RESULTS ON SPLINE-FOURIER AND CIESIELSKI-FOURIER SERIES

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Dedicated to Professor Zbigniew Ciesielski on his 70-th birthday

Abstract. Some recent results on spline-Fourier and Ciesielski-Fourier series are summarized. The convergence of spline Fourier series and the basis properties of the spline systems are considered. Some classical topics, that are well known for trigonometric and Walsh-Fourier series, are investigated for Ciesielski-Fourier series, such as inequalities for the Fourier coefficients, convergence a.e. and in norm, Fejér and \( \vartheta \)-summability, strong summability and multipliers. The connection between Fourier series and Hardy spaces is studied.

1. Introduction. It is known that the spline Fourier series of \( f \in L^p \) converges a.e. and in \( L^p \) norm to \( f \), whenever \( 1 \leq p < \infty \). The maximal operator of the partial sums with respect to the spline (or unbounded Ciesielski) systems of order \((m, k)\) is bounded from the Hardy space \( H_p \) to \( L_p \) \( (1/(m - k + 2) < p < \infty) \) and is of weak type \((1, 1)\). If \( 1 < p < \infty \) then \( H_p \) is equivalent to \( L_p \). Moreover, the spline systems are unconditional and equivalent bases to the Haar system in \( H_p \) \( (1/(m - k + 2) < p < \infty) \).

We investigate also the bounded Ciesielski systems, which can be obtained from the spline systems in the same way as the Walsh system from the Haar. Some results, that are well known for trigonometric and Walsh-Fourier series, are extended to Ciesielski-Fourier series. Paley and Hardy-Littlewood type inequalities are shown for Ciesielski-Fourier coefficients. If \( f \in L_p \) \( (1 < p < \infty) \) then the Ciesielski-Fourier series of \( f \) converges

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a.e. and in $L^p$-norm to $f$. It is proved that a lacunary sequence of the Ciesielski-Fourier series of $f \in H_1$ converges almost everywhere to $f$. The maximal operator of the Fejér means of the Ciesielski-Fourier series is bounded from $H^p$ to $L^p$ ($1/2 < p < \infty$) and it is of weak type $(1, 1)$. As a consequence we obtain that the Fejér means of the Ciesielski-Fourier series of a function $f \in L^1$ converges a.e. to $f$. Some other summability methods, called $\theta$-summability are considered as well. $\theta$-summability includes the well known summations of Weierstrass, Picard, Bessel, Riesz, de La Vallée-Poussin, Rogosinski and Riemann. The analogue of the strong convergence result is also given. It is proved that the maximal strong operator is of weak type $(1, 1)$ and the Ciesielski-Fourier series of $f \in L^1$ is strong summable to $f$. Some multiplier theorems, e.g. the Marcinkiewicz multiplier theorem is shown for Ciesielski-Fourier series. The results can be extended to the multi-dimensional case, too.

2. Hardy spaces on the unit interval. We consider the unit interval $[0, 1)$ with the Lebesgue measure $\lambda$. We briefly write $L^p$ instead of the real $L^p([0, 1), \lambda)$ equipped with the norm (or quasinorm) $\|f\|_p := (\int_0^1 |f|^p \, d\lambda)^{1/p}$ ($0 < p \leq \infty$). The space $L^p, \infty = L^p, \infty([0, 1), \lambda)$ ($0 < p < \infty$) consists of all measurable functions $f$ for which

$$\|f\|_{p, \infty} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L^{\infty, \infty} = L^\infty$. Note that $L^{p, \infty}$ is a quasi-normed space. It is easy to see that

$$L^p \subset L^{p, \infty} \quad \text{and} \quad \|\cdot\|_{p, \infty} \leq \|\cdot\|_p$$

for each $0 < p \leq \infty$. The space $L^p(l_r)$ ($1 \leq p, r < \infty$) consists of all sequences $f := (f_n, n \in \mathbb{N})$ of functions for which

$$\|f\|_{L^p(l_r)} := \left(\left(\sum_{n \in \mathbb{N}} |f_n|^r\right)^{1/r}\right)_p < \infty.$$

In order to have a common notation for the dyadic and classical Hardy spaces we define the Poisson kernels $P^{(m,k)}_t$. If $k \leq m$ then we introduce $P^{(m,k)}_t$ by

$$P^{(m,k)}_t(x) := \frac{ct}{(t^2 + |x|^2)} \quad (x \in \mathbb{R}, t > 0).$$

If $k = m + 1$ then we define $P^{(m,k)}_t$ as follows. For a fixed $t > 0$, if $n \leq t < n + 1$ for some $n \in \mathbb{N}$ then let

$$P^{(m,k)}_t(x) := 1_{[0,2^{-n})}(x) \quad (x \in \mathbb{R}).$$

For a tempered distribution $f$ the non-tangential maximal function is defined by

$$f_\ast^{(m,k)}(x) := \sup_{t > 0} \|f * P^{(m,k)}_t(x)\|_p \quad (x \in \mathbb{R})$$

where $*$ denotes the convolution.

For $0 < p < \infty$ the Hardy space $H^{(m,k)}_p(\mathbb{R})$ consists of all tempered distributions $f$ for which

$$\|f\|_{H^{(m,k)}_p(\mathbb{R})} := \|f_\ast^{(m,k)}\|_p < \infty.$$

Now let

$$H_p := H^{(m,k)}_p([0, 1)) := \{f \in H^{(m,k)}_p(\mathbb{R}) : \text{supp } f \subset [0, 1)\}.$$
Obviously, \( H_p \) is the dyadic Hardy space if \( k = m + 1 \). It is known (see Stein [56]) that the space \( H_p \) coincides with \( L_p \) if \( 1 < p < \infty \).

The atomic decomposition is a basic tool in proving the results below. A function \( a \in L_\infty \) is called a \( p \)-atom if there exists an interval \( I \subset [0, 1) \) such that

(i) \( \text{supp} \ a \subset I \),

(ii) \( \|a\|_\infty \leq |I|^{-1/p} \),

(iii) \( \int_I a(x)x^j \, dx = 0 \) where \( j \in \mathbb{N} \) and \( j \leq \lfloor 1/p - 1 \rfloor \).

Note that \( [x] \) denotes the integer part of \( x \in \mathbb{R} \).

In the dyadic case, i.e. if \( k = m + 1 \), we consider only dyadic intervals \( I \) and instead of (iii) we assume

\[ (iii') \int_I a(x) \, dx = 0. \]

The following result shows that for the boundedness of a sublinear operator \( V \) from the Hardy space \( H_p \) to \( L_p \), it is enough to check the operator on \( p \)-atoms (Weisz [61]).

**Theorem 1.** Suppose that the operator \( V \) is sublinear and

\[ \int_{[0,1)\setminus 16I} |Va|^p \, d\lambda \leq C_p \]

for every \( p \)-atom \( a \) with support \( I \), where \( 0 < p \leq 1 \). If \( V \) is bounded from \( L_{p_1} \) to \( L_{p_1} \) for some \( 1 < p_1 \leq \infty \) then

\[ \|Vf\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p). \]

**3. Spline systems.** We are going to introduce the spline systems as in Ciesielski [12].

Let us denote by \( D \) the differentiation operator and define the integration operators

\[ Gf(t) := \int_0^t f \, d\lambda, \quad Hf(t) := \int_t^1 f \, d\lambda. \]

Define the \( \chi_n, n = 1, 2, \ldots, \) Haar system by \( \chi_1 := 1 \) and

\[ \chi_{2^n+k}(x) := \begin{cases} 2^{n/2}, & \text{if } x \in ((2k-2)2^{-n-1}, (2k-1)2^{-n-1}), \\ -2^{n/2}, & \text{if } x \in ((2k-1)2^{-n-1}, (2k)2^{-n-1}), \\ 0, & \text{otherwise}, \end{cases} \]

for \( n, k \in \mathbb{N} \), \( 0 < k \leq 2^n \), \( x \in [0, 1] \).

Let \( m \geq -1 \) be a fixed integer. Applying the Schmidt orthonormalization to the linearly independent functions

\[ 1, t, \ldots, t^{m+1}, G^n \chi_n(t), \quad n \geq 2, \]

we get the spline system \( \{f_n^{(m)}(t), n \geq -m\} \) of order \( m \). For \( 0 \leq k \leq m + 1 \) and \( n \geq k - m \) define the splines

\[ f_n^{(m,k)} := D^k f_n^{(m)}, \quad g_n^{(m,k)} := H^k f_n^{(m)} \]

of order \((m,k)\). Let us normalize these functions and introduce a more unified notation,

\[ h_n^{(m,k)} := \begin{cases} f_n^{(m,k)} \|f_n^{(m,k)}\|^{-1} & \text{for } 0 \leq k \leq m + 1 \\ g_n^{(m,-k)} \|f_n^{(m,-k)}\|^{-1} & \text{for } 0 \leq -k \leq m + 1. \end{cases} \]
We get the Haar system if \( m = -1, k = 0 \) and the Franklin system if \( m = 0, k = 0 \). The systems \((h^{(m,k)}_i, i \geq |k| - m)\) and \((h^{(m,-k)}_j, j \geq |k| - m)\) are biorthogonal, i.e.

\[
(h^{(m,k)}_i, h^{(m,-k)}_j) = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j,
\end{cases}
\]

where \((f, g)\) denotes the usual scalar product \( \int_{[0,1]} fg \, d\lambda \).

Ciesielski \([11, 12]\) proved that the spline functions have the nice property

\[
|D^N h^{(m,k)}_{2^u + \nu}(t)| \leq C2^{((N+1/2)\mu q^{2^{\nu}|t-\nu/2^u|},}
\]

which is used several times in the proofs, where \( m \geq -1, |k| \leq m + 1, k + N \leq m + 1, \mu \in \mathbb{N} \) and \( \nu = 1, \ldots, 2^\mu \).

In this paper the constants \( C \) and \( q \) depend only on \( m \) and the constants \( C_p \) depend only on \( p \) and \( m \). The symbols \( C, q \) and \( C_p \) may denote different constants in different contexts, however, \( q \) denotes constants for which \( 0 < q < 1 \).

**3.1. Spline-Fourier series.** The partial sums and the maximal operator of the partial sums of the spline-Fourier series are defined by

\[
P_n^{(m,k)} f := \sum_{i=|k|-m}^n (f, h^{(m,k)}_i) h^{(m,-k)}_i = \int_0^1 F_n^{(m,k)}(t, x) f(t) \, dt
\]

and

\[
P_s^{(m,k)} f := \sup_{n \in \mathbb{N}} |P_n^{(m,k)} f|
\]

respectively, where \( m \geq -1 \) and \( |k| \leq m + 1 \). Using (1), Domsta and Ciesielski \([7]\) (see also Ciesielski \([12]\)) proved that

\[
|F_n^{(m,k)}(t, x)| := \left| \sum_{j=|k|-m}^n h^{(m,k)}_j(t) h^{(m,-k)}_j(x) \right| \leq Cn q^n|t-x|.
\]

Since

\[
\int_0^1 |F_n^{(m,k)}(t, x)| \, dt \leq C, \quad \int_0^1 |F_n^{(m,k)}(t, x)| \, dx \leq C,
\]

we have

\[
\|P_n^{(m,k)} f\|_p \leq C_p\|f\|_p \quad (f \in L_p)
\]

for \( p = \infty \) and \( p = 1 \). By interpolation we get (3) for all \( 1 < p < \infty \). Note that \( C \) and \( C_p \) are independent of \( n \).

**Corollary 1.** Assume that \( m \geq -1 \) and \( |k| \leq m + 1 \). If \( f \in L_p \) for some \( 1 \leq p \leq \infty \) then \( P_n^{(m,k)} f \to f \) in \( L_p \) norm as \( n \to \infty \).

This means that the spline systems are bases in \( L_p \) \((1 \leq p \leq \infty)\). There is also a vector valued version of (3) (see Weisz \([69]\)).

**Theorem 2.** Assume that \( m \geq -1, |k| \leq m + 1 \) and \( f = (f_i, i \in \mathbb{N}) \in L_p(l_r) \) \((1 < p, r < \infty)\). If \( n(i) \) is an arbitrary natural number for each \( i \in \mathbb{N} \) then

\[
\int_0^1 \left( \sum_{i=0}^\infty |P_{n(i)}^{(m,k)} f_i|^r \right)^{p/r} \, d\lambda \leq C_{p,r} \int_0^1 \left( \sum_{i=0}^\infty |f_i|^r \right)^{p/r} \, d\lambda.
\]
Ciesielski [11, 12, 9] has generalized (3) and showed that $P^{(m,k)}_*$ can be estimated by the Hardy-Littlewood maximal operator $M$. Hence $P^{(m,k)}_*$ is bounded on $L_p$ ($1 < p \leq \infty$) and is of weak type $(1,1)$.

**Theorem 3.** If $m \geq -1$ and $|k| \leq m + 1$ then
\[
\|P^{(m,k)}_* f\|_p \leq C_p \|f\|_p \quad (f \in L_p)
\]
for all $1 < p \leq \infty$. In particular, if $f \in L_1$ then
\[
\lambda(P^{(m,k)}_* f > \rho) \leq \frac{C}{\rho} \|f\|_1 \quad (\rho > 0).
\]

The last weak type inequality and the usual density theorem due to Marcinkiewicz and Zygmund [29] imply

**Corollary 2.** If $m \geq -1$, $|k| \leq m + 1$ and $f \in L_1$ then $P^{(m,k)}_n f \to f$ a.e. as $n \to \infty$.

Theorem 3 can be extended to Hardy spaces as follows (see Sjölin [55] and Weisz [58, 61], for $p = 1$ Schipp and Simon [39]). Let
\[
p_0 := \begin{cases} 
1/(m - k + 2), & \text{if } k \leq m, \\
0, & \text{if } k = m + 1.
\end{cases}
\]

**Theorem 4.** If $m \geq -1$ and $|k| \leq m + 1$ then
\[
\|P^{(m,k)}_* f\|_{p_0} \leq C_{p_0} \|f\|_{H_{p_0}} \quad (f \in H_{p_0})
\]
for all $p_0 < p < \infty$. If $k \leq m$ then
\[
\|P^{(m,k)}_* f\|_{p_0, \infty} \leq C_{p_0} \|f\|_{H_{p_0}} \quad (f \in H_{p_0}).
\]

**3.2. Equivalent bases.** The spline systems have very nice bases properties. To see this we will investigate the conditional partial sums and the square function of the spline-Fourier series:
\[
T_n^{(m,k)} f := \sum_{i=|k|}^{n} \epsilon_i (f, h^{(m,k)}_i) h^{(m,-k)}_i
\]
and
\[
Q^{(m,k)} f := \left( \sum_{i=|k|}^{\infty} |(f, h^{(m,k)}_i) h^{(m,-k)}_i|^2 \right)^{1/2},
\]
respectively, where $m \geq -1$, $|k| \leq m + 1$ and $\epsilon_i = \pm 1$. Of course, if each $\epsilon_i = 1$ then $T_n^{(m,k)} = P_n^{(m,k)}$. Let $T^{(m,k)} f := \lim_{n \to \infty} T_n^{(m,k)} f$, if the limit does exist.

Bočkariev [3] proved that the Franklin system is an unconditional basis in $L_p$ ($1 < p < \infty$). Some years later Ciesielski [12] verified that the spline systems are equivalent and unconditional bases in $L_p$ ($1 < p < \infty$).

**Theorem 5.** Assume that $m \geq -1$, $|k| \leq m + 1$ and $1 < p < \infty$. Then the systems $(h^{(m,-k)}_n)$ are unconditional bases in $L_p$ and
\[
\|T_n^{(m,k)} f\|_p \leq C_p \|f\|_p \quad (f \in L_p),
\]
uniformly in $n \in \mathbb{N}$. 

Theorem 6. If \( m \geq -1, \ |k| \leq m + 1, \ 1 < p < \infty \) and \( f \in L_p \) then
\[
C_p^{-1} \|f\|_p \leq \|Q^{(m,k)} f\|_p \leq C_p \|f\|_p.
\]

Corollary 3. Assume that \( m \geq -1, \ |k| \leq m + 1 \) and \( 1 < p < \infty \). Then the Haar system \((\chi_n)\) and the spline system \((h_n^{(m,-k)})\) are equivalent bases in the \( L_p \) space, i.e. the following two series
\[
\sum_{n=1}^{\infty} a_n \chi_n \quad \text{and} \quad \sum_{n=|k|-m}^{\infty} a_{(n+m+1-|k|)} h_n^{(m,-k)}
\]
are equiconvergent in \( L_p \).

The spline systems are also equivalent in \( L_p(l_r) \) (see Figiel and Wojtaszczyk [17, 18]). Actually, Figiel and Wojtaszczyk proved the equivalence in more general \( UMD \) spaces.

Corollary 4. Assume that \( m \geq -1, \ |k| \leq m + 1 \) and \( 1 < p, r < \infty \). Then the Haar system \((\chi_n)\) and the spline system \((h_n^{(m,-k)})\) are equivalent bases in the \( L_p(l_r) \) space.

Maurey [31] proved that the classical \( H_1 \) space has an unconditional basis. His proof is non-constructive, he proved that \( H_1 \) is linearly isomorphic to the dyadic \( H_1 \) in which the Haar system is an unconditional basis. Carleson [5] constructed an unconditional basis in \( H_1 \) and Wojtaszczyk [71] verified that the Franklin system is an unconditional basis in \( H_1 \). Later Sjölin and Strömberg [52] and Wojtaszczyk [72] extended these results to \( H_p \) spaces as follows. Let
\[
p_1 := \begin{cases} 1/(m - k + 2), & \text{if } m \geq 0, \\ 0, & \text{if } m = -1. \end{cases}
\]

Theorem 7. If \( m \geq -1, \ 0 \leq k \leq m + 1 \) and \( p_1 < p < \infty \) then
\[
C_p^{-1} \|f\|_{H_p} \leq \|Q^{(m,k)} f\|_p \leq C_p \|f\|_{H_p}, \quad (f \in H_p).
\]

The next corollary follows easily from this.

Corollary 5. Assume that \( m \geq -1, \ 0 \leq k \leq m + 1 \) and \( p_1 < p < \infty \). Then the systems \((h_n^{(m,-k)})\) are unconditional bases in \( H_p \) and
\[
\|T_n^{(m,k)} f\|_{H_p} \leq C_p \|f\|_{H_p}, \quad (f \in H_p),
\]
uniformly in \( n \in \mathbb{N} \).

It is proved in Sjölin and Strömberg [53] and Wojtaszczyk [72] that the spline systems are not bases in \( H_p \), if \( p < p_1 \). Moreover, the spline systems are not bases in the Hardy spaces if \( k \) is negative (see Wojtaszczyk [72]).

The next result shows that the dyadic and classical Hardy spaces have equivalent bases (cf. Maurey [31]).

Corollary 6. Assume that \( m \geq -1, \ 0 \leq k \leq m + 1 \) and \( p_1 < p < \infty \). Then the Haar system \((\chi_n)\) and the spline system \((h_n^{(m,-k)})\) are equivalent bases in the dyadic and classical \( H_p \) spaces.
The following result is due to Chang and Ciesielski [6] for $p = 1$. For general $p$ see Weisz [58, 61].

**Corollary 7.** Assume that $0 \leq k \leq m + 1$ and $p_1 < p < \infty$. Then the following properties are equivalent:

1. $f \in H_p$,
2. $Q^{(m,k)} f \in L_p$,
3. $\sup_\epsilon \|T^{(m,k)} f\|_p < \infty$,
4. $P^{(m,k)}_n f$ converges unconditionally in $L_p$.

Furthermore, the norms corresponding to (i), (ii) and (iii) are equivalent norms.

**4. Ciesielski systems.** First we define the Walsh system. Let

$$r(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}), \\ -1 & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1. The Rademacher system $(r_n, n \in \mathbb{N})$ is defined by

$$r_n(x) := r(2^nx) \quad (x \in [0, 1), n \in \mathbb{N}).$$

The Walsh functions are given by

$$w_n(x) := \prod_{k=0}^\infty r_k(x)^{n_k} \quad (x \in [0, 1), n \in \mathbb{N})$$

where $n = \sum_{k=0}^\infty n_k2^k$, ($n_k = 0$ or $n_k = 1$).

Starting with the spline system $(h^{(m,k)}_n, n \geq |k| - m)$ we define the Ciesielski system $(c^{(m,k)}_n, n \geq |k| - m)$ in the same way as the Walsh system arises from the Haar system, namely,

$$c^{(m,k)}_n := h^{(m,k)}_n \quad (n = |k| - m, \ldots, 1)$$

and

$$c^{(m,k)}_{2\nu+i} := \sum_{j=1}^{2\nu} A^{(\nu)}_{i,j} h^{(m,k)}_{2\nu+j} \quad (1 \leq i \leq 2\nu),$$

where

$$A^{(\nu)}_{i,j} = A^{(\nu)}_{j,i} = 2^{-\nu/2}w_{i-1}(\frac{2j - 1}{2\nu + 1})$$

(see Ciesielski [10], Ciesielski, Simon, Sjölin [8] or Schipp, Wade, Simon, Pál [40]). We get immediately that

$$h^{(m,k)}_{2\nu+j} := \sum_{i=1}^{2\nu} A^{(\nu)}_{i,j} c^{(m,k)}_{2\nu+i} \quad (1 \leq j \leq 2\nu).$$

As mentioned before,

$$c^{(-1,0)}_n = w_{n-1} \quad (n \geq 1)$$

is the usual Walsh system. The system $(c^{(m,k)}_n)$ is uniformly bounded and it is biorthogonal to $(c^{(m,-k)}_n)$ whenever $|k| \leq m + 1$ (see Ciesielski [10]).
4.1. Inequalities for Ciesielski-Fourier coefficients. In this subsection we assume that $k = 0$ and use the notation $c_n^{(m,0)} = c_n^{(m)}$. For simplicity we suppose here that $(f, c_n^{(m)}) = 0$ for $-m \leq n \leq 1$. Of course, the theorems of this section can be shown without the last condition, too.

Paley [33] proved the inequality

$$\left( \sum_{i=0}^{\infty} |(f, c_{2i}^{(m)})|^2 \right)^{1/2} \leq C_p \|f\|_{H^1}$$

for Walsh-Fourier coefficients (see also Coifman and Weiss [13]). A similar result was shown in the trigonometric case by Gundy and Varopoulos [20]. Recently the inequality was extended to unbounded Vilenkin systems and to all $p$ by Simon and Weiss [46] (see also Simon [47]). We generalized this inequality for Ciesielski-Fourier coefficients in [63].

**Theorem 8.** Let $m \geq -1$ and $A_n \subset [2^n + 1, 2^{n+1}]$ be sets of integers. Then the following two conditions are equivalent.

(i) For all $f \in H_p \cap L_1$

$$\left( \sum_{n=0}^{\infty} 2^{n(2-2/p)} \sum_{k \in A_n} |(f, c_k^{(m)})|^2 \right)^{1/2} \leq C_p \|f\|_{H_p} \quad (p_1 < p \leq 1),$$

(ii) $\sup_{n \in \mathbb{N}} |A_n| < \infty$, where $|A_n|$ denotes the number of the elements of $A_n$.

By interpolation and duality argument we obtain a new version of Khinchine’s inequality:

**Corollary 8.** If $m \geq -1$, $1 < p \leq 2$, $A_n \subset [2^n + 1, 2^{n+1}]$ and $\sup_{n \in \mathbb{N}} |A_n| < \infty$ then

$$\|f\|_p \sim \left( \sum_{n=0}^{\infty} \sum_{k \in A_n} |(f, c_k^{(m)})|^2 \right)^{1/2} \sim \|f\|_{BMO} \quad (0 < p < \infty)$$

for $f = \sum_{n=0}^{\infty} \sum_{k \in A_n} (f, c_k^{(m)}) c_k^{(m)}$, where $BMO$ is the dual space of $H_1$.

Now we formulate the Hardy-Littlewood inequality for Ciesielski-Fourier coefficients (see Weisz [68]).

**Theorem 9.** If $m \geq -1$ and $f \in H_p \cap L_1$ then

$$\left( \sum_{n=2}^{\infty} \frac{|(f, c_n^{(m)})|^p}{n^{2-p}} \right)^{1/p} \leq C_p \|f\|_{H_p} \quad (p_1 < p \leq 2).$$

This inequality is due to Hardy and Littlewood [22] for the trigonometric system (see also Coifman and Weiss [13]) and to Ladhawala [27] for the Walsh system. The inequality is also known for Fourier transforms (see Jawerth and Torchinsky [23]).

4.2. Convergence of Ciesielski-Fourier series. The partial sums and the maximal operators of the Ciesielski-Fourier series are defined by

$$s_n^{(m,k)} f(x) := \sum_{i=|k|-m}^{n} (f, c_i^{(m,k)}) c_i^{(m,-k)}(x) = \int_0^1 D_n^{(m,k)}(t, x) f(t) \, dt$$
and
\[ s_n^{(m,k)} f := \sup_{n \in \mathbb{N}} |s_n^{(m,k)} f|, \]
respectively, where \( m \geq -1 \) and \( |k| \leq m + 1 \). Here
\[ D_n^{(m,k)}(t, x) := \sum_{i=|k|-m}^{n} c_i^{(m,k)}(t)c_i^{(m,-k)}(x), \]
is the \( n \)th Dirichlet kernel.

In contrast to the spline system, the Lebesgue functions are not bounded in this case. Ciesielski [10] proved that, similarly to the trigonometric and Walsh system,
\[ \int_0^1 |D_n^{(m,k)}(t, x)| \, dt \leq C \log n \quad (n \in \mathbb{N}, x \in [0, 1)). \]
The next sharper result, which is well known for the trigonometric, Walsh- and Vilenkin systems, is due to the author [66].

**Theorem 10.** If \( m \geq -1 \), \( |k| \leq m + 1 \) and \( n \geq 0 \) then
\[ |D_n^{(m,k)}(t, x)| \leq \frac{C}{|t - x|} \quad (t, x \in [0, 1), t \neq x). \]

The following theorem was proved by Carleson [4] for the trigonometric system, by Billard [1] and Sjölin [54] (see also Schipp [44]) for the Walsh system and by Schipp [43] and Ciesielski [11, 12] for the Ciesielski system.

**Theorem 11.** If \( m \geq -1 \) and \( |k| \leq m + 1 \) then
\[ \|s_n^{(m,k)} f\|_p \leq C_p \|f\|_p \quad (f \in L_p) \]
for all \( 1 < p < \infty \).

This means that \( s_n^{(m,k)} \) is uniformly bounded on \( L_p \) \((1 < p < \infty)\). The result corresponding to Theorem 2 holds for Ciesielski systems as well ([69]).

**Theorem 12.** Assume that \( m \geq -1 \), \( |k| \leq m + 1 \) and \( f \in L_p(I_r) \) \((1 < p, r < \infty)\). If \( n(i) \) is an arbitrary natural number for each \( i \in \mathbb{N} \) then
\[ \int_0^1 \left( \sum_{i=0}^{\infty} |s_{n(i)}^{(m,k)} f_i|^r \right)^{p/r} \, d\lambda \leq C_{p,r} \int_0^1 \left( \sum_{i=0}^{\infty} |f_i|^r \right)^{p/r} \, d\lambda. \]

This was proved by Marcinkiewicz and Zygmund for trigonometric Fourier series (see e.g. Zygmund [76, II. p. 225]) and by Sunouchi [57] for Walsh-Fourier series.

Theorem 11 imply

**Corollary 9.** Assume that \( m \geq -1 \) and \( |k| \leq m + 1 \). If \( f \in L_p \) for some \( 1 < p < \infty \) then \( s_n^{(m,k)} f \to f \) a.e. and in \( L_p \) norm as \( n \to \infty \).

Ciesielski, Simon and Sjölin [8] verified the a.e. convergence also for functions from \( L(\log L)^3 \).

Since the Ciesielski systems are uniformly bounded, due to a theorem of Bočkariev [2], this theorem does not hold for functions in \( L_1 \). Moreover, there is \( f \in L_1 \) such that the Ciesielski-Fourier series diverges a.e. (see Kazarian and Sargsian [24]).
The convergence result is not true even for Hardy spaces. Ladhawala and Pankratz [26] verified for Walsh-Fourier series that there exists \( f \in H_1 \) such that the partial sums diverge everywhere (see also Schipp and Simon [37]). However, if \( f \in H_1 \) then a lacunary sequence of the partial sums converge a.e. to \( f \). This was proved for Walsh-Fourier series by Ladhawala and Pankratz [26], for trigonometric Fourier series by Zygmund [76, II. p. 235] and for Vilenkin-Fourier series by Young [73]. Now we extend this result to Ciesielski-Fourier series (see Weisz [62]).

We say that a sequence \((n_i, i \in \mathbb{N})\) of positive integers is lacunary if \( n_{i+1}/n_i > \alpha > 1 \) for all \( i \in \mathbb{N} \).

**Theorem 13.** Assume that \( m \geq -1 \), \(|k| \leq m + 1 \) and \((n_i, i \in \mathbb{N})\) is a lacunary sequence of positive integers. If \( f \in H_1 \) then \( \lim_{i \to \infty} s_n^{(m,k)} f = f \) a.e.

**4.3. Fejér summability.** Taking a suitable summability method, we can obtain a.e. convergence for \( f \in L_1 \) functions, too. It was proved by Lebesgue [28] that the Fejér means of the trigonometric Fourier series of an integrable function \( f \) converge a.e. to \( f \), i.e.

\[
\frac{1}{n} \sum_{k=0}^{n-1} \left( s_k f(x) - f(x) \right) \to 0 \quad \text{as} \quad n \to \infty
\]  

(4)

for a.e. \( x \in [-\pi, \pi] \), where \( s_k f \) denotes the \( k \)th partial sum of the Fourier series of \( f \). Fejér summability for Walsh-Fourier series is due to Fine [19] (see also Schipp [42]).

The Fejér means and the maximal Fejér operator of the Ciesielski-Fourier series are given by

\[
\sigma_n^{(m,k)} f(x) := \frac{1}{n} \sum_{j=1}^{n} s_j^{(m,k)}(x) = \int_0^1 K_n^{(m,k)}(t, x) f(t) \, dt
\]

and

\[
\sigma_*^{(m,k)} f := \sup_{n \in \mathbb{N}} |\sigma_n^{(m,k)} f|,
\]

where \( m \geq -1 \), \(|k| \leq m + 1 \) and

\[
K_n^{(m,k)}(t, x) := \frac{1}{n} \sum_{j=1}^{n} D_j^{(m,k)}(t, x)
\]

is the \( n \)th Fejér kernel. The \( L_1 \) norms of the Fejér kernels are bounded (see Weisz [59]), which implies the boundedness of \( \sigma_*^{(m,k)} \) on \( L_\infty \).

**Theorem 14.** If \( m \geq -1 \) and \(|k| \leq m + 1 \) then

\[
\int_0^1 |K_n^{(m,k)}(t, x)| \, dt \leq C \quad (n \in \mathbb{N}, x \in [0, 1)).
\]

Using Theorem 1 we have proved in [59, 67] that \( \sigma_*^{(m,k)} \) is bounded from the Hardy space \( H_p \) to \( L_p \) \((1/2 < p \leq \infty)\). From this it follows by interpolation that \( \sigma_*^{(m,k)} \) is also of weak type \((1,1)\). More exactly, we have

**Theorem 15.** If \( m \geq -1 \) and \(|k| \leq m + 1 \) then

\[
\|\sigma_*^{(m,k)} f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p)
\]
for all \(1/2 < p < \infty\). In particular, if \(f \in L_1\) then
\[
\lambda(\sigma^{(m,k)}_n f > \rho) \leq \frac{C}{\rho} \|f\|_1 \quad (\rho > 0).
\]
Moreover,
\[
\|\sigma^{(m,k)}_n f\|_{1/2, \infty} \leq C \|f\|_{H_{1/2}} \quad (f \in H_{1/2}).
\]

The weak type \((1, 1)\) inequality and the usual density theorem imply the analogue to (4):

**Corollary 10.** If \(m \geq -1\) and \(|k| \leq m + 1\) then \(f \in L_1\) implies
\[
\sigma^{(m,k)}_n f \to f \quad \text{a.e. as } n \to \infty.
\]

Simon [51] gave a counterexample which shows the following result for the Walsh-Fourier series.

**Theorem 16.** The operator \(\sigma^{(-1,0)}_n\) is not bounded from \(H_p\) to \(L_p\), if \(0 < p < 1/2\).

### 4.4. Other summability methods

It is easy to see that the Fejér means can also be given by
\[
\sigma^{(m,k)}_n f = \sum_{j \geq |k| - m} (f, c_j^{(m,k)}) c_j^{(m,-k)} + \sum_{j=1}^{n} \left(1 - \frac{j - 1}{n}\right)(f, c_j^{(m,k)}) c_j^{(m,-k)}.
\]
As a generalization of Fejér summability we introduce the \(\theta\)-summation. Assume that \(\theta \in L_1(\mathbb{R})\) is even and continuous, \(\theta(0) = 1\),
\[
\left(\theta\left(\frac{k}{n+1}\right)\right)_{k \in \mathbb{Z}} \in \ell_1, \lim_{x \to \infty} \theta(x) = 0,
\]
\(\theta\) is twice continuously differentiable on \(\mathbb{R}\) except at finitely many points, \(\theta'' \neq 0\) except at finitely many points and finitely many intervals, the left and right limits \(\lim_{x \to y} x \theta'(x) \in \mathbb{R}\) exist at each point \(y \in \mathbb{R}\), \(\lim_{x \to \infty} x \theta'(x) = 0\).

Note that the second condition of (5) is satisfied if \(\theta\) is non-increasing on \((c, \infty)\) for some \(c \geq 0\) or if it has compact support.

The \(\theta\)-means and the maximal \(\theta\)-operator of \(f \in L_1\) are defined by
\[
\sigma^{(m,k), \theta}_n f(x) := \sum_{j \geq |k| - m} (f, c_j^{(m,k)}) c_j^{(m,-k)} + \sum_{j=1}^{n} \theta\left(\frac{j - 1}{n}\right)(f, c_j^{(m,k)}) c_j^{(m,-k)}
\]
and
\[
\sigma^{(m,k), \theta}_* f := \sup_{n \in \mathbb{N}} \|\sigma^{(m,k), \theta}_n f\|.
\]
Obviously, if \(\theta(x) := (1 - |x|) \vee 0\), then we get the Fejér means.

We have seen in [66] that if \(\sigma^{(m,k)}_n\) is bounded on a quasi-normed space then so is \(\sigma^{(m,k), \theta}_n\). Let \(X\) and \(Y\) be two complete quasi-normed spaces of measurable functions and \(L_\infty\) be continuously and densely embedded into \(X\). Suppose that if \(0 \leq f \leq g\), \(f, g \in Y\) then \(\|f\|_Y \leq \|g\|_Y\). Moreover, if \(f_n, f \in Y\), \(f_n \geq 0\) (\(n \in \mathbb{N}\)) and \(f_n \not\to f\) a.e. as \(n \to \infty\), then \(\|f - f_n\|_Y \to 0\). Note that the spaces \(L_p\) and \(L_{p,\infty}\) (\(0 < p \leq \infty\)) satisfy these properties.
Theorem 17. If (5) is satisfied and \( \sigma^{(m,k)} : X \to Y \) is bounded, then \( \sigma^{(m,k),\theta} \) is also bounded, i.e.,
\[
\|\sigma^{(m,k),\theta} f\|_Y \leq C\|f\|_X \quad (f \in X).
\]

The next result follows easily from Theorem 15 (see [66]).

Theorem 18. If \( m \geq -1 \), \( |k| \leq m + 1 \) and (5) is satisfied then
\[
\|\sigma^{(m,k),\theta} f\|_p \leq C_p\|f\|_{H_p} \quad (f \in H_p)
\]
for all \( 1/2 < p \leq \infty \),
\[
\|\sigma^{(m,k),\theta} f\|_{1/2,\infty} \leq C\|f\|_{H_{1/2}} \quad (f \in H_{1/2})
\]
and
\[
\sup_{\rho>0} \rho \lambda(\sigma^{(m,k),\theta} f > \rho) \leq C\|f\|_1 \quad (f \in L_1).
\]

Corollary 11. If \( m \geq -1 \), \( |k| \leq m + 1 \) and (5) is satisfied then \( f \in L_1 \) implies
\[
\sigma^{(m,k)} f \to f \quad \text{a.e. as } n \to \infty.
\]

One can show easily that \( \theta \)-summation includes Weierstrass, Picar, Bessel, Riesz, de La Vallée-Poussin, Rogosinski and Riemann summations. For the details see Weisz [66].

4.5. Strong summability. Taking absolute value in (4) we obtain the strong summability. More generally, we consider the convergence of the means
\[
(\frac{1}{n} \sum_{k=0}^{n-1} |s_k f(x) - f(x)|^r)^{1/r}.
\]

Strong summability was considered first by Hardy and Littlewood [21] for trigonometric Fourier series. They verified that these means tend to 0 a.e. as \( n \to \infty \), whenever \( f \in L_p \) (\( 1 < p < \infty \)). This result was generalized for \( L_1 \) functions and for \( r = 2 \) by Marcinkiewicz [30] and for all \( r > 0 \) by Zygmund [75].

For Walsh-Fourier series the strong summability was shown by Schipp [41, 45] for \( r = 2 \) and by Rodin [35, 34] for \( r > 0 \) and for \( BMO \) means. Now we formulate these results for Ciesielski-Fourier series.

Let
\[
S^{(m,k),(r)} \nu := \left( \frac{1}{n} \sum_{j=1}^{n} |s_j^{(m,k)} f(x)|^r \right)^{1/r} \quad (n \in \mathbb{N})
\]
be the strong means and
\[
S^{(m,k),(r)} \ast f := \sup_{n \geq 1} S^{(m,k),(r)} \nu f
\]
be the strong maximal operator, where \( 0 < r < \infty \). In [64] we extended strong summability to Ciesielski-Fourier series.

Theorem 19. If \( m \geq -1 \), \( |k| \leq m + 1 \) and \( 0 < r \leq 2 \) then
\[
\|S^{(m,k),(r)} \ast f\|_{L_\infty} \leq C\|f\|_\infty \quad (f \in L_\infty)
\]
and
\[
\sup_{\rho>0} \rho \lambda(S^{(m,k),(r)} \ast f > \rho) \leq C\|f\|_1 \quad (f \in L_1).
\]
The next corollary follows easily by interpolation.

**Corollary 12.** If \( m \geq -1, \; |k| \leq m + 1, \; 0 < r \leq 2 \) and \( 1 < p < \infty \) then
\[
\| S_s^{(m,k),(r)} f \|_p \leq C_p \| f \|_p \quad (f \in L_p).
\]

Note that the strong maximal operator is not bounded from \( H_1 \) to \( L_1 \) (see Schipp and Simon [38]). The weak type \((1,1)\) inequality in Theorem 17 and the usual density argument imply

**Corollary 13.** If \( m \geq -1, \; |k| \leq m + 1 \) and \( 0 < r \leq 2 \) then \( f \in L_1 \) implies
\[
\left( \frac{1}{n} \sum_{j=1}^{n} |s_j^{(m,k)} f(x) - f(x)|^r \right)^{1/r} \to 0 \quad \text{for a.e. } x \in [0,1) \quad \text{as } n \to \infty.
\]

**4.6. Multiplier theorems.** For simplicity, in this subsection we suppose that
\[
(f, c_j^{(m,k)}) = 0 \quad \text{for} \quad j = |k| - m, \ldots, 1.
\]

The multiplier operator generalizes the operator \( T_n^{(m,k)} \). For a given multiplier \( \lambda = (\lambda_j, j = 2, \ldots) \) where the \( \lambda_j \)'s are real numbers, the **multiplier operators** are defined by
\[
T_\lambda^{(m,k)} f := \sum_{j=2}^{\infty} \lambda_j (f, c_j^{(m,k)}) c_j^{(m,-k)}
\]
if the sum exists and by
\[
T_{\lambda,n}^{(m,k)} f := \sum_{j=2}^{n} \lambda_j (f, c_j^{(m,k)}) c_j^{(m,-k)} \quad (n \in \mathbb{N}),
\]
where \( f \in L_1 \).

The Marcinkiewicz multiplier theorem is generalized for Ciesielski systems in the next theorem (see Weisz [69]).

**Theorem 20.** Assume that \( m \geq -1, \; |k| \leq m + 1 \) and \( f \in L_p \) \((1 < p < \infty)\). If
\[
|\lambda_i| \leq C, \quad \sum_{j=2^i+1}^{2^i+1} |\lambda_j - \lambda_{j+1}| \leq C \quad (i \in \mathbb{N}) \quad (6)
\]
then \( T_\lambda^{(m,k)} f \in L_p \) and
\[
\| T_\lambda^{(m,k)} f \|_p \leq C_p \| f \|_p.
\]

The theorem can be proved by using Theorem 12 similarly to the trigonometric system (see Zygmund [76, II. p. 232]). The result for Vilenkin-Fourier series is due to Young [74].

The multiplier theorem is extended to Hardy spaces in [69]. Note that (6) follows from (7).

**Theorem 21.** Assume that \( m \geq -1 \) and \( |k| \leq m + 1 \). If
\[
|\lambda_n| \leq C, \quad \sup_{2^n+1 \leq j \leq 2^n+1} j|\lambda_j - \lambda_{j+1}| \leq C \quad (n \in \mathbb{N}) \quad (7)
\]
and
\[ \sum_{j=2^n+1}^{2^{n+1}-2} j|\lambda_j - 2\lambda_{j+1} + \lambda_{j+2}| \leq C \quad (n \in \mathbb{N}) \]
then, for \(1/2 < p < \infty\),
\[ \| \sup_{N \in \mathbb{N}} |T_{\lambda, 2^N}^{(m,k)} f|\|_p \leq C_p \| f \|_{H_p} \quad (f \in H_p). \] (8)

Some other multiplier theorems for the Walsh system and for Hardy spaces can be found in Kitada [25], Onnewer and Quek [32], Simon [48, 49, 50] and Daly and Fridli [14, 16, 15].

If the multiplier \(\lambda\) is piecewise linear then we can prove a stronger result ([69]).

**Theorem 22.** Assume that \(m \geq -1\) and \(|k| \leq m + 1\). If (7) is satisfied and
\[ \lambda_j - 2\lambda_{j+1} + \lambda_{j+2} = 0 \quad \text{for all} \quad j = 2^n + 1, \ldots, 2^{n+1} - 2 \quad (n \in \mathbb{N}) \]
then (8) holds for all \(p_0 < p < \infty\).

For examples of multipliers we introduce
\[ \lambda_j^{(1)} := \frac{j - 1}{2^n} \quad \text{if} \quad 2^n + 1 \leq j \leq 2^{n+1} \quad (n \in \mathbb{N}) \]
and
\[ \lambda_j^{(2)} := \frac{2^n}{j - 1} \quad \text{if} \quad 2^n + 1 \leq j \leq 2^{n+1} \quad (n \in \mathbb{N}). \]

For more general multipliers we refer to [69]. It is easy to see that \(\lambda^{(1)}\) satisfies the conditions of Theorems 21 and 22 and, moreover, \(\lambda^{(2)}\) fulfills the conditions in Theorem 21. These two multipliers are used in [69] to prove some inequalities for the Sunouchi operators.

**Remark.** For the multi-dimensional versions of the theorems presented in this paper see [58, 63, 68, 60, 66, 65, 70, 61]. To shorten the paper we do not give these extensions here.

**References**


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