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# A REMARKABLY INTERESTING, SIMPLE P.D.E.

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**Abstract.** In this note, I will summarize and make a couple of small additions to some results which I obtained earlier with David Williams in [1]. Williams and I hope to expand and refine these additions in a future paper based on work that is still in process.

**0. Introduction.** The problem under consideration is that of understanding solutions $u : (0,\infty) \times [0,\infty) \to \mathbb{R}$ to the boundary value problem

$$\dot{u} = \tfrac{1}{2}u'' + \mu u' \quad \text{with} \quad \dot{u}(t,0) = -u'(t,0), \tag{1}$$

where $\dot{u} \equiv \partial_t u$ and $u' = \partial_x u$ denote differentiation in, respectively, the time and space directions. An interesting aspect of equation (1) is the observation that the minimum principle is absent. That is, just because $f \geq 0$, it is not necessarily true that $u \geq 0$.

Throughout, we will assume that solutions belong to the class $U$ of $u \in C^{1,2}\big((0,\infty) \times [0,\infty);\mathbb{R}\big)$ (i.e., functions which have one continuous derivative with respect to time and two continuous derivatives with respect to space, and all these existing and continuous up to and including the spacial boundary) with the properties that, for each $0 < T_1 < T_2$,

$$\|u\|_{C^{1,2}_{\mathrm{b}}([T_1,T_2]\times[0,\infty))} \equiv \sup_{(t,x)\in[T_1,T_2]\times[0,\infty)} |u(t,x)| \vee |\dot{u}(t,x)| \vee |u''(t,x)| < \infty$$

and $u \restriction (0,1] \times [0,\infty)$ is bounded.

The following theorem describes the basic facts about existence and uniqueness of solutions $u \in U$ to (1). In its statement, $F$ is used to denote the class of bounded functions $f : [0,\infty) \to \mathbb{R}$ which are continuous on $(0,\infty)$ but not necessarily at 0.

THEOREM 1. *If $u \in U$ satisfies* (1) *and, as $t \searrow 0$, $u(t,\cdot)$ converges uniformly on compact subsets of* $(0,\infty)$, *then* $\lim_{t\searrow 0} u(t,0)$ *exists. In fact, for each $f \in F$, there is a unique*

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*solution* $u_{f,\mu}$ *which satisfies, as* $t \searrow 0$, *the initial conditions* $u(t,0) \to f(0)$ *and* $u(t,\,\cdot\,) \to f$ *uniformly on compact subsets of* $(0,\infty)$.

The essential ingredient in the proof of Theorem 1 is the introduction of the function

$$J_\mu(x) = \begin{cases} 2e^{-2x} & \text{if } \mu \ge -1, \\ 2e^{2\mu x} & \text{if } \mu \le -1. \end{cases} \tag{2}$$

The role played by $J_\mu$ comes from the observation that when $u \in U$ satisfies (1), then integration by parts shows that

$$\begin{aligned}
\frac{d}{dt}\langle u(t), J_\mu\rangle &= \langle \tfrac12 u''(t) + \mu u'(t), J_\mu\rangle \\
&= \langle u(t), \tfrac12 J_\mu'' - \mu J_\mu'\rangle - \tfrac12 u'(t,0)J_\mu(0) + \tfrac12 u(t,0)J_\mu'(0) - \mu u(t,0)J_\mu(0) \\
&= 2(\mu+1)^+\big(\langle u(t), J_\mu\rangle - u(t,0)\big) + \dot u(t,0),
\end{aligned}$$

where $\langle \varphi, \psi\rangle \equiv \int_{(0,\infty)} \varphi(x)\psi(x)\,dx$. Hence,

$$\langle u(t), J_\mu\rangle - u(t,0) = e^{2(\mu+1)^+(t-s)}\big(\langle u(s), J_\mu\rangle - u(s,0)\big) \quad \text{for } 0 < s \le t. \tag{3}$$

To give an example of how (3) gets used, take $t = 1$ and note that if, as $s \searrow 0$, $u(s,\,\cdot\,)$ converges uniformly on compact subsets of $(0,\infty)$, then (3) makes it clear that $\lim_{s\searrow 0} u(s,0)$ must exist. That is, (3) leads immediately to the first assertion in Theorem 1. The role that (3) plays in the second part is that it allows one to formulate (1) as an integral equation in which (3) makes it possible to hide the boundary condition in an expression which is less singular than one might have expected. Namely, if $g(t,x) = (2\pi t)^{-\frac12}\exp\big(-\frac{x^2}{2t}\big)$,

$$\begin{aligned}
Q^0_\mu(t,x,y) &= e^{-\mu x - \frac{\mu^2 t}{2}}\big(g(t,x-y) - g(t,x+y)\big)e^{\mu y}, \\
q_\mu(t,x) &= \frac{x}{t}g(t, x+\mu t),
\end{aligned}$$

and $h_{f,\mu}(t,x) = \int_{(0,\infty)} Q^0_\mu(t,x,y)f(y)\,dy$, then $u$ solves (1) with initial data $f \in F$ if and only if

$$\begin{gathered}
u(t,x) = h_{f,\mu}(t,x) + \beta_{f,\mu}\int_0^t e^{2(\mu+1)^+\tau} q_\mu(t-\tau,x)\,d\tau + \int_0^t q_\mu(t-\tau,x)\langle u(\tau), J_\mu\rangle\,d\tau \\
\text{where } \beta_{f,\mu} \equiv f(0) - \langle f, J_\mu\rangle.
\end{gathered} \tag{4}$$

**1. The absence of the minimum principle.** As we said earlier, solutions to (1) do not satisfy the minimum principle. In fact, a central result in [1] is the following

THEOREM 2. *The solution* $u_{f,\mu}$ *is non-negative if and only if* $f \restriction (0,\infty) \ge 0$ *and* $f(0) \ge \langle f, J_\mu\rangle$.

The "if" part of Theorem 2 is quite easy. Indeed, because of (3), it is clear that (4) leads to

$$\langle u_{f,\mu}(t), J_\mu\rangle \ge \langle h_{f,\mu}(t), J_\mu\rangle + \int_0^t \langle q_\mu(t-\tau), J_\mu\rangle\langle u_{f,\mu}(\tau), J_\mu\rangle\,d\tau.$$

Hence, there cannot be a last $t > 0$ at which $\langle u_{f,\mu}(t), J_\mu\rangle \geq 0$, and so, again by (3), $u_{f,\mu}(t,0) \geq 0$ for all $t \geq 0$. But, because

$$u_{f,\mu}(t,x) = h_{f,\mu}(t,x) + \int_0^t q_\mu(t-\tau,x)u_{f,\mu}(\tau,0)\,d\tau,$$

this means that $u_{f,\mu} \geq 0$ everywhere. The argument which justifies the "only if" assertion is much more involved. Namely, it relies on the probabilistic interpretation of solutions to (1). To describe this interpretation, let $\{B(t) : t \geq 0\}$ be a standard, $\mathbb{R}$-valued Brownian motion, set $L_\mu(t) = \max\{(B(s) + \mu s)^- : s \in [0,t]\}$, $X_\mu(t) = B(t) + \mu t + L_\mu(t)$ and $\Psi_\mu(t) = L_\mu(t) - t$.[1] Then, $t \rightsquigarrow X_\mu(t)$ is "reflecting Brownian motion with drift $\mu$" and, because $X_0 = 0$, $\Psi_\mu(t)$ will be strictly positive for $t$ in the interval $(0, \zeta_\mu)$, where $\zeta_\mu \equiv \inf\{t > 0 : \Psi_\mu(t) = 0\}$ is strictly positive with probability 1. Moreover, if $u \in U$ is a solution to (1), then $t \in (0,\infty) \mapsto u\big(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu)\big) \in \mathbb{R}$ will be a local martingale. Hence, if $u_{f,\mu} \geq 0$, then $t \rightsquigarrow u_{f,\mu}\big(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu)\big)$ is a non-negative supermartingale, and so

$$\begin{aligned} f(0) &= \lim_{t\searrow 0}\mathbb{E}\big[u_{f,\mu}(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu)\big] \\ &\geq \lim_{t\nearrow\infty}\mathbb{E}\big[u_{f,\mu}(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu))\big] \geq \mathbb{E}\big[f\big(X_\mu(\zeta_\mu)\big),\, \zeta_\mu < \infty\big]. \end{aligned}$$

Thus, we will be done once we show that[2]

$$\mathbb{E}\big[f\big(X_\mu(\zeta_\mu)\big),\, \zeta_\mu < \infty\big] = \langle f, J_\mu\rangle. \tag{5}$$

To prove (5), we use the following lemma.

Lemma 1. *If $\mu < -1$, then, for every $f \in F$,*

$$\|u_{f,\mu}\|_{\mathrm{u}} \leq \big(\sup_{x\in(0,\infty)}|f(x)|\big) \vee \frac{|f(0) - \langle f, J_\mu\rangle|}{1 - \langle 1, J_\mu\rangle} < \infty,$$

*and $\lim_{t\to\infty} u_{f,\mu}(t,\,\cdot\,) = 0$ uniformly on compact subsets of $[0,\infty)$ if $f(0) = \langle f, J_\mu\rangle$. If $\mu \geq -1$ and $f(0) = \langle f, J_\mu\rangle$, then $\|u_{f,\mu}\|_{\mathrm{u}} \leq \|f\|_{\mathrm{u}}$.*[3]

*Proof.* Although all but the very first of these assertions are covered by the results in [1], we will prove them all here, by a slightly different line of reasoning.

To begin with, suppose that $\mu < -1$, and set $A = (\sup_{x\in(0,\infty)}|f(x)|) \vee \frac{|f(0)-\langle f,J_\mu\rangle|}{1-\langle 1,J_\mu\rangle}$. Then $A \pm f(x) \geq 0$ on $(0,\infty)$ and $(A\pm f)(0) \geq \langle A \pm f, J_\mu\rangle \geq 0$. Hence, by Theorem 1, $A \pm u_{f,\mu} = u_{A\pm f,\mu} \geq 0$.

Next, again assume that $\mu < -1$ and, in addition, that $f(0) = \langle f, J_\mu\rangle$. In this case (4) leads to

$$\langle u_{f,\mu}(t), J_\mu\rangle = \langle h_{f,\mu}(t), J_\mu\rangle + \int_0^t \langle q_\mu(t-\tau), J_\mu\rangle\langle u_{f,\mu}(\tau), J_\mu\rangle\,d\tau.$$

[1] For the reader who has looked at [1], it may be helpful to point out that the $\Psi_\mu(t)$ here is $-1$ times the $\Phi_t$ there.

[2] Note that $\zeta_\mu < \infty \;\Rightarrow\; X_\mu(\zeta_\mu) \in (0,\infty)$.

[3] $\|\cdot\|_{\mathrm{u}}$ is the uniform, or supremum, norm.

Therefore, since $\int_{(0,\infty)} \langle q_\mu(\tau), J_\mu\rangle\,d\tau < 1$,

$$\langle u_{f,\mu}(t), J_\mu\rangle = \sum_{m=0}^{\infty} w_m(t)$$

$$\text{with } w_0(t) = \langle h_{f,\mu}(t), J_\mu\rangle \text{ and } w_m(t) = \int_0^t \langle q_\mu(t-\tau), J_\mu\rangle w_{m-1}(\tau)\,d\tau,$$

where the convergence is uniform on $[0,\infty)$. Hence, because, by induction on $m \geq 0$, $\lim_{t\to\infty} w_m(t) = 0$ for each $m \geq 0$, it follows first that $\lim_{t\to\infty}\langle u_{f,\mu}(t), J_\mu\rangle = 0$ and then, by (4), that $u_{f,\mu}(t,\,\cdot\,) \to 0$ uniformly on compacts.

Finally, assume that $\mu \geq -1$ and that $f \in F$ satisfies $f(0) = \langle f, J_\mu\rangle$. To see that $u_{f,\mu}$ is bounded, set $M = \|f\|_{\mathrm{u}}$ and observe that, because $M \pm u_{f,\mu} = u_{M\pm f,\mu}$ and that, because now $\langle \mathbf{1}, J_\mu\rangle = 1$, $M \pm f(0) = \langle M \pm f, J_\mu\rangle$. Hence, by the "if" part of Theorem 2, both $M + u_{f,\mu}$ and $M - u_{f,\mu}$ are non-negative. ∎

In order to complete the proof of (5), we still need to know that

$$\begin{aligned}&\mu < -1 \;\Rightarrow\; \lim_{t\to\infty} \Psi_\mu(t) = \infty \quad \text{almost surely,}\\ &\mu \geq -1 \;\Rightarrow\; \varliminf_{t\to\infty} \Psi_\mu(t) = -\infty \quad \text{almost surely.}\end{aligned} \tag{6}$$

The proof of (6) is based on the "strong law"

$$\lim_{t\to\infty} \frac{1}{t} \sup_{\tau\in[0,t]} |B(\tau)| = 0 \quad \text{almost surely} \tag{7}$$

for Brownian motion. Indeed, if $\mu \neq -1$, then (7) makes it completely obvious that, depending on whether $\mu < -1$ or $\mu > 1$, as $t \to \infty$, $\Psi_\mu(t)$ tends almost surely to $\infty$ or $-\infty$. When $\mu = -1$, it is no longer true that $\Psi_\mu(t)$ converges almost surely. Instead, $\varlimsup_{t\to\infty} \Psi_{-1}(t) = \infty$ and $\varliminf_{t\to\infty} \Psi_{-1}(t) = -\infty$ almost surely. A proof of these can be found in [1], especially part (D) of the final section there.

We now have all the ingredients needed to prove (5). Indeed, if $\mu < -1$ and therefore $u_{f,\mu}(t,\,\cdot\,) \to 0$ boundedly and uniformly on compacts when $f(0) = \langle f, J_\mu\rangle$, it follows immediately that, for such $f$'s, $u_{f,\mu}\big(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu)\big)$ is a bounded martingale and, by (6), $\lim_{t\to\infty} u_{f,\mu}\big(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu)\big) = 0$ almost surely on $\{\zeta_\mu = \infty\}$. Hence,

$$\langle f, J_\mu\rangle = f(0) = \lim_{t\to\infty} \mathbb{E}\big[u_{f,\mu}\big(\Psi_\mu(t\wedge\zeta_\mu), X_\mu(t\wedge\zeta_\mu)\big)\big] = \mathbb{E}\big[f\big(X_\mu(\zeta_\mu)\big),\, \zeta_\mu < \infty\big].$$

That is, we have proved that (5) holds when $\mu < -1$. Similarly, because, for any $f \in F$ which satisfies $f(0) = \langle f, J_\mu\rangle$, the corresponding $u_{f,\mu}$ is bounded, we can repeat the preceding argument, apply (6) to see that $\mathbb{P}(\zeta_\mu < \infty) = 1$ when $\mu \geq -1$, and thereby conclude that (5) holds for all $\mu \in \mathbb{R}$. Finally, as was pointed out just above (5), this also means that we have completed the proof that $u_{f,\mu} \geq 0 \;\Rightarrow\; f(0) \geq \langle f, J_\mu\rangle$.

Before concluding, it may be worth observing that the arguments given allow us to prove the following statement.

THEOREM 3. *If $\mu < -1$ and $f \in F$, then $u_{f,\mu}$ is always bounded and $f(0) = \langle f, J_\mu\rangle \;\Leftrightarrow\; \lim_{t\to\infty} u_{f,\mu}(t,\,\cdot\,) = 0$ uniformly on compacts. On the other hand, if $\mu \geq -1$ and $f \in F$, then $f(0) = \langle f, J_\mu\rangle$ if and only if $u_{f,\mu}$ is bounded. In fact, when $\mu > -1$, then $f(0) = \langle f, J_\mu\rangle$ whenever $t \rightsquigarrow u(t,0) - \langle u(t), J_\mu\rangle$ is bounded.*

*Proof.* The only part of the case $\mu < -1$ that is not covered by Lemma 1 is the assertion that $f(0) = \langle f, J_\mu \rangle$ when $u_{f,\mu}(t,\,\cdot\,) \to 0$ uniformly on compacts. However, because $u_{f,\mu}$ is bounded, the argument used to prove (5) shows that $f(0) = \mathbb{E}\big[f\big(X_\mu(\zeta_\mu)\big),\, \zeta_\mu < \infty\big]$ if $u_{f,\mu}(t,\,\cdot\,) \to 0$, which, together with (5), implies that $f(0) = \langle f, J_\mu \rangle$.

When $\mu \geq -1$, we have to check that $f(0) = \langle f, J_\mu \rangle$ when $u_{f,\mu}$ is bounded. But, by the argument with which we proved (5) for $\mu \geq -1$, we again conclude that $f(0) = \mathbb{E}\big[f\big(X_\mu(\zeta_\mu)\big),\, \zeta_\mu < \infty\big]$, which, by (5), again leads to $f(0) = \langle f, J_\mu \rangle$. Alternatively, in the case when $\mu > -1$, one need only observe that, because, by (3), $f(0) - \langle f, J_\mu \rangle = e^{-2(\mu+1)t}\big(u(t,0 - \langle u(t), J_\mu \rangle\big)$, $f(0) - \langle f, J_\mu \rangle$ must vanish if $t \rightsquigarrow u(t,0) - \langle u(t), J_\mu \rangle$ is bounded. ∎

## References

[1] D. Stroock and D. Williams, *A simple PDE and Wiener–Hopf Riccati equations*, Comm. Pure Appl. Math., to appear.