v} &= \sum_{|\ell_1|>0} h_{u,v,\ell_1,0}\,, \qquad g_{3,u,v} = \sum_{|\ell_2|>0} h_{u,v,0,\ell_2}\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
\Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} g_{1,u,v} \Big| L_p(\mathbb{T}^2)\Big\| \le c \Big(& \Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{|\ell_1| \in J_{u-m}} \sum_{|\ell_2| \in J_v} |h_{u,v,\ell_1,\ell_2}| \Big| L_p(\mathbb{T}^2)\Big\| \\
& + \Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} \sum_{|\ell_1| \in J_u} \sum_{|\ell_2| \in J_{v-m}} |h_{u,v,\ell_1,\ell_2}| \Big| L_p(\mathbb{T}^2)\Big\| \Big)
\end{aligned}$$

and now we can continue as in Substep 2.1.1. Also the estimates of

$$\Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} g_{2,u,v} \Big| L_p(\mathbb{T}^2) \Big\| \qquad \text{and} \qquad \Big\| \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} g_{3,u,v} \Big| L_p(\mathbb{T}^2) \Big\|$$

can be done in this way. This proves

$$\| T_2 \,| L_p(\mathbb{T}^2) \| \le c_3 \, m^{1/q'} \, 2^{-mr} \, \| f \,| S^r_{p,q} F(\mathbb{T}^2) \| \, . \tag{14}$$

Inequalities (12) and (13) and (14) yield the estimate of $\| I - B_m : S^r_{p,q}F(\mathbb{R}^2) \to L_p(\mathbb{R}^2) \|$ from above.

Step 3. Estimate from below. We employ lacunary series as test functions. Let

$$f_m(x_1, x_2) := \sum_{u=2}^{m-1} e^{i2^u x_1 + i2^{m-u+1} x_2} \, , \qquad m = 3, 4, \ldots \, . \tag{15}$$

Then

$$B_m f_m(x_1, x_2) = -(m-2) \, e^{-i(x_1+x_2)} + \sum_{u=2}^{m-1} e^{i2^u x_1 - ix_2} + \sum_{u=2}^{m-1} e^{-ix_1 + i2^{m-u+1} x_2} \, .$$

Obviously

$$\| f_m \,| S^r_{p,q} F(\mathbb{T}^2) \| \quad \sim \quad m^{1/q} \, 2^{mr} \, . \tag{16}$$

To calculate the L_p-norm of f_m and B_m we shall use the following Littlewood-Paley assertion, cf. [9]. There exist positive constants A_p and B_p such that

$$A_p \, \| f \,| L_p(\mathbb{T}^2) \| \le \Big\| \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^2) \Big\| \le B_p \, \| f \,| L_p(\mathbb{T}^2) \|$$

holds for all $f \in L_p(\mathbb{T}^2)$ $(1 < p < \infty)$. This yields

$$\| f_m \,| L_p(\mathbb{T}^2) \| \sim m^{1/2} \, , \tag{17}$$

$$\| B_m f_m \,| L_p(\mathbb{T}^2) \| \sim m \, , \tag{18}$$

if $1 < p < \infty$. Combining (16) with (17) and (18) the estimate from below follows. The proof is complete. ∎

Remark 1. Lemma 1(ii),(iii) suggests to compare $f - B_m f$ with $f - S^H_m f$, where

$$S^H_m f(x) := \sum_{k \in H_m} c_k(f) \, e^{ikx} \, ,$$

is the partial sum of the Fourier series with respect to the hyperbolic cross H_m. It is known that if $1 < p < \infty$ and $r > 0$, then

$$\| I - S^H_m \,| S^r_{p,q} F(\mathbb{T}^2) \to L_p(\mathbb{T}^2) \| \quad \sim \quad \begin{cases} 2^{-mr} & \text{if} \quad 1 < q \le 2 \, , \\ m^{\frac{1}{2} - \frac{1}{q}} \, 2^{-mr} & \text{if} \quad 2 < q < \infty \, , \end{cases}$$

holds, cf. [12] for a proof in the nonperiodic situation (but the arguments carry over). This implies

$$\frac{\| I - B_m \,|\, S^r_{p,q} F(\mathbb{T}^2) \to L_p(\mathbb{T}^2) \, \|}{\| I - S^H_m \,|\, S^r_{p,q} F(\mathbb{T}^2) \to L_p(\mathbb{T}^2) \, \|} \quad \sim \quad \begin{cases} m^{1-1/q} & \text{if} \quad 1 < q \le 2 \, , \\ m^{1/2} & \text{if} \quad 2 < q < \infty \, , \end{cases}$$

at least if $1 < p < \infty$ and $r > 1$. Hence, one has to pay a price for using the operator B_m (based on the function values of f) instead of the operator S_m^H (based on integrals). This does not have a counterpart in the one-dimensional case.

Remark 2. From the density of the trigonometric polynomials in $S^r_{p,q}F(\mathbb{T}^2)$ it follows that

$$\lim_{m\to\infty} \| f - S_m^H f \,|S^r_{p,q}F(\mathbb{R}^2)\| = 0\,.$$

From this, Proposition 2 and $B_m(S_m^H f) = S_m^H f$, see Lemma 1(iii), we conclude that

$$\lim_{m\to\infty} m^{-1+1/q}\, 2^{mr}\, \| f - B_m f \,|L_p(\mathbb{R}^2)\| = 0$$

for each $f \in S^r_{p,q}F(\mathbb{R}^2)$, $1 < p, q < \infty$ and $r > 1$.

Remark 3. For Besov spaces of dominating mixed smoothness the picture is a bit different. For $1 < p < \infty$ and $r > 0$ we have

$$\| I - S_m^H \,|S^r_{p,q}B(\mathbb{T}^2) \to L_p(\mathbb{T}^2)\| \sim \begin{cases} 2^{-mr} & \text{if} \quad 1 \le q \le \min(p,2)\,, \\ m^{\frac{1}{2}-\frac{1}{q}}\, 2^{-mr} & \text{if} \quad 2 < p < \infty \quad \text{and} \quad q > 2 \\ m^{\frac{1}{p}-\frac{1}{q}}\, 2^{-mr} & \text{if} \quad 1 < p \le 2 \quad \text{and} \quad p \le q \le \infty\,. \end{cases}$$

This has been known for the Nikol'skij-Besov spaces $S^r_{p,\infty}(\mathbb{T}^2)$ for a long time, see the papers of Bugrov [2], Nikol'skaya [8] or [21, Theorem III.3.3]. For $1 \le q \le \infty$ the problem has been treated by Kamont [6] (in the context of spline approximation on the unit cube) and in [12]. In view of this Proposition 1 yields

$$\frac{\| I - B_m \,|\, S^r_{p,q}B(\mathbb{T}^2) \to L_p(\mathbb{T}^2)\, \|}{\| I - S_m^H \,|\, S^r_{p,q}B(\mathbb{T}^2) \to L_p(\mathbb{T}^2)\, \|} \sim \begin{cases} m^{1-1/p} & \text{if} \quad 1 < p \le 2 \quad \text{and} \quad p \le q \le \infty\,, \\ m^{1/2} & \text{if} \quad 2 < p < \infty \quad \text{and} \quad q > 2\,, \\ m^{1-1/q} & \text{if} \quad 1 \le q \le \min(p,2)\,, \end{cases}$$

at least if $1 < p < \infty$ and $r > 1$.

4.2. *The approximation power of B_m for functions belonging to the Sobolev classes of dominating mixed smoothness.* Of course, by means of the equality $S^r_{p,2}F(\mathbb{T}^2) = S^r_pW(\mathbb{T}^2)$ we immediately derive some assertions about B_m and its approximation power for functions taken from Sobolev spaces. However, the restriction $r > 1$ in Proposition 2 is not satisfactory.

Theorem 1. *Suppose* $1 < p < \infty$ *and* $r > \max(1/p, 1/2)$. *Then*

$$\| I - B_m \,:\, S^r_pW(\mathbb{R}^2) \to L_p(\mathbb{R}^2)\| \sim m^{1/2}\, 2^{-mr}\,. \tag{19}$$

Proof. Step 1. Estimate from below. It is enough to observe that the restriction $r > 1$ has not been used in Step 3 of the proof of Proposition 2.

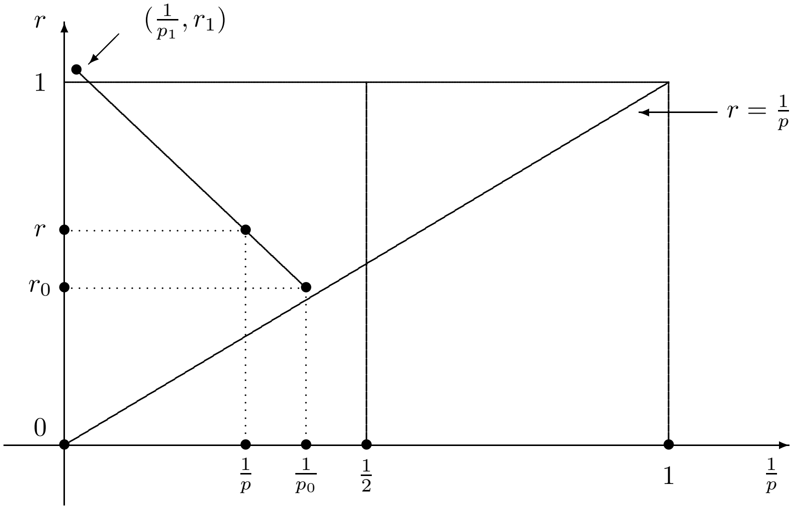
Step 2. Estimate from above. We use Proposition 1, Proposition 2 and complex interpolation.

Step 2.1. As long as $r > 1$ we have nothing to do because of $S^r_{p,2}F(\mathbb{R}^2) = S^r_pW(\mathbb{R}^2)$ (equivalent norms), cf. Proposition 2.

Step 2.2. In case $1 < p \le 2$ and $r > 1/p$ we use the continuous embedding $S^r_{p,2}W(\mathbb{R}^2) \hookrightarrow S^r_{p,2}B(\mathbb{R}^2)$ and Proposition 1.

Step 2.3. Let $2 < p < \infty$ and let $1/p < r \leq 1$. If we proved the estimate from above in (19) for some $r_0 < 1$, then by complex interpolation with fixed p we would get the estimate from above for all $r \geq r_0$. So we concentrate on the smallest r.

For this we proceed as demonstrated in the figure below, that means we use (5) with p_1 close to infinity, q_1 close to 1, r_1 close to 1, and r_0 close to $1/p_0$.



To simplify the considerations we formally work with the limit case. Finally we shall use the argument that we can come arbitrarily close to the following constellation of the parameters:

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} = \frac{1-\Theta}{p_0}$$

and

$$\frac{1}{q} = \frac{1-\Theta}{p_0} + \frac{\Theta}{q_1} = \frac{1-\Theta}{p_0} + \Theta = \frac{1}{p} + \Theta\,.$$

It follows that

$$r = (1-\Theta)\,r_0 + \Theta\,r_1 = \frac{1-\Theta}{p_0} + \Theta = \frac{1}{p} + \Theta = \frac{1}{q}\,.$$

Since we want to have $q = 2$ we arrive at $r = 1/2$ independent of p. The interpolation property of the complex method, Proposition 1 with respect to $S^{r_0}_{p_0,p_0}B(\mathbb{R}^2) = S^{r}_{p_0,p_0}F(\mathbb{R}^2)$, and Proposition 2 with respect to $S^{r_1}_{p_1,q_1}F(\mathbb{R}^2)$ yield the desired conclusion. ■

REMARK 4. As in Remark 1 we conclude

$$\frac{\|\, I - B_m \,|\, S^r_p W(\mathbb{T}^2) \mapsto L_p(\mathbb{T}^2)\,\|}{\|\, I - S^H_m \,|\, S^r_p W(\mathbb{T}^2) \mapsto L_p(\mathbb{T}^2)\,\|} \quad \sim \quad m^{1/2}$$

if $1 < p < \infty$ and $r > 1/\min(2,p)$.

5. Approximate optimal recovery. We study the effectiveness of the approximation by generalized sampling operators. Let F be a class of continuous periodic function defined on $\mathbb{T}^2 = [0, 2\pi)^2$. Then, following [21, Chapter 4, Section 5], we consider for fixed m,

$\xi = (\xi^1, \xi^2, \dots, \xi^m)$, $\xi^j \in \mathbb{T}^2$, $j = 1, \dots, m$, and $\psi_1(x_1, x_2), \dots, \psi_m(x_1, x_2)$ the linear operator

$$\Psi_m(f, \xi)(x_1, x_2) := \sum_{j=1}^{m} f(\xi^j)\, \psi_j(x_1, x_2)$$

and define the quantities

$$\Psi_m(F, \xi, L_p(\mathbb{T}^2)) := \sup_{f \in F} \| \Psi_m(f, \xi) - f \,|L_p(\mathbb{T}^2)\|$$

and

$$\varrho_m(F, L_p(\mathbb{T}^2)) := \inf_{\psi_1, \dots, \psi_m} \inf_{\xi} \Psi_m(F, \xi, L_p(\mathbb{T}^2)) \,.$$

Hence $\varrho_m(F, L_p(\mathbb{T}^2))$ measures the optimal approximate recovery of the functions from F. Here we are interested in the case when F is the unit ball in a Lizorkin-Triebel space $S^r_{p,q}F(\mathbb{T}^2)$ of dominating mixed smoothness. As a consequence of Lemma 1(i) and Proposition 2 we obtain the following.

Theorem 2. *Let $1 < p < \infty$.*

(i) *Let $1 < q < \infty$ and $r > 1$. Let F be the unit ball in $S^r_{p,q}F(\mathbb{T}^2)$. For any natural number m there exists a system of points $\xi^1, \dots, \xi^m \in \mathbb{T}^2$, a collection of trigonometric polynomials $\psi_1(x_1, x_2), \dots, \psi_m(x_1, x_2)$ and a constant C (independent of m) such that*

$$\sup_{f \in F} \| \Psi_m(f, \xi) - f \,|L_p(\mathbb{T}^2)\| \le C\, m^{-r} \, (\log m)^{r+1-1/q} \,. \tag{20}$$

(ii) *Let $r > \max(1/2, 1/p)$. Let F be the unit ball in $S^r_pW(\mathbb{T}^2)$. For any natural number m there exists a system of points $\xi^1, \dots, \xi^m \in \mathbb{T}^2$, a collection of trigonometric polynomials $\psi_1(x_1, x_2), \dots, \psi_m(x_1, x_2)$ and a constant C (independent of m) such that*

$$\sup_{f \in F} \| \Psi_m(f, \xi) - f \,|L_p(\mathbb{T}^2)\| \le C\, m^{-r} \, (\log m)^{r+1/2} \,. \tag{21}$$

Remark 5. Theorem 2(ii) improves an estimate given by Temlyakov in [19], see also [21, 4.5]. However, let us mention that Temlyakov has treated the general d-dimensional case in his papers.

References

[1] G. Baszenski and F. J. Delvos, *A discrete Fourier transform scheme for Boolean sums of trigonometric operators*, in: International Series of Numerical Mathematics, Birkhäuser, Basel, 1989, 15–24.

[2] Ya. S. Bugrov, *Approximation of a class of functions with a dominant mixed derivative*, Mat. Sbornik 64 (106) (1964), 410–418.

[3] F.-J. Delvos and W. Schempp, *Boolean Methods in Interpolation and Approximation*, Longman Scientific & Technical, Harlow, 1989.

[4] V. N. Hristov, *On convergence of some interpolation processes in integral descrete norms*, in: Constructive Theory of Functions ’81, BAN, Sofia, 1983, 185–188.

[5] K. G. Ivanov, *On the rates of convergence of two moduli of functions*, Pliska Stud. Math. Bulgar. 15 (1983), 97–104.

[6] A. Kamont, *On hyperbolic summation and hyperbolic moduli of smoothness*, Constr. Approx. 12 (1996), 111–126.

[7] P. I. Lizorkin, *On Fourier multipliers in spaces $L_{p,\Theta}$*, Trudy Mat. Inst. Steklov. 89 (1967), 231–248.

[8] N. S. Nikol'skaya, *Approximation of differentiable functions of several variables by Fourier sums in the L_p metric*, Sibirsk. Mat. Zh. 15 (1974), 395–412.

[9] S. M. Nikol'skij, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer, Berlin, 1975.

[10] H.-J. Schmeisser, *An unconditional basis in periodic spaces with dominating mixed smoothness properties*, Anal. Math. 13 (1987), 153–168.

[11] H.-J. Schmeisser and W. Sickel, *On strong summability of multiple Fourier series and smoothness properties of periodic functions*, Anal. Math. 8 (1982), 57–70.

[12] H.-J. Schmeisser and W. Sickel, *Spaces of functions of mixed smoothness and their relations to approximation from hyperbolic crosses*, JAT 128 (2004), 115–150.

[13] H.-J. Schmeisser and H. Triebel, *Topics in Fourier Analysis and Function Spaces*, Wiley, Chichester, 1987.

[14] W. Sickel, *Some remarks on trigonometric interpolation on the n-torus*, Z. Analysis Anwendungen 10 (1991), 551–562.

[15] W. Sickel, *Approximation on sparse grids and Sobolev spaces of dominating mixed smoothness*, in: Constructive Theory of Functions DARBA, 2002, 404–411.

[16] W. Sickel and F. Sprengel, *Interpolation on sparse grids and Nikol'skij-Besov spaces of dominating mixed smoothness*, J. Comp. Anal. Appl. 1 (1999), 263–288.

[17] S. A. Smolyak, *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Dokl. Akad. Nauk 148 (1963), 1042–1045.

[18] F. Sprengel, *A tool for approximation in bivariate periodic Sobolev spaces*, in: Approximation Theory IX, Vol. 2, Vanderbilt Univ. Press, Nashville, 1999, 319–326.

[19] V. N. Temlyakov, *Approximate recovery of periodic functions of several variables*, Mat. Sbornik 128 (1985), 256–268.

[20] V. N. Temlyakov, *On approximate recovery of functions with bounded mixed derivatives*, J. of Complexity 9 (1993), 41–59.

[21] V. N. Temlyakov, *Approximation of Periodic Functions*, Nova Science, New York, 1993.

[22] V. M. Tikhomirov, *Approximation theory*, in: Encyclopaedia of Math. Sciences 14, Springer, Berlin, 1990, 93–244.

[23] G. Wasilkowski and H. Woźniakowski, *Explicit cost bounds of algorithms for multivariate tensor product problems*, J. of Complexity 11 (1995), 1–56.

[24] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1959.