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RESEARCH PROBLEMS OF JERZY ZABCZYK

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Abstract. In the paper we present a selected variety of problems studied by Professor Jerzy Zabczyk. Important part of Prof. Zabczyk's scientific activity was devoted to his PhD students. He has promoted 9 PhD students: Tomasz Bielecki, Jarosław Sobczyk, Łukasz Stettner and Gianmario Tessitore work mostly in control and its applications to mathematical finance, whereas the research of Anna Chojnowska-Michalik, Wojciech Jachimiak, Anna Milian, Szymon Peszat and Anna Rusinek is concentrated mostly on stochastic evolution equations. Description of research problems of Prof. Jerzy Zabczyk starts from infinite-dimensional (deterministic) control problems, stochastic control problems: optimal choice problems, stopping time problems, to come first to mathematics of finance and then to stochastic analysis: stochastic evolution and partial differential equations. We show main results and point out their importance. The paper is based on the presentation of the authors during a special session at the conference.

1. Infinite-dimensional control problems. We start our description from two papers of Professor Zabczyk devoted to discrete time linear quadratic control problems on Hilbert spaces. Namely, in [63] and [64] the discrete time linear controlled system

$$x_{i+1} = \Phi x_i + D u_i \tag{1}$$

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is studied. Above x_i and u_i take values in a Hilbert spaces H and U, respectively, and Φ and D are linear bounded operators. The system is considered with the cost functional

$$J_N(x_0, u_0, u_1, \ldots, u_{N-1}) = \sum_{i=0}^{N-1} \left[(Qx_i, x_i) + (Ru_i, u_i)\right]. \tag{2}$$

The operators Q and R are linear bounded self-adjoint, positive semidefinite and the operator R is invertible. Given initial state x_0 and time horizon N the goal is to find an optimal strategy; i.e. a control sequence $(u_0, u_1, \ldots, u_{N-1})$ taking values in U for which the cost functional (2) is minimal. The main result of [63] concerns limit properties of solution to the control problem when time horizon increases to infinity. The solution of infinite horizon problem is closely related with solution K to the algebraic Riccati equation

$$K = \Phi^* K(I + DR^{-1}D^*K)^{-1}\Phi + Q. \tag{3}$$

Existence of a solution to (3) is obtained in [63] using a nice Krasnoselskii method. In the paper [63] necessary and sufficient conditions for the existence of positive semidefinite solutions to algebraic Riccati equations are given. In the next paper [64] a stochastic version of the linear quadratic problem is considered. In the system three kinds of noises are introduced: control dependent, state dependent and purely additive noises. Infinite horizon control problem as in the deterministic case leads to the algebraic Riccati equation, which is again solved using Krasnoselskii's method. Next natural step was to study continuous time models in Hilbert spaces. Namely, in [66] Prof. Zabczyk studies the linear equation

$$\dot{x} = Ax + Bu, \qquad x(0) \in H. \tag{4}$$

Above A with domain $\mathcal{D}_A$, is the generator of a C_0-semigroup (T_t), and B is a bounded linear operator from the Hilbert space U of control parameters into H. The algebraic Riccati equation corresponding to linear quadratic problem is of the form

$$2(PAx, x) + (Cx, x) - (PBR^{-1}B^*Px, x) = 0, \qquad x \in \mathcal{D}_A, \tag{5}$$

where C and R are linear bounded operators with R additionally positive definite. We are looking for a positive semidefinite linear operator P satisfying (5). We say that the pair (A, B) is *stabilizable* iff $A - BK$ is stable for some linear operator K. The pair (C, A) is *detectable* iff $A - SC$ is stable for some linear operator S. The main result of [66] states that

Theorem 1.1. *If the pair (C, A) is detectable then equation (5) has at most one solution and if P is the solution to (5), then the operator $A - BR^{-1}B^*P$ is stable. If, in addition, the pair (A, B) is stabilizable then equation (5) has exactly one solution.*

The study of a C_0-semigroup (T_t) with the generator A leads Prof. J. Zabczyk to an interesting example presented in [65]. Below $\sigma(A)$ denotes the spectrum of the operator A.

Theorem 1.2. *For any two real numbers $\omega_s < \omega_0$ there exists a C_0-semigroup (T_t) on a Hilbert space H such that*

$$\omega_s = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \quad \textit{and} \quad \|T_t\| = \mathrm{e}^{\omega_0 t},\ t \geq 0. \tag{6}$$

Theorem 1.2 shows that a well known result which says that if the generator A is a bounded operator, then the semigroup $T_t = \mathrm{e}^{At}$ satisfies $\|T_t\| \le M\mathrm{e}^{\omega t}$ for any $\omega > \omega_s$, cannot be extended to all C_0-semigroups.

Further studies of Prof. J. Zabczyk concern stabilizability of infinite-dimensional systems (see papers: [54] with A. J. Pritchard and [75]). Consider dynamical system (4) with the so-called mild solution $(x(t))$ given by the formula

$$x(t) = T_t x(0) + \int_0^t T_{t-s} Bu(s)\,\mathrm{d}s. \tag{7}$$

The system is said to be *weakly* (*asymptotically, exponentially*) *stabilizable* if there exists a bounded linear operator $K \in L(H, U)$ such that the system with control $u(t) = Kx(t)$ is weakly (asymptotically, exponentially) stable. We have (see [54]):

THEOREM 1.3. *The following conditions are equivalent*:

(a) *The system is exponentially stabilizable.*
(b) *For every initial condition $x(0) \in H$ there exists a control $u(\cdot)$ such that*
$$\int_0^\infty (\|x(t)\|^2 + \|u(t)\|^2)\,\mathrm{d}t < \infty.$$
(c) *There exists a nonnegative operator P satisfying the Riccati equation*
$$2(PAx, x) + (x, x) - (PZx, x) = 0, \quad x \in \mathcal{D}(A).$$
(d) *For every initial condition $x(0) \in H$ there exists a control $u(\cdot)$ such that the control $u(t)$ and the corresponding solution $x(t)$ converge to 0 exponentially as $t \to \infty$.*

Assume now that $H = H_1 \oplus H_2$, and that a C_0-semigroup (T_t) is invariant in H_1 and H_2; i.e. $T_t H_1 \subset H_1$, $T_t H_2 \subset H_2$, for $t \ge 0$. Let $T_t^1 = T_{t|H_1}$, $T_t^2 = T_{t|H_2}$ and A_1 be the generator of (T_t^1). We have (see [54]):

THEOREM 1.4. *If (T_t^2) is exponentially stable, then $(x(t))$ is exponentially stabilizable iff $\dot{x}_1 = A_1 x_1 + B_1 u$ is exponentially stabilizable. Furthermore, if the linear system on H is exactly null-controllable, then it is also exponentially stabilizable. If, in addition, the operator A generates a group, then exact null-controllability is a necessary and sufficient condition for exponential stabilizability with an arbitrary prefixed exponential decay rate.*

Linear systems often are used to approximate nonlinear systems. In [75] Professor J. Zabczyk is studying stabilizability of a nonlinear system using its linearization. More precisely, consider the nonlinear system on $\mathbb{R}^n$,

$$\dot{x} = f(x, u), \tag{8}$$

and its linearization

$$\dot{x} = Ax + Bu, \tag{9}$$

where $A = D_x f(0,0)$ and $B = D_u f(0,0)$. Let $x(t; x_0, u)$ denote the value of solution of (8) with initial condition x_0. System (8) is said to be *exponentially stabilizable* if there exist an admissible feedback control $u(\cdot)$ and constants $\alpha, M, \delta > 0$ such that, for all $t \to 0$ and all x_0, $|x_0| \le \delta$, we have

$$|x(t; x_0, u)| \le M\mathrm{e}^{-\alpha t}|x_0|, \qquad t \ge 0. \tag{10}$$

We have the following (for more details see [75]):

THEOREM 1.5. *System* (8) *is exponentially stabilizable if and only if its linearization* (9) *is exponentially stabilizable. Moreover, if system* (9) *is stabilizable then as a stabilizing feedback for* (8) *we can always take a linear one. In addition, for an arbitrary feedback* $u(\cdot)$ *exponentially stabilizing* (8)*, such that* (10) *holds, we have*

$$\frac{\alpha}{M} \leq \frac{|A|}{d}, \tag{11}$$

where $d = \sup_{x\in R(A), \|x\|=1} \rho(x, R(B))$, *with* $R(A)$ *and* $R(B)$ *denoting ranges of transformations* A *and* B *respectively, and using the convention* $0/0 = a/0 = +\infty$ *for* $a > 0$.

We say that 0 is a *global attractor* for (8), a *Lyapunov stable equilibrium* for (8), or an *asymptotically stable equilibrium* for (8), if there exists a feedback $u(\cdot)$ such that 0 is, respectively, a global attractor, a Lyapunov stable equilibrium, or an asymptotically stable equilibrium for the resulting closed-loop system (8). We have (see [75]):

THEOREM 1.6. *If* 0 *is a global attractor for* (8)*, an asymptotically stable equilibrium for* (8)*, or a Lyapunov stable equilibrium for* (8) *and* $n \leq 2$*, then* f *transforms an arbitrary neighborhood of* $(0,0) \in \mathbb{R}^n \times U$ *onto a neighborhood of* 0 *in* $\mathbb{R}^n$.

Further contributions of Prof. J. Zabczyk concern the so-called null controllability with vanishing energy. We consider again system (4) with initial condition $x(0) = x$ and mild solution (7) the value of which at time t we denote by $x(t; x, u)$ to point out initial state x and chosen control u. We say that the system is *null controllable with vanishing energy* (NCVE for short) if it is null controllable and for any $x \in H$ there exist a sequence of controls u_n and a sequence of times t_n such that $x(t_n; x, u_n) = 0$ for any positive integer n and $\int_0^{t_n} |u_n(s)|^2\, ds \to 0$ as $n \to \infty$.

In the paper [50] (jointly with E. Priola) the following set of assumptions called *Hypothesis 1.1* is introduced:

(i) There exists a sequence $(\lambda_n) \subset \sigma(A)$ such that each λ_n is isolated in $\sigma(A)$, and

$$\lim_{n\to\infty} \mathrm{Re}(\lambda_n) = s(A) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\},$$

(ii) There exist closed linear subspaces H_s and H_u invariant for e^{tA}, $t \geq 0$, such that their direct sum is H, and, moreover,

(a) the semigroup (e^{tA}) restricted to H_s is exponentially stable on H_s,

(b) the set of all generalized eigenvectors of A (i.e. such x that for some positive integer k we have $(\lambda - A)^k x = 0$) contained in H_u is linearly dense in H_u.

The following three theorems are shown:

THEOREM 1.7. *Assume that system* (4) *is null controllable and Hypothesis 1.1 holds. Then the following statements are equivalent*:

(a) *the system is NCVE*;
(b) $s(A) \leq 0$.

THEOREM 1.8. *Assume that the system is exactly controllable and Hypothesis 1.1 holds. Then the following statements are equivalent*:

(a) *the system is exactly controllable with vanishing energy* (*ECVE*),
(b) $\sigma(A) \subset \{i\lambda : \lambda \in R\}$.

THEOREM 1.9. *A null controllable system* (4) *is NCVE if and only if the algebraic Riccati equation $PA + A^*P - PBB^*P = 0$, $P \geq 0$, has a unique bounded nonnegative solution $P = 0$.*

These results are then extended to boundary control systems in the joint paper with L. Pandolfi and E. Priola [38].

2. Stochastic control theory. Contributions of Prof. J. Zabczyk to stochastic control theory consist in results concerning optimal selection problems, stopping problems, filtering theory and stochastic controllability of linear systems. In the paper [13] written together with R. Cowan the following problem is studied. We want to select an apartment in fixed time T. Opportunities to inspect apartments occur at the epochs of a stationary Poisson process with known intensity λ and we have to decide at that epoch whether or not to accept the apartment. All permutations of apartments ranks are equally likely and independent of the Poisson process. The objective is to maximize the chance of selecting the best apartment from those (if any) available in the interval $[0, T]$. We call the m-th epoch *serious* if the apartment which has occurred at this epoch is better comparing to previous apartments. The optimal selection is characterized by the following (for details see [13]):

THEOREM 2.1. *For the optimal selection of an apartment one must accept the first apartment whose decision variable* (m, t) (*the m-th epoch is serious and t is the time that remains*) *satisfies the inequality $\lambda t < x_m$, where x_m is the unique solution of the equation*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!(m+n)} = \sum_{n=1}^{\infty} \frac{x^n}{n!(m+n)} \sum_{k=1}^{n} \frac{1}{k+m-1}\,. \tag{12}$$

Optimal selection problems were earlier studied for discrete time. Paper [13] originated intensive studies on various versions of selection problems in continuous time. The solution to the problem was based on optimal stopping of a Markov chain. More general stopping problem was considered in the paper with Ł. Stettner [56].

Assume (g_t) is a right continuous bounded process adapted to complete right continuous filtration (F_t), and $\alpha > 0$. We want to characterize the right continuous version of the process

$$v_t := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\big\{\mathrm{e}^{-\alpha(\tau - t)} g_\tau \mid F_t\big\}, \tag{13}$$

where essential supremum is over all (F_t) stopping times. Notice that $\mathrm{e}^{-\alpha t} v_t$ is the so-called Snell envelope of the process $\mathrm{e}^{-\alpha t} g_t$. In [56] the process (v_t) is approximated by using generalized penalty equation, which consists in finding for $\beta > 0$ a right continuous process (v_t^β) satisfying the equation

$$v_t^\beta = \mathbb{E}\Big\{\int_0^\infty \mathrm{e}^{-\alpha s}\big[\beta(g_{s+t} - v_{s+t}^\beta)^+\big]\,\mathrm{d}s \;\Big|\; F_t\Big\}. \tag{14}$$

We have (see [56]):

THEOREM 2.2. *For each $\beta > 0$ there is a unique solution (v_t^β) to equation* (14) *and moreover, $v_t^\beta \uparrow v_t$, $\mathbb{P}$-a.e. as $\beta \to \infty$.*

Another result of Prof. J. Zabczyk (see [70]) concerns the so-called Dynkin game for symmetric Markov processes. Assume that we are given a symmetric Markov process on a state space E with reference measure m and that g, h are elements of suitable C_0-regular Dirichlet space $\mathcal{E}$ such that $g(x) \leq h(x)$, m-a.e. We would like to characterize

$$v(x) = \inf_{\tau} \sup_{\sigma} \mathbb{E}_x \big\{ \mathrm{e}^{-\alpha \tau \wedge \sigma} \big(\chi_{\{\tau \leq \sigma\}} h(x_\tau) + \chi_{\{\sigma < \tau\}} g(x_\sigma) \big) \big\} \tag{15}$$

where inf and sup are over all stopping times. Assume furthermore the following *separability condition*: there exist two α-potentials w_1, w_2 such that $g \leq w_1 - w_2 \leq h$, m-a.e. Then using a suitable version of the penalty method introduced above we can show that (see [70]):

THEOREM 2.3. *For $x \in E \setminus N$, where N is properly exceptional, $v(x)$ is the value function of the stopping game and optimal saddle point strategies are of the form $\hat{\tau} = \inf\big\{ s \geq 0 : v(x_s) = h(x_s) \big\}$, $\hat{\sigma} = \inf\big\{ s \geq 0 : v(x_s) = g(x_s) \big\}$.*

In 1983 Prof. J. Zabczyk delivered a section lecture at the International Congress of Mathematicians in Warsaw. His talk was dedicated to stopping problems in stochastic control. Significant part of this talk was devoted to continuity of the optimal stopping value function (see [71]). Given a Feller Markov process (x_t) on a locally compact separable metric space, a continuous bounded function f and $\alpha > 0$, we consider

$$v(x) = \inf_{\tau} \mathbb{E}_x \left\{ \mathrm{e}^{-\alpha \tau} f(x_\tau) \right\}, \tag{16}$$

where infimum is over all stopping times. Using the methods of discretization considered by V. Mackevičius (1973), duality method of M. Bismut (1982), penalty method introduced for Markov processes by M. Robin (1978) and then extended in [56] or variational inequalities we can show continuity of the function v. This is a major step to study the so-called impulse control problems which consist of a sequence of pairs (τ_n, ξ_n), where $\tau_1 \leq \tau_2 \leq \tau_3 \leq \ldots$, is a sequence of stopping times and $\xi_n \in \Gamma(x_{\tau_n})$ is a sequence of decision variables or shifts of the process (x_t) at these random times. With each shift from x to $y \in \Gamma(x)$ we associate the cost $c(x, y)$. Furthermore we have also a running cost f. The problem is to minimize discounted cost functional. The corresponding Bellman equation is of the form

$$v(x) = \inf_{\tau} \mathbb{E}_x \Big\{ \int_0^{\tau} \mathrm{e}^{-\alpha s} f(x_s)\, \mathrm{d}s + \mathrm{e}^{-\alpha \tau} M v(x_\tau) \Big\} \tag{17}$$

with $Mh(x) = \inf_{y \in \Gamma(x)} (c(x, y) + h(y))$. Assuming that $\alpha > 0$, functions f and $c \geq 0$ are continuous and bounded, c is separated from 0 plus suitable continuity of $\Gamma(x)$, we can show that the value function is a continuous solution to (17) and with the use of (17) we can construct optimal impulse strategies. It is worth to point out that the results are no longer true whenever $\alpha = 0$, or the state space is not locally compact. The case of stopping problems on Polish spaces was studied by Prof. J. Zabczyk in [77]. Natural situation of stopping problem on a Polish space appears in the case when we have partial observation. Assume we control using impulses a Feller Markov process (x_t) on locally compact separable metric space E with partial observation; i.e. we observe the process

$$y_t = \int_0^t f(x_s^V)\, \mathrm{d}s + W_t, \tag{18}$$

where (x_t^V) denotes the trajectory of the controlled process and W_t is an independent Brownian motion. Consequently, our impulse strategy $V = (\tau_n, \xi_n)$ consists of an increasing sequence of stopping times (τ_n) adapted to the filtration $Y^t := \sigma(y_s, s \le t)$ and $\xi_n \in \Gamma(x_{\tau_n})$ are Y^{τ_n} measurable. We want to minimize the cost functional

$$J(V) = \mathbb{E}_x\Big\{\int_0^\infty \mathrm{e}^{-\alpha s} f(x_s^V)\,\mathrm{d}s + \sum_{i=1}^\infty c(x_{\tau_i}^V, \xi_i)\Big\}. \tag{19}$$

It is shown in the paper with G. Mazziotto, Ł. Stettner and J. Szpirglas [31] that whenever $\alpha > 0$ and functions f, h and $c \ge 0$ are continuous bounded, c is separated from 0 plus suitable continuity of $\Gamma(x)$, the value function for partially observed impulse control problem is continuous. Furthermore there are optimal strategies, which can be constructed with the use of a suitable version of Bellman equation (17) defined on the space $\mathcal{P}(E)$ of probability measures on E, where the state process is replaced by filtering process (π_t) by taking values in $\mathcal{P}(E)$ and defined by the formula

$$\pi_t(A) = \mathbb{P}\big\{x_t \in A \,|\, Y^t\big\} \tag{20}$$

for any Borel subset A of E. The filtering process (π_t) plays a fundamental role in the studies on partially observed nonlinear models. In the joint paper with N. U. Ahmed and M. Fuhrman [1] it is assumed that the state process X is in a Hilbert space H with given a finite-dimensional observation process Y. Let

$$\mathrm{d}X = (AX + F(X))\,\mathrm{d}s + B\,\mathrm{d}W(t) \tag{21}$$

with $X(0) = X_0 \in H$, where A generates C_0-semigroup (S_t), such that $\|S_t\| \le M\mathrm{e}^{-\omega t}$, $\int_0^\infty S_t B B^* S_t^*\,\mathrm{d}t$ is nuclear, and F is Lipschitz continuous. We assume that finite-dimensional observation $(Y(t))$ is given by the formula

$$Y(t) = \int_0^t G(X(s))\,\mathrm{d}s + CW(t), \tag{22}$$

where G is bounded continuous and C is a surjective continuous linear operator. In the paper [1] the nonlinear filtering and Zakai equations for the above partially observed model are studied. It is shown that the process (π_t) is a solution to the nonlinear filtering equation and given solution (σ_t) to the Zakai equation we obtain filtering process $\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}$ for any bounded Borel measurable function f.

We complete the section devoted to stochastic control with a result of Prof. Zabczyk concerning controllability of linear stochastic systems (see [68]). Consider the following linear stochastic system in $\mathbb{R}^n$,

$$\mathrm{d}x = Ax\,\mathrm{d}t + Bu\,\mathrm{d}t + C\,\mathrm{d}w_t \tag{23}$$

with $x(0) = x_0$, where (w_t) is a Brownian motion. The system is *weakly controllable* if for any x_0 and any nonempty open set V there is a control (u_t) such that

$$\mathbb{P}_{x_0}\{x(t) \in V \quad \text{for some } t > 0\} > 0. \tag{24}$$

The system is *strongly controllable* iff $\mathbb{E}_{x_0} T_V < \infty$, where $T_V = \inf\{s \ge 0 : x_s \in V\}$.

The main result of [68] is:

THEOREM 2.4. *The system is weakly controllable iff*

$$\operatorname{rank}\left\{B, AB, A^2B, \dots, A^{n-1}B, C, AC, A^2C, \dots, A^{n-1}C\right\} = n. \tag{25}$$

The system is strongly controllable iff it is weakly controllable and the matrix A is stable.

3. Mathematics of finance. The interest of Prof. J. Zabczyk in mathematics of finance started in late 80-ties although he published papers in this area much later. Significant contribution of Prof. J. Zabczyk concerns term structure models with jumps. In the paper [24] (with J. Jakubowski) a generalization of the so-called Heath–Jarrow–Morton model (HJM) of forward rate curve is studied. Namely, let $P(t,\theta)$ be the market price at time t of a bond paying 1 at maturity time θ. Assume that $P(t,\theta) = \exp\bigl(-\int_t^\theta f(t,s)\,\mathrm{d}s\bigr)$, where $f(t,s)$ is called a *forward rate curve*. We assume that forward rate curve is a solution to the equation

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)\,\mathrm{d}s + \int_0^t (\sigma(s,\theta),\,\mathrm{d}Z_s) \tag{26}$$

with (Z_t) being a Lévy process taking values in a separable Hilbert space. Let $r(\theta) := f(\theta,\theta)$ be a short term interest rate and let $B_t = \exp\bigl(\int_0^t r(v)\,\mathrm{d}v\bigr)$ be the value at time t of a unit located on a bank account at time 0. The *HJM model postulate* is the requirement that the discounted bond price processes

$$\hat{P}(t,\theta) = \frac{P(t,\theta)}{B_t}, \qquad \theta \in [0,T], \tag{27}$$

are local martingales. We impose the following two assumptions:

(H1) The processes α and σ are predictable and with probability one they have bounded trajectories (however the bound can depend on ω).
(H2) For arbitrary $r > 0$, the function b (Laplace transformation of jump measure ν of Z restricted to the complement of unit ball) is bounded on $\{u : |u| \le r, b(u) < \infty\}$.

We have (see [24]):

THEOREM 3.1. *Assume that* (H1) *holds.*

(i) *If the HJM postulate holds, then for arbitrary $\theta \le T$,*

$$(\star) \quad \int_T^\theta \sigma(t,v)\,\mathrm{d}v \in B := \{u : b(u) < \infty\}, \quad \mathbb{P}\text{-}a.s. \text{ for almost all } t \in [0,\theta].$$

(ii) *Assume* (H2) *and that* $(\star)$ *holds for all $\theta \le T$. Then the HJM postulate is satisfied if and only if the following HJM condition holds*:

$$\int_t^\theta \alpha(t,v)\,\mathrm{d}v = J\Bigl(\int_t^\theta \sigma(t,v)\,\mathrm{d}v\Bigr) \tag{28}$$

for almost all $t \in [0,\theta]$, where $J(u) := \frac{1}{t}\log \mathbb{E}\,\mathrm{e}^{-(u,Z(t))}$ is the Lévy exponent of Z.

In the paper with M. Barski and J. Jakubowski [3] a bond market is considered in which the forward rate curve $f(t,\theta)$ is the solution to the equation

$$\mathrm{d}f(t,\theta) = \alpha(t,\theta)\,\mathrm{d}s + \sum_{i=1}^{\infty} \sigma^i(t,\theta)\,\mathrm{d}W_t^i, \qquad t \le \theta \le \bar{T}, \tag{29}$$

where $(W^i(t))$ is a sequence of independent Brownian motions. We say that the market is *complete* on $[0,T]$ whenever any bounded contingent claim adapted to the filtration generated by sequence $(W^i(t))$ till time T can be replicated; i.e. one can construct a portfolio consisting of bonds the value of which at time T is equal to that contingent claim. The main result of [3] is stated as follows:

THEOREM 3.2. *The bond market is not complete on* $[0,T]$ *for any* $T \le \bar{T}$.

The authors formed a counterexample which was rather surprising since in the other papers, under different set of assumptions, the completeness of such market was shown.

In the paper with M. Barski [4] the interest rate model with forward rate $f(s,\theta)$ satisfying equation (26) with $\sigma(s,\theta) = \lambda(s,\theta)f(s-,\theta)$ is studied. It was known that when the Lévy process L is a Brownian motion, then equation (26) does not have a global solution. If L is a Lévy process then under two sets of assumptions: one in terms of certain properties of the derivative of J defined in Theorem 3.1 and the other requiring absence of a Gaussian part and positive jumps we can find a global solution to (26).

In the next paper (see [5]), joint with M. Barski, a general equation for forward rate is studied. It is of the form

$$\mathrm{d}f(t,\theta) = \Big[\frac{\mathrm{d}}{\mathrm{d}\theta}f(t,\theta) + F(f(t))(\theta)\Big]\,\mathrm{d}t + G(f(t-)(\theta)\,\mathrm{d}L(t). \tag{30}$$

It is assumed that $G(f)(\theta) := g(\theta, f(\theta))$,

$$F(f)(\theta) := J'\Big(\int_0^{\theta} g(v, f(v))\,\mathrm{d}v\Big) g(\theta, f(\theta))$$

with

$$J'(z) = -az + \int_{\mathbb{R}} \big(\mathrm{e}^{-zy} - 1 + \chi_{\{(-1,1)\}}(y)zy\big)\,\nu(\mathrm{d}y), \qquad z \in \mathbb{R}, \tag{31}$$

$a \in \mathbb{R}$, $\int_{\mathbb{R}}(y^2 \wedge 1)\,\nu(\mathrm{d}y) < \infty$. In the paper the existence of local and global solutions of equation (30) is studied in various function spaces under the assumptions that g is general or g is linear with respect to the second variable (in particular, when we have the so-called Musiela parametrization).

4. Stochastic analysis and SEEs. Professor J. Zabczyk exerted a huge influence on the development of Stochastic Evolution Equations (SEEs for short). He is the author of many research papers and five books. In particular, together with Giuseppe Da Prato he wrote the first monograph [15] on SEEs, then again with Giuseppe Da Prato a book [17] on ergodic properties of SEEs, and more recently with Szymon Peszat the first monograph [47] on SEEs with Lévy noise. In this section we would like to sketch the main achievements of Prof. J. Zabczyk as well as present the works of his PhD students devoted to SEE which were supervised by Prof. J. Zabczyk. It is our pleasure to admit that the problems suggested by Prof. J. Zabczyk were natural and important. Similar problems still attract some attention.

4.1. Generalized Ornstein–Uhlenbeck equation. Prof. J. Zabczyk's PhD thesis was prepared in 1969 under supervision of Prof. Zbigniew Ciesielski in University of Gdańsk. It focussed on certain issues of the potential theory for processes with independent increments. To some extent publication [69] of Prof. J. Zabczyk from 1983 is related to the topic of his thesis. Namely he found necessary and sufficient conditions for the existence of an invariant probability for a Markov solution of the stochastic ordinary differential equation (generalized Ornstein–Uhlenbeck equation with jump noise)

$$\mathrm{d}X = AX\,\mathrm{d}t + B\,\mathrm{d}Z, \tag{32}$$

where $A \in M(d\times d)$, $B \in M(d\times n)$ are matrices, and Z is a Lévy process in $\mathbb{R}^n$. Moreover, the set of all such measures is described. The relation between stationarity and stability of A is investigated. For example, if the semigroup e^{At}, $t \geq 0$, is stable and $\mathbb{E}\,|Z_1| < \infty$, then there exists a unique invariant probability measure.

The Ornstein–Uhlenbeck equation (32) still is a source of nice problems. For example, in the recent joint work with E. Priola [52] the following result on absolute continuity of $X(t)$, $t > 0$, is shown.

THEOREM 4.1. *Assume that the following rank condition is satisfied*

$$\operatorname{rank}[B, AB, \ldots, A^{d-1}B] = d.$$

Assume also that the Lévy measure μ of Z is infinite and there is an $r > 0$ such that μ restricted to the ball $\{y \in \mathbb{R}^n : |y| \leq r\}$ is absolute continuous. Then, for any time $t > 0$ and initial value x, the law of the solution to (32) *starting at time* 0 *from x is absolutely continuous.*

The investigations of Prof. J. Zabczyk's paper [69] were developed by Anna Chojnowska-Michalik who considered the existence of invariant measure to the generalized Ornstein–Uhlenbeck equation

$$\mathrm{d}X = AX\,\mathrm{d}t + B\,\mathrm{d}Z, \tag{33}$$

where A is the generator of a C_0-semigroup S on a Hilbert space H, Z is a Lévy process taking values in a Hilbert space U, and B is a linear operator from U to H. Since A is in general an unbounded linear operator a proper definition of the solution should be formulated. In fact, usually in the theory of SEEs one considers the so-called *mild solution*. It is given by the *variation of constants* or *Duhamel formula*

$$X(t) = S_t X(0) + \int_0^t S_{t-s} B\,\mathrm{d}Z(s), \qquad t \geq 0.$$

If Z is a Wiener process with covariance Q, then an invariant measure exists iff

$$\int_0^\infty \|S_t B Q^{1/2}\|_{HS}^2\,\mathrm{d}t < \infty,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm (see paper [72] of Prof. J. Zabczyk). This is in fact iff condition for the existence of the integral $\int_0^\infty S_t B\,\mathrm{d}Z(t)$. Moreover, see [72, 15] for more details, any invariant measure is of the form

$$\rho * \mathcal{L}\Big(\int_0^\infty S_t B\,\mathrm{d}Z(t)\Big), \tag{34}$$

where ρ is any invariant measure for the semigroup S, $*$ is the convolution operator, and $\mathcal{L}(Y)$ denotes the law of a random element Y. A. Chojnowska-Michalik proved (see [11, 12]) that in the general Lévy case an invariant measure exists iff $\int_0^\infty S_t B\,\mathrm{d}Z(t)$ exists and then any invariant measure is of the form (34) for a large class of semigroups S. Moreover, if the semigroup is exponentially stable and B is a bounded operator, then $\int_0^\infty S_t B\,\mathrm{d}Z(t)$ exists if

$$\int_U \log_+ |x|\,\mu(\mathrm{d}x) < \infty,$$

where μ is the Lévy measure of Z.

Analytical properties of solutions to generalized Ornstein–Uhlenbeck equation (33) is one of the favorite research topics of Prof. J. Zabczyk. In particular he investigated the strong Feller property of the corresponding transition semigroup, its irreducibility, equivalence of transition probabilities, and recurrence. Let us recall that a transition semigroup (P_t) is *strong Feller* at $t_0 > 0$ if $P_{t_0} : B_b(H) \to C_b(H)$. If the transition semigroup is strong Feller and *irreducible at* t_0: $P_{t_0}\chi_O(x) > 0$ for all $x \in H$ and open O, then it is *regular*: that is for all $t, s \geq t_0$ and all $x, \tilde{x}$ the transition probability $P_t(x, \mathrm{y})$ is absolutely continuous with respect to the transition probability $P_s(\tilde{x}, \mathrm{y})$, and there is at most one invariant measure. The following result holds (see [17, 72]).

Theorem 4.2. *Assume that the system is linear with Gaussian noise. Let* $t_0 > 0$. *Then the following conditions are equivalent*:

(i) *the semigroup* (P_t) *is* t_0*-regular,*
(ii) $P_{t_0} : B_b(E) \mapsto C_b(E)$, *that is* (P_t) *is* t_0*-strong Feller,*
(iii) $\operatorname{Im} S_{t_0} \subset \operatorname{Im} Q_{t_0}^{1/2}$, *where* Q_t *is the covariance operator of* $P_t(x, \cdot)$.

Characteristics of linear system (33) with Gaussian noise are given often in terms of the corresponding control problem

$$\mathrm{d}Y_x^\psi(t) = \big(AY_x^\psi(t) + B\psi(t)\big)\,\mathrm{d}t,\ Y_x^\psi(0) = x, \qquad x \in H,\ \psi \in L^2(0,T;U). \tag{35}$$

The key idea of Prof. J. Zabczyk [72], then developed by G. Da Prato and J. Zabczyk [15], is to relate the *Reproducing Kernel Hilbert Space* $\mathcal{H}$ of X, RKHS for short, with (35). Let us recall that RKHS of a centered Gaussian r.v. X is the Hilbert space $\mathcal{H}$ such that $X = \sum_k X_k e_k$, where (e_k) is an orthonormal basis of $\mathcal{H}$ and (X_k) are independent standard real-valued Gaussian random variables. Roughly speaking X considered on RKHS has identity covariance. Covariance operator depends on the space on which X is considered, whereas its RKHS does not. Thus RKHS is the intrinsic characteristic of a Gaussian random variable. It is shown in [15, 72] that if X is the solution to the Ornstein–Uhlenbeck equation with Wiener noise, then, provided that X takes valued in a Hilbert space H, the random element $(X(t); t \geq 0)$ is Gaussian in $L^2(0,T;H)$. Moreover its RKHS is equal to the range of the control operator $J : L^2(0,T;U) \ni \psi \mapsto Y_0^\psi \in C([0,T];H)$, where Y_0^ψ is the solution to (35) starting from $x = 0$. Finally the scalar product on $\mathcal{H}$ is induced by the pseudo inverse operator J^{-1}:

$$\langle u, v\rangle_{\mathcal{H}} = \langle J^{-1}u, J^{-1}v\rangle_{L^2(0,T;U)}, \qquad u, v \in \mathcal{H} = J(H).$$

As a consequence of the representation above, X is irreducible if and only if the associated deterministic system is approximately controllable. Exact controllability is necessary and sufficient for the equivalence of transition probabilities of the stochastic system defined by (33).

Another problem studied by Prof. J. Zabczyk was the so-called law equivalence of stochastic linear systems. Namely, one would like to find conditions for equivalence of distributions (laws) on the space of trajectories of two linear systems

$$\mathrm{d}X = AX\,\mathrm{d}t + B\,\mathrm{d}W, \qquad \mathrm{d}\tilde{X} = \tilde{A}\tilde{X}\,\mathrm{d}t + B\,\mathrm{d}W,$$

The starting point was the result of S. M. Kozlov [28, 29], where $B = I$, and $A, \tilde{A}$ were differentiable operators of orders $2l$ and $2\tilde{l}$ on $\mathbb{T}^d$. S. M. Kozlov proved that the laws are mutually absolutely continuous iff $l = \tilde{l}$ and $A - \tilde{A}$ of order $l - d/2$. This result was generalized by S. Peszat [39, 40], J. Zabczyk [76], B. Goldys and S. Peszat [19], and more recently Ł. Kuciński [30].

Structural properties (for example strong Feller property) of linear equations with Lévy noise are investigated in joint works of Prof. J. Zabczyk with E. Priola. In [53] the following version of (33) is considered: A is a self-adjoint operator, B equals identity operator, $Z = \sum_n \beta_n Z_n e_n$, where (Z_n) are independent real valued α-stable processes, (β_n) is a sequence of real numbers, and (e_n) is an orthonormal basis of H of eigenvectors of A, and $Ae_n = -\gamma_n e_n$, with $\gamma_n > 0$. In this case (33) can be written as a sequence of independent one-dimensional equations

$$\mathrm{d}X_n = -\gamma_n X_n\,\mathrm{d}t + \beta_n\,\mathrm{d}Z_n, \qquad n \in \mathbb{N}. \tag{36}$$

Obviously $H \ni x \mapsto (\langle x, e_n\rangle_H) \in l^2$ defines an isometry between H and l^2. Given $x = (x_n) \in l^2$, let $X_x = (X_{n,x})$, where $X_{n,x}(0) = x_n$. In [53] the following result is shown:

Theorem 4.3.

(i) *For any $x \in l^2$, the process $X_x = (X_{x,n})$ defined by* (36) *lives in $H \equiv l^2$ iff*

$$\sum_n \frac{\beta_n^\alpha}{\gamma_n} < \infty. \tag{37}$$

Moreover, if (37) *holds, then X_x is stochastically continuous in l^2 and has trajectories in $L^p(0,T;l^2)$ for any $0 < p < \alpha$. Finally,* (37) *guarantees that for arbitrary $x \in l^2$, $T > 0$, and $0 < p < \alpha$, the support of $(X_x(T), (X_x(t); t \in [0,T]))$ is equal to $l^2 \times L^p(0,T;l^2)$.*

(ii) *Assume* (37)*. If*

$$\forall\, t > 0 \qquad \sup_{n\in\mathbb{N}} \frac{\mathrm{e}^{-\gamma_n t}\gamma_n^{1/\alpha}}{\beta_n} < \infty, \tag{38}$$

then for any $t > 0$ and $x, y \in l^2$ the distributions of $X_x(t)$ and $X_y(t)$ are absolutely continuous. Moreover, if (38) *does not hold for a certain $t > 0$, then there is an $x \in l^2$ such that the distribution of $X_x(t)$ is not absolutely continuous with respect to the distribution of $X_0(t)$.*

Prof. J. Zabczyk has been interested in harmonic analysis since his PhD thesis. In particular, in his joint paper with E. Priola [51] he studies the problem of the existence of a non-constant bounded harmonic function for the non-local operator

$$\begin{aligned}Lu(x) &= \frac{1}{2}\operatorname{Trace} QD^2u(x) + \langle Ax+a, Du(x)\rangle \\ &\quad + \int_{\mathbb{R}^d} \big(u(x+y)-u(x)-\chi_{\{|y|\le 1\}}\langle y, Du(x)\rangle\big)\,\mu(\mathrm{d}y), \qquad x\in\mathbb{R}^d,\ u\in C_b^2(\mathbb{R}^d).\end{aligned}$$

Note that L can be seen as the generator of the generalized Ornstein–Uhlenbeck process defined by (32) for suitably chosen B and Z. We say that $u\in C_b^2(\mathbb{R}^d)$ is *harmonic* for L iff $Lu\equiv 0$. Furthermore, we say that L has *Liouville property* iff any harmonic (bounded) function u is constant. Liouville property has a probabilistic interpretation. Roughly speaking, it implies that for the corresponding Markov family (X_x), any *absorption function*

$$u_D(x) := \mathbb{P}\big\{\omega\in\Omega : \exists\, \tau_x(\omega) : X_x(t)\in D \text{ for all } t\ge \tau_x(\omega)\big\}, \qquad x\in\mathbb{R}^d,\ D\in\mathcal{B}(\mathbb{R}^d),$$

is identically equal to 0 or to 1. The following result is shown in [51].

Theorem 4.4. *Assume that*

$$\int_{\{|y|\le 1\}} |y|^2\,\mu(\mathrm{d}y) + \int_{\{|y|>1\}} \log|y|\,\mu(\mathrm{d}y) < \infty.$$

Then if all eigenvalues of A have negative real parts, then L has Liouville property. If at least one eigenvalue of A has positive real part, then L does not have the Liouville property. If, moreover,

$$\operatorname{rank}[Q, AQ, \ldots, A^{d-1}Q] = d,$$

then L has the Liouville property iff

$$\max\{\operatorname{Re}\lambda : \lambda\in\sigma(A)\}\le 0.$$

In [51] the Liouville property of the generator of a Markov family defined by a nonlinear SOE is studied as well. Moreover, an extension of the Bismut–Elworthy–Li formula is given.

As we see a significant part of scientific activity of Prof. J. Zabczyk is devoted to stochastic linear equations. One of the reason is the clarity and beauty of the problems and conditions. On the other hand, linear stochastic evolution equations have some universal feature. Namely they can be seen as models for Markov families with Gaussian distributions. In fact in the joint work of Prof. J. Zabczyk with B. Goldys and S. Peszat (see [20]) it is shown that any time homogeneous Gaussian Markov family on a Hilbert space H can be seen as solution to the SEE

$$\mathrm{d}X = (AX+a)\,\mathrm{d}t + B\,\mathrm{d}W, \qquad X(0)=x,$$

where B is a possibly unbounded linear operator on H and $a\in V$, where $V\hookleftarrow H$.

4.2. Different concepts of solutions. Anna Chojnowska-Michalik who was the first PhD student of Prof. J. Zabczyk prepared her thesis in 1977. She investigated different concepts of solutions (strong, weak, mild) to the SEE

$$\mathrm{d}X = (AX+F(X))\,\mathrm{d}t + \mathrm{d}M$$

driven by infinite-dimensional semimartingale M possibly dependent on X. Her main result (see [8]) was the equivalence of weak and mild solutions which required some version of the stochastic Fubini theorem (studied in details in [15]). This can be now considered as a very basic and fundamental problems in SEEs. Another problem which attracted attention of Prof. J. Zabczyk and also Anna Chojnowska-Michalik in her thesis was the semigroup approach to the stochastic delay equation (see [9, 10, 15] for more details)

$$\mathrm{d}X(t) = \left(AX(t) + \int_{-\theta}^{0} v(\mathrm{d}\rho)X(t+\rho) + f(X(t))\right)\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t). \tag{39}$$

The problem is to write (39) as SEE with linear part being the generator of a C_0-semigroup. This enables us to consider Markov solutions corresponding to (39). The semigroup approach was initiated in the 1970s by M. C. Delfour and A. Manitius. It is still promising; see the works of S. Cox and M. Górajski on solutions to stochastic delay equations in L^p-spaces, $p \geq 1$. It is worth to point out that M. Górajski was a PhD student of A. Chojnowska-Michalik.

4.3. Equations with multilinear noise. One of the main motivation to study SEEs comes from filtering. It leads to the famous Zakai equation

$$\mathrm{d}X = AX\,\mathrm{d}t + \sum_k B_k X\,\mathrm{d}W_k, \tag{40}$$

where typically A is a second order differential operator and B_k are first order differential operators. The systematic study of this type of equation was initiated by E. Pardoux, B. Rozowski, N. Krylov. It still attracts a lot of attention (see e.g. the works of F. Flandoli, Z. Brzeźniak, K. Twardowska). Important problems are: existence of the solution and the stability of 0-solution (see the works of R. Datko, U. G. Haussman, A. Ichikawa, R. Z. Hasminski). The main achievements of Prof. J. Zabczyk are sufficient conditions for the existence of solution (see Chapter 6 of [15]) and iff conditions for the mean-square stability of 0-solution. Namely, see Chapter 11 of [15], $\mathbb{E}|X(t)|^2 \to 0$ iff the Lyapunov equation

$$A^*R + RA + \sum_k B_k^* R B_k + I = 0$$

has a symmetric positive-definite solution R. This general abstract criterion leads to very concrete applications. The following example comes from Prof. J. Zabczyk's paper [67], see also Example 11.18 from [15]. Namely, let (c_k) and (b_k) be sequences of elements of a Hilbert space H. Let $B_1, \ldots, B_m$, $m < +\infty$, be given by

$$B_k u = b_k \langle u, c_k\rangle_H, \qquad k = 1, \ldots, m.$$

Then if the semigroup generated by A is stable, then $X(t) \to 0$ in L^2 if and only if the eigenvalues of the matrix $M = [M_{i,j}]$,

$$M_{ij} := \int_0^{+\infty} \langle S_t c_i, b_j\rangle_H^2\,\mathrm{d}t,$$

are all of modulus less than 1. Various types of stability were and are studied (see e.g. works by E. Pardoux and V. Wihstutz, A. A. Kwiecińska, B. Xie, T. Caraballo, K. Liu and X. Mao, K. Liu, H. Crauel, T. Damm, and A. Ilchmann, D. Applebaum and M. Siakalli).

4.4. Time and space regularity. A basic problem of SEEs is the time and space regularity of stochastic convolutions

$$Y(t) := \int_0^t S_{t-s}\,\mathrm{d}M(s), \qquad t \geq 0, \tag{41}$$

driven by a semimartinagle M. Note that Y is (usually) not a semimartingale! Therefore maximal inequalities for semimartingales cannot be applied. To see that t in the integrand makes a big difference let us recall a little surprising result from the joint work of Prof. J. Zabczyk with Z. Brzeźniak and S. Peszat [7], which says that for a standard real-valued Brownian motion W and a typical (i.e. of the second Baire category) deterministic and continuous function $f : \mathbb{R} \to \mathbb{R}$ the trajectories of the process

$$[0,1] \ni t \mapsto \int_0^t f(t-s)\,\mathrm{d}W(s) \in \mathbb{R}$$

are unbounded $\mathbb{P}$-a.s. Prof. J. Zabczyk has some very important achievements in the study of the regularity of solutions to SEEs or stochastic convolutions. First of all, we cite his beautiful and extremely important work with G. Da Prato and S. Kwapień on factorization [14]. In some sense it plays a similar role as the classical Sobolev imbedding. Namely, it says that, provided that the integrals below are well-defined, the stochastic convolution process Y defined by (41) can be written as

$$Y(t) = C_\alpha \int_0^t (t-s)^{-\alpha} S_{t-s} Y_\alpha(s)\,\mathrm{d}s, \qquad t \geq 0,$$

where

$$Y_\alpha(t) := \int_0^t (t-s)^{\alpha-1} S_{t-s}\,\mathrm{d}M(s), \qquad t \geq 0.$$

The analogy with Sobolev imbeddings is obvious. Instead of showing directly the continuity of stochastic convolution which requires some estimates on $\mathbb{E}\sup_{t\leq T}|Y(t)|$ one needs to estimate $\mathbb{E}\int_0^T |Y_\alpha(t)|^p\,\mathrm{d}t$, where Y_α is a "more singular" stochastic convolution. In particular, to show that Y has continuous trajectories in a space H it is enough to show that the process Y_α has trajectories in $L^p(0,T;H)$ with $p > 1/\alpha$.

For Lévy process the factorization does not work, as the convolution can be only càdlàg. The following trick invented by E. Hausenblas and J. Seidler [21] can be used. Namely if the semigroup S generated by A is the semigroup of *generalized contractions*, i.e. $\|S_t\| \leq \mathrm{e}^{\omega t}$ for a certain not necessarily negative ω, then by the Nagy dilation theorem there is a bigger space $\mathcal{H} \hookleftarrow H$ and a C_0-group T_t, $t \in \mathbb{R}$, on $\mathcal{H}$ such that $S_t = PT_t$, where P is a continuous projection of $\mathcal{H}$ onto H. Then

$$Y(t) := \int_0^t S_{t-s}\,\mathrm{d}M(s) = PT_t \int_0^t T_{-s}\,\mathrm{d}M(s) = PT_t R(t), \qquad t \geq 0.$$

The process R is a semimartingale in $\mathcal{H}$, and thus has continuous or càdlàg trajectories in $\mathcal{H}$. Since P and T_t are continuous, Y has trajectories of desired regularity in H. In this way one can obtain the classical Kotelenez maximal inequality for contraction semigroups. The trick of E. Hausenblas and J. Seidler was used many times in the book [47] of Prof. J. Zabczyk with S. Peszat. Recently, however, it was applied by Prof. J. Zabczyk and

S. Peszat (see [49]) to study the time regularity of solutions to the stochastic Volterra equation

$$X(t) = X_0 + \int_0^t v(t-s)AX(s)\,\mathrm{d}s + M(t), \qquad t \in [0,T]. \tag{42}$$

The main result of [49] is valid for an arbitrary semimartingale M and can be stated as follows.

THEOREM 4.5. *Let $X_0 \in H$. Assume that A is a self-adjoint negative semi-definite operator on a Hilbert space H and that a locally integrable function v of at most exponential growth at infinity satisfies one of the following conditions*

(a) $v: (0,+\infty) \to \mathbb{R}$ *is nonincreasing and positive,*
(b) $v: [0,+\infty) \to \mathbb{R}$ *is a function of locally bounded variation,* $v(0) \geq 0$, *and the generated measure* $\mathrm{d}v$ *is positive definite.*

If M is càdlàg (or continuous) semimartingale in H, then the weak solution to (42) *exists, is unique, and has a càdlàg (resp. continuous) modification.*

EXAMPLE 4.6. Let $v(t) = t^{-\alpha}$ for some $\alpha \in (0,1)$. Then v is locally integrable, strictly decreasing, and positive. Therefore it is of the type (a) from Theorem 4.5.

EXAMPLE 4.7. Let $v(t) = t$, $t \geq 0$. Then $v' \equiv 1$ is positive definite, and thus v is of type (b). In this case the solution is given in the explicit form

$$X(t) = \cos\big(\sqrt{-A}t\big)X_0 + \int_0^t \cos\big(\sqrt{-A}(t-s)\big)\,\mathrm{d}M(s), \qquad t \geq 0.$$

EXAMPLE 4.8. Assume that $v \in C^1([0,+\infty))$, v' is non-negative, non-increasing and concave. Then, by the Bochner theorem, $t \mapsto v'(|t|)$ is positive definite as required. In particular, if $v(0) \geq 0$, $v'(t) \geq 0$, $v''(t) \leq 0$ and $v'''(t) \geq 0$, then v is of type (b).

EXAMPLE 4.9. Let $a, b > 0$, and let

$$v'(t) = \begin{cases} a - \frac{a}{b}t, & t \in [0,b], \\ 0, & t \geq b. \end{cases}$$

Then v' is non-negative, non-increasing and concave. Note that

$$v(t) = \begin{cases} v(0) + at - \frac{a}{2b}t^2, & t \in [0,b], \\ v(0) + \frac{ab}{2}, & t \geq b, \end{cases}$$

is of type (b) provided that $v(0) \geq 0$.

The limitation of the above trick of E. Hausenblas and J. Seidler is that the noise M has to be H-valued. This eliminates the so-called cylindrical noise. In fact, if the noise lives in $U: H \hookrightarrow U$, and it has jumps from $U \setminus H$, then the solution is not càdlàg in H, see the book of Prof. J. Zabczyk and S. Peszat [47]. A natural example of a noise living in a bigger space but with jumps in a smaller is the following (see also (36))

$$Z(t) = \sum_k \beta_k Z_k(t) e_k,$$

where (Z_k) are independent real-valued Lévy noises, (e_k) is an orthonormal basis of H and (β_k) is a sequence of real numbers. Note that due to the independence of Z_k, the Lévy measure of Z is concentrated on

$$\bigcup_k \{ae_k : a \in \mathbb{R}\} \subset H.$$

It turns out that in this case different scenarios are possible. For example the following result holds (see Z. Brzeźniak, B. Goldys, P. Imkeller, S. Peszat, E. Priola, J. Zabczyk [6]).

THEOREM 4.10. *Let S be the C_0-semigroup on a Hilbert space H generated by an operator A. Assume that (Z_k) are independent identically distributed Lévy processes with the Lévy measure ν such that $\nu(\mathbb{R}\setminus\{0\}) \neq 0$. Assume that the process*

$$X(t) = \int_0^t S_{t-s}\,\mathrm{d}Z(s), \qquad t \geq 0, \tag{43}$$

is H-valued and that the elements of the basis (e_k) belong to the domain $D(A^*)$ of the operator A^* adjoint to A. If β_k do not converge to 0, then, with probability 1, trajectories of X have no point $t \in [0, +\infty)$ at which the left limit $X(t-) \in H$ or the right limit $X(t+) \in H$ exists.

COROLLARY 4.11. *Assume that the hypotheses of Theorem 4.10 hold. Then the process X has no càdlàg modification in H.*

Note that X given by (43) is the solution to the generalized Ornstein–Uhlenbeck equation

$$\mathrm{d}X = AX\,\mathrm{d}t + \mathrm{d}Z, \qquad X(0) = 0.$$

In S. Peszat and J. Zabczyk [48] it is shown that it can happen that Z is not H-valued, but X has càdlàg trajectories in H. In fact the following result holds. In its formulation H_ρ, $\rho > 0$, is the domain of $(-A)^\rho$ equipped with the norm $|z|_\rho := |(-A)^\rho z|_H$. If $\rho < 0$, then H_ρ is the dual space to $H_{-\rho}$ where the duality is given by the identification $H = H^*$.

THEOREM 4.12. *Assume that there are $\rho < 1/2$ and $\varepsilon > 0$ such that Z takes values in $H_{-\rho}$, the Lévy measure ν of Z satisfies $\nu(H_{-\rho}\setminus H) = 0$, and*

$$\int_{\{|z|_{-\rho}\leq R\}} |z|_\varepsilon^4\,\nu(\mathrm{d}z) < \infty, \qquad \forall\, R > 0.$$

Then X has a càdlàg modification in H.

The following example comes from [48].

EXAMPLE 4.13. Let $Z_k = \sigma_k L_k$, $k \in \mathbb{N}$, where L_k are independent and identically distributed Lévy processes of type

$$Z_k(t) = \int_0^t \int_{\{|z|>1\}} z\,\pi_k(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\{|z|\leq 1\}} z\,\widehat{\pi}_k(\mathrm{d}s, \mathrm{d}z),$$

where each π_k is a Poisson random measure with intensity measure μ satisfying $\int_{\mathbb{R}} |z|^2 \wedge 1\,\mu(\mathrm{d}z) < \infty$, and (σ_k) is a sequence of strictly positive numbers. Assume that the Lévy measure μ of L_k has finite moments up to order 4. Then the Lévy measure μ_n

of $Z_n = \sigma L_n$ equals $\mu(\cdot/\sigma_n)$. Consequently, if there is an $\varepsilon > 0$ such that

$$\sum_{n=1}^{\infty}\left[\gamma_n^{\varepsilon-1}\sigma_n^2 + \gamma_n^{\varepsilon}\sigma_n^4\right] < \infty,$$

then by Theorem 4.12, X has a càdlàg modification in l^2. In particular, if $\gamma_n \asymp n^{\alpha}$, $\sigma_n \asymp n^{-\kappa}$, then X has a càdlàg modification in l^2 provided that $\alpha > 1-2\kappa$ and $\kappa > 1/4$. Assume now that each L_n is a Poisson process with intensity 1. Then $\mu = \delta_1$ and $\sigma_n L_n$ has the Lévy measure $\mu_n = \delta_{\sigma_n}$. Therefore $Z = (\sigma_n L_n)$ lives in l^2 if and only if

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}} x_n^2 \wedge 1\, \mu_k(\mathrm{d}x_n) = \sum_{n=1}^{\infty}\sigma_n^2 \wedge 1 < \infty.$$

Hence, if $\sigma_n \asymp n^{-\kappa}$, then Z takes values in l^2 if and only if $\kappa > 1/2$. Summing up, if $1/4 < \kappa < 1/2$ and $\alpha > 1-2\kappa$, then Z does not take values in l^2 but the solution X has a càdlàg modification in l^2.

In [48] different concepts of càdlàg property (cylindrical and weak) are investigated as well.

4.5. Strong Feller property for nonlinear equations. For nonlinear equations the strong Feller property is shown by Prof. J. Zabczyk and S. Peszat (see [43]) by using a very nice Bismut–Elworthy–Li formula. Namely, assume that the transition semigroup (P_t) corresponds to the equation

$$\mathrm{d}X = (AX + F(X))\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W.$$

Then

$$\langle \nabla P_t\psi(x), h\rangle_H = \frac{1}{t}\,\mathbb{E}\Big\{\psi(X(t,x))\int_0^t \big\langle \sigma^{-1}(X(s,x))D_xX(s,x)h, \mathrm{d}W(s)\big\rangle_H\Big\}.$$

For the case of Lévy noise see the more recent paper of E. Priola and J. Zabczyk [53].

4.6. Large deviation. Szymon Peszat who was a PhD student of Prof. J. Zabczyk prepared his thesis in 1992 on the large deviation principle (Wentzel–Freidlin estimates) for SEEs. Exit problems were studied in details in [15] following [73, 74]. More recently A. Święch and J. Zabczyk [56] established large deviation principle for a class of SEEs with Lévy noise. They prove a large deviation principle result for solutions of abstract stochastic evolution equations perturbed by small Lévy noise. They used general large deviations theorems of Varadhan and Bryc coupled with the techniques of Feng and Kurtz, viscosity solutions of integro-partial differential equations in Hilbert spaces, and deterministic optimal control methods. The Laplace limit is identified as a viscosity solution of a Hamilton–Jacobi–Bellman equation of an associated control problem. They also establish exponential moment estimates for solutions of stochastic evolution equations driven by Lévy noise. General results are applied to stochastic hyperbolic equations perturbed by subordinated Wiener process.

4.7. Dissipative systems. A significant part of the G. Da Prato and J. Zabczyk book [17] on ergodic properties of SPDEs (or SEEs) is devoted to the so-called *dissipative systems.* The book summarizes the results of several publications of the authors.

Roughly speaking, the system

$$\mathrm{d}X = (AX + F(X))\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W,$$

is *dissipative* if

$$2\big\langle A(x-y) + F(x) - F(y), x-y\big\rangle + \|\sigma(x)-\sigma(y)\|_{HS}^2 \le -\omega|x-y|_H^2.$$

G. Da Prato and J. Zabczyk showed (see [17]) that if the system is dissipative, then there is a unique invariant measure μ, and moreover

$$|P_t\psi(x) - (\psi,\mu)| \le c(x)\mathrm{e}^{-\omega t}\|\psi\|_{\mathrm{Lip}}$$

and

$$\|P_t(x,\cdot) - \mu\|_{FM} \le c(x)\mathrm{e}^{-\omega t},$$

where $\|\cdot\|_{FM}$ is the so-called Fortet–Mourier norm. For G. Da Prato and J. Zabczyk motivations to study dissipative systems come from chemistry (reaction diffusion equations where A is a second order differential operator and F is a polynomial) and statistical physics where the general results for dissipative systems were applied to the so-called lattice systems:

$$\mathrm{d}X_k = \Big(\sum_{j\in\mathbb{Z}^d} a_{k,j}X_j + f(X_k)\Big)\,\mathrm{d}t + g(X_k)\,\mathrm{d}W_k.$$

Lattice systems with Lévy noise and interest rate models with Lévy noise can be seen as dissipative systems. They are the topics of the most recent Prof. J. Zabczyk's PhD student Anna Rusinek (see [55]), see also [47].

4.8. Homogeneous noise. The starting point for this research was the paper [18] of D. A. Dawson and H. Salehi. It was concerned with the following equation on $\mathbb{R}^d$:

$$\mathrm{d}X = AX\,\mathrm{d}t + X\,\mathrm{d}W.$$

Above $W(t,x)$ is a Gaussian random field with the covariance

$$\mathbb{E}\,W(t,x)W(s,y) = t\wedge s\,\Gamma(x-y).$$

The first question was whether the equation belongs to the class studied in the book of G. Da Prato and J. Zabczyk [15].

In [18] only the case of W being a random field is considered. In S. Peszat and J. Zabczyk [44] the case of generalized space homogeneous random fields is considered. More precisely, it is assumed that W is a process taking values in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. It has the following properties

(i) W is a Wiener process; that is, for any finite set of test functions $\psi_1,\dots,\psi_n \in \mathcal{S}(\mathbb{R}^d)$, the $\mathbb{R}^n$-valued process

$$(t,\omega)\mapsto \big((W(t;\omega),\psi_1),\dots,(W(t;\omega),\psi_n)\big)$$

is Wiener.

(ii) W is space homogeneous; that is for any $t\ge 0$, the distribution of $W(t)$ in $\mathcal{S}'(\mathbb{R}^d)$ is invariant with respect to the group of spatial translations τ_x, $x\in\mathbb{R}^d$.

The second condition can be stated directly as follows

$$\mathbb{P}\big(W(t)\in\tau_x^{-1}(\Gamma)\big) = \mathbb{P}(W(t)\in\Gamma), \qquad \forall\, x\in\mathbb{R}^d,\ \Gamma\in\mathcal{B}\big(\mathcal{S}'(\mathbb{R}^d)\big),$$

where

$$(\tau_x\xi, \psi) = (\xi, \psi(x + \cdot)), \qquad \xi \in \mathcal{S}'(\mathbb{R}^d),\ \psi \in \mathcal{S}(\mathbb{R}^d).$$

It is shown in [44] that the law of W is characterized by its *spectral measure* μ on $\mathbb{R}^d$:

$$\mathbb{E}(W(t), \psi)(W(s), \phi) = t \wedge s \int_{\mathbb{R}^d} \widehat{\psi}(x)\overline{\widehat{\phi}(x)}\, \mu(\mathrm{d}x), \qquad t, s \geq 0,\ \psi, \phi \in \mathcal{S}(\mathbb{R}^d),$$

where $\widehat{\psi}$ is the Fourier transform of ψ. The spectral measure has to be a tempered distribution. If μ is a finite measure, then W can be identified with a Gaussian stationary in x random field $W(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$,

$$(W(t), \psi) = \int_{\mathbb{R}^d} W(t, x)\psi(x)\, \mathrm{d}x, \qquad \psi \in \mathcal{S}(\mathbb{R}^d).$$

Moreover,

$$\mathbb{E}\, W(t, x)W(s, y) = t \wedge s\, \Gamma(x - y), \qquad t, s \geq 0,\ x, y \in \mathbb{R}^d,$$

where the *spatial correlation* Γ is the Fourier transform of the spectral measure μ.

The main result from [44] says that any spatially homogeneous Wiener process W is a cylindrical Wiener process with RKHS

$$\mathcal{H}_W = \left\{\widehat{\psi\mu} : \psi \in L^2_{(s)}(\mu)\right\},$$

where $L^2_{(s)}(\mu)$ is the space of all square integrable with respect to μ complex-valued functions ψ satisfying $\psi(x) = \overline{\psi(-x)}$, $x \in \mathbb{R}^d$. Moreover, the scalar product on $\mathcal{H}_W$ is induced by the map $L^2_{(s)}(\mu) \ni \psi \mapsto \psi\mu \in \mathcal{H}_W$. Finally, the SPDE

$$\mathrm{d}X(t, x) = \big(AX(t, x) + f(X(t, x))\big)\, \mathrm{d}t + \sigma(X(t, x))\, \mathrm{d}W(t, x)$$

can be treated as SEE in the sense of [15] in some weighted L^2-spaces. In particular, with approach of [44] (and consequently [15]) the following heat and wave equations can be treated

$$\frac{\partial X}{\partial t}(t, x) = \Delta X(t, x) + f(X(t, x)) + \sigma(X(t, x))\frac{\partial W}{\partial t}(t, x), \tag{44}$$

$$\frac{\partial^2 X}{\partial t^2}(t, x) = \Delta X(t, x) + f(X(t, x)) + \sigma(X(t, x))\frac{\partial W}{\partial t}(t, x). \tag{45}$$

The main achievements of Prof. J. Zabczyk in this topic is a criterion for a function valued solution either to heat or wave equation with Lipschitz continuous f and σ. The criterion is in terms of the spectral measure μ of the noise W:

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}y)}{1 + |x - y|^2} < \infty,$$

This condition follows from [25, 27, 26, 45, 42, 41]. This topic is studied e.g. by R. Dalang, N. Frangos, M. Sanz-Solé, A. Millet, C. Mueller.

4.9. Viability. J.-P. Aubin and G. Da Prato [2] initiated the study of the so-called *stochastic Nagumo viability theorem*. Roughly speaking, we are interested whether a domain $D \subset H$ is invariant, that is $X(t) \in D$, $t > 0$, if $X(0) \in D$. For the diffusion

$$\mathrm{d}X = F(X)\, \mathrm{d}t + \sigma(X)\, \mathrm{d}W,$$

the condition is $\sigma(x) = 0$ for $x \in \partial D$, and $F(x)$ at most parallel to ∂D. Significant results in this fields were obtained by Anna Milian (see [32, 33, 34, 36, 37]), W. Jachimiak [22], G. Tessitore and J. Zabczyk [59], and J. Zabczyk (see e.g. [47]). Let us recall that Anna Milian, Wojciech Jachimiak and Gianmario Tessitore were PhD students of Prof. J. Zabczyk.

4.10. Comments. Prof. J. Zabczyk is the author or co-author of about 130 research papers or books. His works have been widely cited (about 2800 citations in MscNet basis). In this review we have decided to concentrate on the topics which are close to our expertise. Thus some research activities (for example Kolmogorov equations) have not been described here.

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