STOCHASTIC ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 105
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2015

A NOTE ON γ-RADONIFYING AND SUMMING OPERATORS

ZDZISŁAW BRZEŹNIAK
Department of Mathematics, University of York
Heslington, York YO10 5DD, UK
E-mail: zdzislaw.brzezniak@york.ac.uk

HONGWEI LONG
Department of Mathematical Sciences, Florida Atlantic University
Boca Raton, Florida 33431, U.S.A.
E-mail: hlong@fau.edu

Abstract. In this note, we discuss certain generalizations of γ-radonifying operators and their applications to the regularity for linear stochastic evolution equations on some special Banach spaces. Furthermore, we also consider a more general class of operators, namely the so-called summing operators and discuss the application to the compactness of the heat semi-group between weighted L^p-spaces.

1. Introduction. It is well-known that γ-radonifying operators play crucial role in developing the theory of Gaussian measures in Banach spaces as well as stochastic integrals on martingale type 2 Banach spaces (cf. Gross [10], Kuo [12], Neidhardt [17], Dettweiler [9], Brzeźniak [1, 2], Brzeźniak and Elworthy [3]). The γ-radonifying operators have been used to characterize some nice geometric structure of the underlying Banach spaces. For example, some embedding theorems related to γ-radonifying operators are established for the underlying Banach spaces to be type 2 or cotype 2 (see van Neerven and Weis [16]). In Kalton et al. [11], these results have been extended to give a characterization for the Banach space being type p or cotype q by embedding vector-valued Besov spaces into spaces of γ-radonifying operators. For a more complete review on γ-radonifying operators, we refer to van Neerven [14].

2010 *Mathematics Subject Classification*: Primary 60H15, 35R60; Secondary 37H10, 34F05.
Key words and phrases: γ-radonifying operators, linear stochastic evolution equations, summing operators, heat semi-groups, compactness, weighted L^p-spaces.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc105-0-3

In Brzeźniak and Peszat [5] and Brzeźniak and van Neerven [4], the authors showed that the γ-radonifying norms reduce to square functions in the special case when the underlying Banach space is an L^p space (see Theorems 2.1 and 2.2 in Section 2). In this note, we shall discuss some generalizations of their results. These results are then used in the study of regularity for stochastic evolution equations (e.g. stochastic heat equations) in the weighted L^p-spaces.

Note that γ-radonifying operators can be regarded as the Gaussian analogues of r-summing operators. It is natural to extend the characterization of γ-radonifying operators to the summing operators setting between different Banach spaces, especially in the L^p spaces. We want to find some sufficient condition which ensures a linear operator between L^r and L^p spaces is p-summing. This result is closely related to the compactness of heat semigroups (generated by Laplacian on $\mathbb{R}^d$) on some weighted L^p-spaces, which will be useful in the study of invariant measures for stochastic heat equations in the L^p-setting. Note that the existence and the uniqueness of mild solutions along with their regularity (so-called space-time continuity) for stochastic partial differential equations on $\mathbb{R}^d$ in the weighted L^p-setting ($p \geq 2$) have been studied by Brzeźniak and Peszat [5] by using the theory of stochastic integrals in martingale type 2 Banach spaces and the factorization method introduced by Da Prato, Kwapień and Zabczyk [7]. Concerning the existence of invariant measures for SEE in Hilbert spaces, Da Prato, Gątarek and Zabczyk [6] introduced the so-called *compactness method* which was further successfully applied to the study of existence of invariant measures for stochastic heat equations in some weighted L^2-spaces by Tessitore and Zabczyk [21]. In the latter, in contrast to the situation considered in [6], the heat semigroups $S(t)$ ($t > 0$) are not compact in any weighted L^2-space. But Tessitore and Zabczyk [21] proved that the heat semigroup is compact from a weighted L^2-space to another one provided that the involved weights satisfy certain integrability condition on $\mathbb{R}^d$. This fact is crucial in their proof of the existence of invariant measures for stochastic heat equations. In this note we want to prove the same property for heat semigroups in weighted L^p-setting. For this, we need to use the notion of summing operators from one Banach space to another Banach space studied first by Pietsch [19], which played an important role in the theory of geometry and probability in Banach spaces (cf. Schwartz [20]). We first provide some sufficient condition for a linear operator between L^r and L^p spaces to be p-summing. Then this result is applied to show the compactness of heat semigroups in weighted L^p spaces.

This note is organized as follows. In Section 2, we discuss some interesting characterizations of γ-radonifying operators and their applications. In Section 3, we discuss further applications to linear stochastic evolution equations driven by spatially homogeneous noise. In Section 4, we study summing operators in some special Banach spaces and discuss the application to the compactness of heat semigroup between weighted L^p spaces.

2. Characterizations of γ-radonifying operators and applications. Let H be a separable Hilbert space and E be a real separable Banach space. We denote by $\gamma(H, E)$

the completion of the finite rank operators from H to E with respect to the norm

$$\Big\|\sum_{i=1}^{k} h_i \otimes x_i\Big\|_{\gamma(H,E)} := \Big(\mathbb{E}\Big\|\sum_{i=1}^{k} \gamma_i x_i\Big\|^2\Big)^{1/2}$$

where $h_1, \dots, h_k$ are orthonormal in H and $\{\gamma_i\}_{i\geq 1}$ is a sequence of independent standard Gaussian random variables defined on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$. A linear operator $L : H \to E$ belonging to $\gamma(H,E)$ is called γ-radonifying. It is known that a bounded linear operator $L \in \gamma(H,E)$ if and only if there exists a centred Gaussian probability measure ν_L on E with covariance LL^*. For $L \in \gamma(H,E)$, we can define

$$\|L\|_{\gamma(H,E)} = \Big(\int_E \|x\|^2 \, d\nu_L(x)\Big)^{1/2},$$

which is a finite number in terms of Fernique's Theorem. Then $\gamma(H,E)$ is a separable Banach space under this norm (cf. Lemma 3.2 of Neidhardt [17]).

The following result is formulated in [5] and proved in [4, Theorem 2.3].

THEOREM 2.1. *Suppose H is a separable real Hilbert space and let $p \in (1,\infty)$ be fixed. Let $(\mathcal{O}, \mathcal{F}, \nu)$ be a σ-finite measure space. For a bounded linear operator $K : H \to L^p(\mathcal{O})$ the following assertions are equivalent*:

(i) *K is γ-radonifying.*
(ii) *There exists a ν-measurable function $\kappa : \mathcal{O} \to H$ with*

$$\int_{\mathcal{O}} |\kappa(x)|_H^p \, d\nu(x) < \infty \tag{1}$$

such that for all ν-almost all $x \in \mathcal{O}$

$$(K(h))(x) = [\kappa(x), h]_H, \quad h \in H. \tag{2}$$

Moreover, there exists a constant $C > 0$ independent of K such that

$$\frac{1}{C}\int_{\mathcal{O}} |\kappa(x)|^p \, d\nu(x) \leq \|K\|^p_{\gamma(H,L^p)} \leq C \int_{\mathcal{O}} |\kappa(x)|^p \, d\nu(x).$$

In the special case when H is equal to $L^2(D)$ over some measure space $(D, \mathcal{C}, \mu)$, the above result reads as follows.

THEOREM 2.2. *Suppose that $(\mathcal{O}, \mathcal{F}, \nu)$ and $(D, \mathcal{C}, \mu)$ are σ-finite measure spaces. Let $p \in (1,\infty)$ be fixed. For a bounded linear operator $K : L^2(D) \to L^p(\mathcal{O})$ the following assertions are equivalent*:

(i) *K is γ-radonifying.*
(ii) *There exists a $\nu \otimes \mu$-measurable function $\kappa : \mathcal{O} \times D \to \mathbb{R}$ with*

$$\int_{\mathcal{O}} \Big[\int_D |\kappa(x,y)|^2 \, d\mu(y)\Big]^{p/2} d\nu(x) < \infty \tag{3}$$

such that for all ν-almost all $x \in \mathcal{O}$

$$(K(h))(x) = \int_D \kappa(x,y) h(y) \, d\mu(y), \quad h \in L^2(D). \tag{4}$$

Moreover, there exists a constant $C>0$ independent of K such that

$$\frac{1}{C}\int_{\mathcal{O}}\Big[\int_D |\kappa(x,y)|^2\,d\mu(y)\Big]^{p/2} d\nu(x) \le \|K\|^p_{\gamma(H,L^p)} \le C\int_{\mathcal{O}}\Big[\int_D |\kappa(x,y)|^2\,d\mu(y)\Big]^{p/2} d\nu(x).$$

Our aim is to present various generalizations of the above theorems. Firstly, we will describe generalization of Theorem 2.1 when the Banach space $L^p(\mathcal{O})$ is replaced by $L^r(\mathcal{O}_1;L^p(\mathcal{O}_2))$, with $1<p,r<\infty$, for two σ-finite measure spaces $(\mathcal{O}_i,\mathcal{F}_i,\nu_i)$, $i=1,2$. We shall prove

Theorem 2.3. *Suppose H is a separable real Hilbert space and let $p,r\in(1,\infty)$. Let $(\mathcal{O}_i,\mathcal{F}_i,\nu_i)$, $i=1,2$, be σ-finite measure spaces. For a bounded linear operator $K: H\to L^r(\mathcal{O}_1;L^p(\mathcal{O}_2))$ the following assertions are equivalent*:

(i) *The linear operator K is γ-radonifying.*
(ii) *There exists a $\nu_1\otimes\nu_2$-measurable function $\kappa:\mathcal{O}_1\times\mathcal{O}_2\to H$ with*

$$\int_{\mathcal{O}_1}\Big[\int_{\mathcal{O}_2}|\kappa(x_1,x_2)|^p_H\,d\nu_2(x_2)\Big]^{r/p} d\nu_1(x_1)<\infty \tag{5}$$

such that for all $\nu_1\otimes\nu_2$-almost all $(x_1,x_2)\in\mathcal{O}_1\times\mathcal{O}_2$

$$(K(h))(x_1,x_2)=[\kappa(x_1,x_2),h]_H,\quad h\in H. \tag{6}$$

Moreover, there exists a constant $C>0$ independent of K such that

$$\frac{1}{C}\int_{\mathcal{O}_1}\Big[\int_{\mathcal{O}_2}|\kappa(x_1,x_2)|^p_H\,d\nu_2(x_2)\Big]^{r/p} d\nu_1(x_1)$$
$$\le \|K\|^p_{\gamma(H,L^r(\mathcal{O}_1;L^p(\mathcal{O}_2)))} \le C\int_{\mathcal{O}_1}\Big[\int_{\mathcal{O}_2}|\kappa(x_1,x_2)|^p_H\,d\nu_2(x_2)\Big]^{r/p} d\nu_1(x_1).$$

Proof. Let $(e_j)_{j=1}^\infty$ be the ONB of the Hilbert space H. Let $(\beta_j)_{j=1}^\infty$ be a sequence of i.i.d. standard mean 0 Gaussian random variables. Then by the Fubini, Kahane and Plancherel theorems for any $n\in\mathbb{N}$

$$\mathbb{E}\int_{\mathcal{O}_1}\Big[\int_{\mathcal{O}_2}\Big|\sum_j^n \beta_j K(e_j)(x_1,x_2)\Big|^p d\nu_2(x_2)\Big]^{r/p} d\nu_1(x_1)$$
$$=\int_{\mathcal{O}_1}\mathbb{E}\Big[\int_{\mathcal{O}_2}\Big|\sum_j^n \beta_j K(e_j)(x_1,x_2)\Big|^p d\nu_2(x_2)\Big]^{r/p} d\nu_1(x_1)$$
$$=\int_{\mathcal{O}_1}\mathbb{E}\Big\|\sum_j^n \beta_j K(e_j)(x_1,\cdot)\Big\|^r_{L^p(\mathcal{O}_2,\nu_2)} d\nu_1(x_1)$$
$$\sim\int_{\mathcal{O}_1}\Big[\mathbb{E}\Big\|\sum_j^n \beta_j K(e_j)(x_1,\cdot)\Big\|^p_{L^p(\mathcal{O}_2,\nu_2)}\Big]^{r/p} d\nu_1(x_1)$$
$$=\int_{\mathcal{O}_1}\Big[\mathbb{E}\int_{\mathcal{O}_2}\Big|\sum_j^n \beta_j K(e_j)(x_1,x_2)\,d\nu_2(x_2)\Big|^p\Big]^{r/p} d\nu_1(x_1)$$

$$= \int_{\mathcal{O}_1} \Big[\int_{\mathcal{O}_2} \mathbb{E} \Big| \sum_j^n \beta_j K(e_j)(x_1,x_2)\, d\nu_2(x_2) \Big|^p \Big]^{r/p} d\nu_1(x_1)$$
$$= c_p^{r/p} \int_{\mathcal{O}_1} \Big[\int_{\mathcal{O}_2} \Big(\sum_j^n \big| K(e_j)(x_1,x_2) \big|^2 \Big)^{p/2} d\nu_2(x_2) \Big]^{r/p} d\nu_1(x_1)$$
$$= c_p^{r/p} \int_{\mathcal{O}_1} \Big[\int_{\mathcal{O}_2} \big(|\kappa(x_1,x_2)|^2 \big)^{p/2} d\nu_2(x_2) \Big]^{r/p} d\nu_1(x_1)$$
$$= c_p^{r/p} \int_{\mathcal{O}_1} \Big[\int_{\mathcal{O}_2} |\kappa(x_1,x_2)|^p \, d\nu_2(x_2) \Big]^{r/p} d\nu_1(x_1).$$

The result then follows by applying Itô–Nisio Theorem, see [13]. ∎

When in Theorem 2.3 we choose $H = L^2(D_1; L^2(D_2)) \equiv L^2(D_1) \otimes L^2(D_2)$ for another pair of σ-finite measure spaces $(D_i, \mathcal{C}_i, \mu_i)$, $i = 1, 2$, we get the following result which is a natural generalization of Theorem 2.2.

THEOREM 2.4. *Assume that $(D_i, \mathcal{C}_i, \mu_i)$, $i = 1, 2$, and $(\mathcal{O}_i, \mathcal{F}_i, \nu_i)$, $i = 1, 2$, be σ-finite measure spaces. Let $p, r \in (1, \infty)$. It could be interesting to note that $L^r(\mathcal{O}_1; L^p(\mathcal{O}_2)) \equiv L^r(\mathcal{O}_1) \otimes L^p(\mathcal{O}_2)$. For a bounded linear operator $K : L^2(D_1; L^2(D_2)) \to L^r(\mathcal{O}_1; L^p(\mathcal{O}_2))$ the following assertions are equivalent*:

(i) *K is γ-radonifying.*
(ii) *There exists a $\nu_1 \otimes \nu_2$-measurable function $\kappa : \mathcal{O}_1 \times \mathcal{O}_2 \to L^2(D_1; L^2(D_2))$ with*
$$\int_{\mathcal{O}_1} \Big\{ \int_{\mathcal{O}_2} \Big[\int_{D_1 \times D_2} |\kappa(x_1,x_2;y_1,y_2)|^2 \, d(\mu_1 \otimes \mu_2)(y_1,y_2) \Big]^{p/2} d\nu_2(x_2) \Big\}^{r/p} d\nu_1(x_1) < \infty$$
such that for any $h \in L^2(D_1; L^2(D_2))$, for all $\nu_1 \otimes \nu_2$-almost all $(x_1,x_2) \in \mathcal{O}_1 \times \mathcal{O}_2$
$$(K(h))(x_1,x_2) = \int_{D_1 \times D_2} \kappa(x_1,x_2;y_1,y_2) h(y) \, d(\mu_1 \otimes \mu_2)(y_1,y_2). \tag{7}$$

Moreover, there exists a constant $C > 0$ independent of K such that
$$\frac{1}{C} \int_{\mathcal{O}_1} \Big\{ \int_{\mathcal{O}_2} \Big[\int_{D_1 \times D_2} |\kappa(x_1,x_2;y_1,y_2)|^2 \, d(\mu_1 \otimes \mu_2)(y_1,y_2) \Big]^{p/2} d\nu_2(x_2) \Big\}^{r/p} d\nu_1(x_1)$$
$$\le \|K\|^p_{\gamma(L^2(D_1;L^2(D_2)), L^r(\mathcal{O}_1; L^p(\mathcal{O}_2)))}$$
$$\le C \int_{\mathcal{O}_1} \Big\{ \int_{\mathcal{O}_2} \Big[\int_{D_1 \times D_2} |\kappa(x_1,x_2;y_1,y_2)|^2 \, d(\mu_1 \otimes \mu_2)(y_1,y_2) \Big]^{p/2} d\nu_2(x_2) \Big\}^{r/p} d\nu_1(x_1).$$

We will apply Theorem 2.4 with $D_1 = \mathcal{O}_1 = (0,T)$ and $D_2 = \mathcal{O}_2 = D$, where $T \in (0, \infty)$ and $D \subset \mathbb{R}^d$ and where $\nu_1 = \mu_1 =$ the Lebesgue measure on $(0,T)$ and $\nu_2 = \mu_2 =$ the Lebesgue measure on D. Suppose that A is a generator of a C_0 semigroup on $L^2(D)$. For $\alpha \in (0,1]$, $\beta \in [0,1)$ we consider the operator $K_{\alpha,\beta}$ defined by
$$(K_{\alpha,\beta}(h))(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (-A)^\beta e^{(t-s)A} h(s)\, ds, \ t \in (0,T), \ h \in L^2(0,T; L^2(D)).$$

Here we assume that the fractional power $(-A)^\beta$ of the operator $-A$ exists (e.g. under the condition that the semigroup $S(t) = e^{tA}$ is of negative type), see [18] and [22].

Let $p_\beta(t,x,y)$ be the integral kernel of the operator $(-A)^\beta e^{tA}$, i.e.

$$[(-A)^\beta e^{tA} g](x) = \int_D p_\beta(t,x,y)g(y)\,dy, \quad x \in D,\ g \in L^2(D).$$

Then, for $(t,x) \in (0,T) \times D$ and $h \in L^2(0,T;L^2(D))$ we have

$$(K_{\alpha,\beta}(h))(t,x) = \int_{(0,T)} \int_D \frac{1}{\Gamma(\alpha)} 1_{(0,t)}(s)(t-s)^{\alpha-1} p_\beta(t-s,x,y)h(s,y)\,dy\,ds.$$

Define

$$\kappa_{\alpha,\beta}(t,x;s,y) = \frac{1}{\Gamma(\alpha)} 1_{(0,t)}(s)(t-s)^{\alpha-1} p_\beta(t-s,x,y), \quad (t,x),(s,y) \in (0,T) \times D.$$

Then we have the equality

$$\begin{aligned}\int_{(0,T)} \Big\{ \int_D \Big[\int_{(0,T)\times D} |\kappa_{\alpha,\beta}(t,x;s,y)|^2\,dy\,ds \Big]^{p/2} dx \Big\}^{r/p} dt \\ = \frac{1}{\Gamma(\alpha)^q} \int_{(0,T)} \Big\{ \int_D \Big[\int_{(0,t)\times D} |(t-s)^{\alpha-1} p_\beta(t-s,x,y)|^2\,dy\,ds \Big]^{p/2} dx \Big\}^{r/p} dt\end{aligned}$$

Hence, we get the following

Corollary 2.5. *Assume that $\alpha \in (0,1]$, $\beta \in [0,1)$ and $p,r \in (1,\infty)$. Then the operator $K_{\alpha,\beta}$ is γ-radonifying from $L^2(0,T;L^2(D))$ to $L^r(0,T;L^p(D))$ iff*

$$\int_{(0,T)} \Big\{ \int_D \Big[\int_{(0,t)\times D} |(t-s)^{\alpha-1} p_\beta(t-s,x,y)|^2\,dy\,ds \Big]^{p/2} dx \Big\}^{r/p} dt < \infty. \tag{8}$$

Moreover, we get the following result in a certain special but important case.

Corollary 2.6. *If in addition A is self-adjoint with eigenvalues $\{-\lambda_j\}_{j=1}^\infty$ and the corresponding set of eigenvectors $\{e_j\}$ (which is also an ONB of $L^2(D)$) then the operator $K_{\alpha,\beta}$ is γ-radonifying from $L^2(0,T;L^2(D))$ to $L^r(0,T;L^p(D))$ iff*

$$\int_{(0,T)} \Big\{ \int_D \Big[\sum_j \lambda_j^{1+2\beta-2\alpha} \Gamma(2\alpha-1, 2\lambda_j t)|e_j(x)|^2\,ds \Big]^{p/2} dx \Big\}^{r/p} dt < \infty,$$

where $\Gamma(z,t)$ is the truncated Euler gamma function defined by

$$\Gamma(z,t) = \int_0^t s^{z-1} e^{-s}\,ds, \quad z > 0.$$

Proof. By the assumptions we have

$$p_\beta(t,x,y) = \sum_j \lambda_j^\beta e^{-\lambda_j t} e_j(x) e_j(y), \quad t > 0,\ x,y \in D,$$

and therefore for $0 \le s < t$ and $x \in D$ we have

$$\int_D |(t-s)^{\alpha-1} p_\beta(t-s,x,y)|^2\,dy = (t-s)^{2\alpha-2} \sum_j \lambda_j^{2\beta} |e_j(x)|^2.$$

Hence,

$$\int_{(0,T)} \Big\{ \int_D \Big[\int_{(0,t)} (t-s)^{2\alpha-2} \sum_j \lambda_j^{2\beta} e^{-2\lambda_j (t-s)} |e_j(x)|^2 \, ds \Big]^{p/2} dx \Big\}^{r/p} dt$$
$$= 2^{r(1/2-\alpha)} \int_{(0,T)} \Big\{ \int_D \Big[\sum_j \lambda_j^{1+2\beta-2\alpha} \Gamma(2\alpha-1, 2\lambda_j t) |e_j(x)|^2 \, ds \Big]^{p/2} dx \Big\}^{r/p} dt$$

and the result follows as we claimed. ∎

Next we shall briefly discuss the application of the previous corollaries, which is concerning the regularity of a stochastic evolution equation in a separable Banach space. We consider the following linear SEE:

$$dX(t) = AX(t)\, dt + B\, dW(t); \quad X(0) = x, \tag{9}$$

where A is a generator of a C_0-semigroup on $H = L^2(D)$ (as well as on $E = L^p(D)$) and B is a bounded operator from $L^2(D)$ to $L^p(D)$, and W is a cylindrical Wiener process on $L^2(D)$. We are interested in the regularity of the (weak) solution of (9).

REMARK 2.7. In this and the next remarks, we suppose that E is a martingale type Banach space, see [2]. Note that as usual we can express the solution of (9) as

$$X(t) = S(t)x + \int_0^t S(t-s)B\, dW(s). \tag{10}$$

As discussed in Section 5.2 of Da Prato and Zabczyk [8], we have the following equivalent formula via factorization

$$X(t) = S(t)x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S(t-s) Y_\alpha(s)\, ds, \quad t \in (0,T], \tag{11}$$

where $\alpha \in (0,1)$ and

$$Y_\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} S(t-s) B\, dW(s).$$

Then following Theorem 5.2.6 of Da Prato and Zabczyk [8], or its generalization to martingale type Banach spaces from [2] we have a similar result by replacing Hilbert space H by a separable martingale type Banach space E.

Assume that

(i) The operator A generates an analytic semigroup $S(t), t \geq 0$ on E such that $\|S(t)\| \leq Me^{-\omega t}$, $t \geq 0$, with ω and M positive numbers.
(ii) There exists $\alpha \in (0, \frac{1}{2})$ such that for any $T > 0$

$$\int_0^T t^{-2\alpha} \|S(t)B\|^2_{\gamma(H,E)}\, dt < \infty.$$

Then, for arbitrary $\beta \in [0, \alpha)$, there exists a version of the stochastic convolution $\int_0^t S(t-s)B\, dW(s)$, which is δ-Hölder continuous with values in $D((-A)^\beta)$ for $\delta \in (0, \alpha - \beta)$. Consequently the (weak) solution has a continuous version.

Remark 2.8. Here we try to follow the ideas in Section 10 of Brzeźniak and van Neerven [4] to discuss the regularity of the solution of (9). It is easy to verify that

$$K_{\alpha,\beta} = K_{\alpha_1,\beta_1} K_{\alpha_2,\beta_2},$$

where $\alpha = \alpha_1 + \alpha_2 \in (0,1]$ and $\beta = \beta_1 + \beta_2 \in [0,1)$. By Corollary 2.5, we know that $K_{\alpha_2,\beta}$ is a γ-radonifying operator from $L^2([0,T];L^2(D))$ to $L^r([0,T];L^p(D))$. By Proposition A.1.1 of Da Prato and Zabczyk [8] in the case when E is a Hilbert space or [2] when E is a martingale type 2 Banach space (which is the case for $E = L^p(D)$ with $p \geq 2$), it follows that $K_{\alpha_1,0}$ is a bounded operator from $L^r([0,T];L^p(D))$ to $C^\delta([0,T];L^p(D))$ with $\delta \in (0, \alpha_1 - \frac{1}{r})$. If we allow r to be sufficiently large, we conclude that the operator $K_{1,0} = (-A)^{-\beta} K_{1,\beta} = (-A)^{-\beta} K_{\alpha_1,0} K_{\alpha_2,\beta}$ is a γ-radonifying operator from $L^2([0,T];L^2(D))$ into $C^\delta([0,T];L^p(D))$ with $\delta \in (0,\alpha_1)$ (here we can choose $\alpha_1 = \frac{\alpha}{2}$ and $\alpha_2 = 1 - \frac{\alpha}{2}$ for $\alpha \in (0,1)$). Consequently, following Theorem 10.6 of [4], we can conclude that the $L^p(D)$-valued solution of (9) has a δ-Hölder continuous version for any $\delta \in (0, \frac{\alpha}{2})$ when $X(0) = 0$.

3. Applications to linear stochastic evolution equations driven by spatially homogeneous noise. In this section, motivated by Brzeźniak and van Neerven [4], we shall use results from Section 2 to study space-time regularity of solutions of the linear stochastic Cauchy problem in $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions on $\mathbb{R}^d$:

$$\begin{aligned} du(t) &= Au(t)\,dt + dW_\mu(t), \quad t \geq 0, \\ u(0) &= 0. \end{aligned} \tag{12}$$

Here A is a pseudodifferential operator on $\mathcal{S}'$ with a symmetric and bounded from above symbol $q : \mathbb{R}^d \to \mathbb{C}$, $W_\mu = \{W_\mu(t)\}_{t\geq 0}$ is a given spatially homogeneous Wiener process with spectral measure μ.

A *weak solution* of (12) is a predictable $\mathcal{S}'$-valued process $\{u(t)\}_{t\geq 0}$ such that for all $\phi \in \mathcal{D}(A)$ we have $s \mapsto \langle A\phi, u(s)\rangle \in L^1_{\mathrm{loc}}[0,\infty)$ a.s. and

$$\langle \phi, u(t)\rangle = \int_0^t \langle A\phi, u(s)\rangle\,ds + \langle \phi, W_\mu(t)\rangle \quad \text{a.s.}, \qquad t \geq 0.$$

For a function $f : \mathbb{R}^d \to \mathbb{C}$ we define $\check{f} : \mathbb{R}^d \ni x \mapsto \overline{f(-x)} \in \mathbb{C}$. We say that f is *symmetric* iff $\check{f} = f$. We also define a closed linear subspace $L^2_{(s)}(\mu)$ of $L^2_{\mathbb{C}}(\mu)$ by

$$L^2_{(s)}(\mu) = \{f \in L^2_{\mathbb{C}}(\mu) : \check{f} = f\}.$$

Throughout this section we assume that a measurable function $q : \mathbb{R}^d \to \mathbb{C}$ satisfies the following conditions:

$$q \text{ is smooth and all of its derivatives have at most polynomial growth,} \tag{13}$$

$$q = \check{q}, \tag{14}$$

$$q^* := \sup_{\xi\in\mathbb{R}^d} \operatorname{Re} q(\xi) \;<\; \infty. \tag{15}$$

Hypothesis (H^α). There exists a constant $C > q^*$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(C - \operatorname{Re} q(\xi))^{1-\alpha}}\,d\mu(\xi) < \infty.$$

Next, we will consider a generalization of Proposition 10.4 from [4] where the case $p = r$ was considered. Let us remind the framework from the last paper. We try in this section to keep the notation from [4] as much as possible.

As in Section 4 of [4] we fix a nonnegative symmetric tempered measure μ on $\mathbb{R}^d$ and denote by H_μ the separable real Hilbert space consisting of all tempered distributions of the form $\mathcal{F}^{-1}(f\mu)$ with $f \in L^2_{(s)}(\mu)$ and endowed with the inner product

$$[\mathcal{F}^{-1}(f\mu), \mathcal{F}^{-1}(g\mu)]_{H_\mu} := [f, g]_{L^2_{(s)}(\mu)}. \tag{16}$$

We also define a semigroup of bounded linear operators $\mathbf{S} = \{S(t)\}_{t\geq 0}$ on H_μ by

$$S(t)\big(\mathcal{F}^{-1}(f\mu)\big) = \mathcal{F}^{-1}\big(e^{tq(\cdot)}f(\cdot)\mu\big), \quad t \geq 0.$$

Since q is symmetric and $\operatorname{Re} q$ is bounded from above, the operators $S(t)$ are well-defined.

Let us take and fix $1 \leq p < \infty$ and a weight function $0 \leq \varrho \in L^1(\mathbb{R}^d)$. For $T > 0$ and $t \in (0, T]$, we may define $(\Lambda_T^{1-\alpha/2} f)(t) \in L^p(\varrho)$ by

$$(\Lambda_T^{1-\alpha/2} f)(t) := \frac{1}{\Gamma(1-\alpha/2)} \int_0^t (t-r)^{-\alpha/2} S_{L^p(\varrho)}(t-r) f(r)\, dr,$$

where $S_{L^p(\varrho)}(t)$ is the operator $S(t)$ considered as an operator in $\mathcal{L}(H_\mu, L^p(\varrho))$.

Let us recall Lemma 10.3 and identity (10.4) from [4].

For $(t, x) \in (0, T) \times \mathbb{R}^d$ let us define $\kappa_t^{1-\alpha/2}(x) \in L^2((0, T); H_\mu)$ by

$$\kappa_t^{1-\alpha/2}(x) = \frac{1}{\Gamma(1-\alpha/2)} 1_{(0,t)}(\cdot)(t-\cdot)^{-\alpha/2} \mathcal{F}\big(e^{i\langle x, \cdot\rangle} e^{(t-\cdot)q(-\cdot)}\mu\big)\mathbf{1}_{(0,t)}(\cdot).$$

Then for all $g \in L^2((0, T); H_\mu)$ we have

$$(\Lambda_T^{1-\alpha/2} fg)(t, x) = [\kappa_t^{1-\alpha/2}(\,\cdot\,), g]_{L^2((0,T);H_\mu)}, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \tag{17}$$

and, for $(t, x) \in (0, T) \times \mathbb{R}^d$,

$$\|\kappa_t^{1-\alpha/2}(x)\|_{L^2((0,T);H_\mu)} = \frac{1}{\Gamma(1-\alpha/2)} \Big(\int_0^t \int_{\mathbb{R}^d} s^{-\alpha} e^{2s \operatorname{Re} q(\xi)}\, d\mu(\xi)\, ds\Big)^{1/2}. \tag{18}$$

Applying Lemma 10.1 from [4] we infer that we can find $M = M(T) > 0$ such that

$$\|\kappa_t^{1-\alpha/2}(x)\|_{L^2((0,T);H_\mu)} \leq M \Big[\int_{\mathbb{R}^d} \frac{1}{(C - \operatorname{Re} q(\xi))^{1-\alpha}}\, d\mu(\xi)\Big]^{1/2}, \quad t \in (0, T). \tag{19}$$

Let us observe that the right hand side of (18) is independent of $(t, x) \in (0, T) \times \mathbb{R}^d$.

Now we are ready to formulate the following generalization of [4, Proposition 10.4].

PROPOSITION 3.1. *Assume that the condition* (H^α) *holds for some* $\alpha \in (0, 1)$ *and let* $1 \leq r, p < \infty$. *Then* $\Lambda_T^{1-\alpha/2}$ *is* γ*-radonifying from* $L^2((0, T); H_\mu)$ *into* $L^r((0, T); L^p(\varrho))$.

Proof. Follows from Theorem 2.3 in view of inequality (19) and the observation made immediately after it. ∎

In what follows we will formulate the generalization of Proposition 10.5 and Theorem 10.6 from [4].

For this aim let us recall the following notation. Given a separable real Banach space E and a real number $\beta \in (0, 1)$, the little Hölder space $c_0^\beta([0, T]; E)$ is the (separable) Banach space of all real-valued continuous functions $f : [0,T] \to E$ such that $f(0) = 0$ and

$$\begin{aligned} \|f\| &:= \sup_{t\in[0,T]} |f(t)| + \sup_{0\le s<t\le T} \frac{|f(t)-f(s)|}{(t-s)^\beta} < \infty, \\ &\lim_{\delta\downarrow 0} \sup_{0<t-s\le\delta} \frac{|f(t)-f(s)|}{(t-s)^\beta} = 0. \end{aligned} \tag{20}$$

Proposition 3.2. *Assume that the symbol q satisfies* (H^α) *for some $\alpha \in (0,1)$. Let $1 \le p < \infty$, $\frac{2}{\alpha} < r < \infty$ and $\beta \in (0, \frac{\alpha}{2} - \frac{1}{r})$ be given and assume that the semigroup $\{S(t)\}_{t\ge 0}$ restricts to a C_0-semigroup on $L^p(\varrho)$. Then the operator $\Lambda_T : L^2((0,T); H_\mu) \to L^r((0,T); L^p(\varrho))$ defined by*

$$(\Lambda_T f)(t) := \int_0^t S_{L^p(\varrho)}(t-\tau) f(\tau)\, d\tau$$

takes values in the space $c_0^\beta([0,T]; L^p(\varrho))$. Λ_T is γ-radonifying as an operator from $L^2((0,T); H_\mu)$ into $c_0^\beta([0,T]; L^p(\varrho))$.

Theorem 3.3. *Assume that the symbol q satisfies* (H^α) *for some $\alpha \in (0,1)$. Let $1 \le p < \infty$ and $0 \le \varrho \in L^1$ be fixed. If the semigroup $\{S(t)\}_{t\ge0}$ restricts to a C_0-semigroup on $L^p(\varrho)$, then for all $\beta \in (0, \frac{\alpha}{2})$ the $L^p(\varrho)$-valued solution of the linear stochastic Cauchy problem* (12) *has a β-Hölder continuous version.*

Remark 3.4. The proofs of Proposition 3.2 and Theorem 3.3 are essentially the same as that of Proposition 10.5 and Theorem 10.6 of Brzeźniak and van Neerven [4] by simply replacing $L^r(\varrho)$ with $L^p(\varrho)$. We omit the details.

4. Summing operators and compactness of heat semigroups. By Theorems 2.1 and 2.2, if $H = L^2(D,\mu)$ and

$$(K\psi)(x) = \int_D k(x,y)\psi(y)\, d\mu(y), \quad x \in \mathcal{O},\ \psi \in L^2(D,\mu),$$

then $K : H \to L^p(\mathcal{O},\nu)$ is γ-radonifying if and only if

$$\int_{\mathcal{O}} \Big[\int_D |k(x,y)|^2\, d\mu(y)\Big]^{p/2} d\nu(x) < +\infty.$$

We can pose the following question: *Suppose one replaces $L^2(D,\mu)$ by $L^r(D,\mu)$ $(r>1)$. Does there exist a condition of the form* (3) *for a linear operator $K : L^r(D,\mu) \to L^p(\mathcal{O},\nu)$ to be p-summing?*

This question is closely related to the compactness of heat semigroups (generated by Laplacian on $\mathbb{R}^d$) on some weighted L^p-spaces. In order to study this question, we first introduce the notion of r-summing operators from one Banach space to another Banach space studied first by Pietsch [19].

Let E and F be two separable Banach spaces. Consider a sequence $e = \{e_i\}_{i\in\mathbb{N}}$ of vectors in E. Let $0 < r < +\infty$. We say that e is in $l^r(E)$ provided that $\sum_{i=1}^\infty \|e_i\|_E^r < +\infty$, and we write $\|e\|_{E,r} = \big(\sum_{i=1}^\infty \|e_i\|_E^r\big)^{1/r}$.

Definition 4.1. The sequence e is *scalarly* l^r, and we write $e \in Sl^r(E)$, provided that for every $\xi \in E^*$ the sequence $\{\langle \xi, e_i \rangle\}_{i\in\mathbb{N}}$ belongs to l^r, i.e.

$$\sum_{i=1}^{\infty} |\langle \xi, e_i \rangle|^r < +\infty.$$

Obviously if $e \in l^r(E)$, then $e \in Sl^r(E)$. But the converse is false. In general, we have that a necessary and sufficient condition for e to be Sl^r is

$$\sup_{\|\xi\|_{E^*}\leq 1} \sum_{i=1}^{\infty} |\langle \xi, e_i \rangle|^r < +\infty.$$

We can define the *scalarly* l^r *norm* by

$$\|e\|^*_{E,r} = \sup_{\|\xi\|_{E^*}\leq 1} \Big(\sum_{i=1}^{\infty} |\langle \xi, e_i \rangle|^r \Big)^{1/r}.$$

Obviously, $\|e\|^*_{E,r} \leq \|e\|_{E,r}$.

Consider a continuous linear map $T : E \to F$. Obviously, T maps $l^r(E)$ sequences to $l^r(F)$ sequences, with $\|T(e)\|_{F,r} \leq \|T\|_{\mathcal{L}(E,F)}\|e\|_{E,r}$. Also, T takes Sl^r sequences to Sl^r sequences.

Definition 4.2. A continuous linear map $T : E \to F$ is called *r-summing* if and only if T maps $Sl^r(E)$ into $l^r(F)$. In this case there exists a constant $C < +\infty$ so that

$$\|T(e)\|_{F,r} \leq C\|e\|^*_{E,r}. \tag{21}$$

The best constant in (21) is denoted by $\pi_r(T)$ and is called the *r-summing norm* of T.

We have the following property of r-summing maps (cf. Schwartz [20]):

Proposition 4.3. *If* $1 \leq r < +\infty$ *and a map* $T : E \to F$ *is* r*-summing, then* T *is weakly compact. If, in addition,* E *is reflexive, then* T *is compact.*

We will answer the aforementioned question positively by proving a more general result. We consider a linear operator $K : E \to L^p(\mathcal{O}, \nu)$, defined by

$$(K\psi)(x) = {}_{E^*}\langle \mathcal{K}(x), \psi \rangle_E, \quad \mathcal{K} : \mathcal{O} \to E^*, \tag{22}$$

where E is a Banach space. We assume that $\|\mathcal{K}(x)\|_{E^*} > 0$ for all $x \in \mathcal{O}$. Then we have the following result:

Proposition 4.4. *If* $\mathcal{K}$ *satisfies the condition*

$$\int_{\mathcal{O}} \|\mathcal{K}(x)\|^p_{E^*}\, d\nu(x) < +\infty, \tag{23}$$

then the operator K *defined in* (22) *is* p*-summing. If, in addition, the Banach space* E *is reflexive, then* K *is compact.*

Proof. Let $e = \{e_i\}_{i=1}^{\infty}$ be a $Sl^p(E)$ sequence. Then

$$\begin{aligned}\sum_{i=1}^{\infty} \|Ke_i\|^p_{L^p(\mathcal{O},\nu)} &= \sum_{i=1}^{\infty} \int_{\mathcal{O}} |{}_{E^*}\langle \mathcal{K}(x), e_i \rangle_E|^p\, d\nu(x) \\ &= \sum_{i=1}^{\infty} \int_{\mathcal{O}} \|\mathcal{K}(x)\|^p_{E^*} \Big|{}_{E^*}\Big\langle \frac{\mathcal{K}(x)}{\|\mathcal{K}(x)\|_{E^*}}, e_i \Big\rangle_E\Big|^p\, d\nu(x)\end{aligned}$$

$$= \int_{\mathcal{O}} \|\mathcal{K}(x)\|_{E^*}^p \sum_{i=1}^{\infty} \Big|_{E^*}\Big\langle \frac{\mathcal{K}(x)}{\|\mathcal{K}(x)\|_{E^*}}, e_i \Big\rangle_E \Big|^p \, d\nu(x)$$

$$\le \int_{\mathcal{O}} \|\mathcal{K}(x)\|_{E^*}^p \cdot \sup_{\xi\in E^*, \|\xi\|_{E^*}\le 1} \Big(\sum_{i=1}^{\infty} \big|_{E^*}\langle \xi, e_i\rangle_E\big|^p \Big) \, d\nu(x)$$

$$\le \int_{\mathcal{O}} \|\mathcal{K}(x)\|_{E^*}^p \, d\nu(x) \cdot (\|e\|_{E,p}^*)^p.$$

Now from condition (23), we conclude that K is p-summing. The compactness of K is obvious from Proposition 4.3 of p-summing operators when E is reflexive. ∎

Now we consider a special case, i.e. $E = L^r(D,\mu)$ $(r > 1)$. In this case, the operator K is defined by

$$(K\psi)(x) = \int_D k(x,y)\psi(y)\, d\mu(y), \quad x \in \mathcal{O}. \tag{24}$$

Then, from Proposition 4.4 and the reflexivity of E, we immediately have the following corollary.

Corollary 4.5. *If the operator K is defined in* (24) *and the integral kernel k satisfies the condition*

$$\int_{\mathcal{O}} \Big(\int_D |k(x,y)|^{r/(r-1)}\, d\mu(y) \Big)^{p(r-1)/r} d\nu(x) < +\infty, \tag{25}$$

then K is p-summing. Moreover, K is compact from $L^r(D,\mu)$ to $L^p(\mathcal{O},\nu)$.

Remark 4.6. Obviously, if $r = 2$, then condition (25) coincides with condition (3). Thus the posed question is confirmatively answered by Corollary 4.5.

Next we consider some concrete measurable spaces. Let $\mathcal{O}$ and D be finite-dimensional spaces. Further, we assume that μ and ν are absolutely continuous measures with densities θ $(\theta > 0)$ and λ with respect to Lebesgue measures on D and $\mathcal{O}$ respectively. Let $E = L^r(D,\mu)$, where $d\mu(x) = \theta(x)\, dx$ (θ is strictly positive on D) and $F = L^p(\mathcal{O},\nu)$ with $d\nu(x) = \lambda(x)dx$. The operator $K : E \to F$ is defined by

$$K\psi(x) = \int_D k(x,y)\psi(y)\, dy, \quad x \in \mathcal{O}, \tag{26}$$

where $k(\cdot,\cdot) : \mathcal{O} \times D \to \mathbb{R}$ satisfies $k(x,\cdot)\theta^{-1} \in E^* = L^{r^*}(D,\mu)$ $(r^* = \frac{r}{r-1})$. Then we have the following result:

Proposition 4.7. *If the integral kernel k satisfies*

$$\int_{\mathcal{O}} \Big(\int_D \big|k(x,y)\cdot\theta(y)^{-1/r}\big|^{r/(r-1)}\, dy \Big)^{p(r-1)/r} \cdot \lambda(x)\, dx < +\infty, \tag{27}$$

then the operator K is p-summing. Moreover, K is compact from E to F (due to the reflexivity of E).

Proof. Note that the operators K defined in (26) and (24) are different. However, the operator K defined in (26) can be rewritten as

$$K\psi(x) = \int_D k(x,y)\theta^{-1}(y)\cdot \psi(y)\, d\mu(y). \tag{28}$$

In this case, the condition (25) in Corollary 4.5 is equivalent to

$$\int_{\mathcal{O}}\Big(\int_D |k(x,y)\theta^{-1}(y)|^{r/(r-1)}\,d\mu(y)\Big)^{p(r-1)/r}\,d\nu(x)<+\infty,$$

which is exactly (27). This completes the proof. ■

Now we turn to heat semigroups on the whole space $\mathbb{R}^d$ ($d\geq 1$). By S we denote the semigroup generated by $\frac{1}{2}\Delta$ on $L^p(\mathbb{R}^d)$ ($p\geq 2$). We have for all $t>0$ and all $\varphi\in L^p(\mathbb{R}^d)$

$$S(t)\varphi=G(t,\cdot)*\varphi \quad\text{and}\quad G(t,x)=\frac{1}{(2\pi t)^{d/2}}\exp\Big(-\frac{|x|^2}{2t}\Big),$$

where $*$ denotes the convolution operator.

An *admissible weight* ϱ is a positive, bounded, continuous function $\varrho\in L^1(\mathbb{R}^d)$ such that for all $t>0$ there exists a constant $C_\varrho(t)$ satisfying

$$G(s,\cdot)*\varrho\leq C_\varrho(t)\varrho,\quad \forall s\in[0,t]. \tag{29}$$

We denote by L^p_ϱ the weighted space $L^p(\mathbb{R}^d,\varrho(x)dx)$ with norm

$$\|\varphi\|_{\varrho,p}=\Big(\int_{\mathbb{R}^d}|\varphi(x)|^p\varrho(x)\,dx\Big)^{1/p}.$$

For simplicity, we also denote $L^p(\mathbb{R}^d)$ by L^p.

We have the following nice property for the heat semigroups on weighted spaces.

PROPOSITION 4.8. *For all admissible weights ϱ, the semigroup S can be extended to a C_0-semigroup on L^p_ϱ. Moreover, if $\hat\varrho$ is another admissible weight such that*

$$\int_{\mathbb{R}^d}\frac{\varrho(x)}{\hat\varrho(x)}\,dx<+\infty, \tag{30}$$

then, for all $t>0$, $S(t)$ is compact from $L^p_{\hat\varrho}$ to L^p_ϱ.

Proof. The first part of Proposition 4.8 has been proved by Brzeźniak and Peszat [5] (see Lemma 2.1 of [5]).

We shall use Proposition 4.7 to show the second part. Let $\mathcal{O}=D=\mathbb{R}^d$. Set $\mu(dx)=\hat\varrho(x)\,dx$ and $\nu(dx)=\varrho(x)\,dx$. Now the integral kernel k is given by $k(x,y)=G(t,x-y)$ ($t>0$). If we can verify that

$$\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\big|G(t,x-y)\cdot\hat\varrho(y)^{-1/p}\big|^{p/(p-1)}\,dy\Big)^{p-1}\cdot\varrho(x)\,dx<+\infty, \tag{31}$$

then the semigroups $S(t)$ ($t>0$) are compact according to Proposition 4.7. In fact, by the Jensen inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|G(t,x-y)\cdot\hat\varrho(y)^{-1/p}|^{p/(p-1)}\,dy\Big)^{p-1}\cdot\varrho(x)\,dx\\
&=\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|G(t,x-y)|^{p/(p-1)}\cdot\hat\varrho^{-1/(p-1)}(y)\,dy\Big)^{p-1}\cdot\varrho(x)\,dx\\
&=\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|G(t,x-y)|^{1/(p-1)}\cdot\hat\varrho^{-1/(p-1)}(y)G(t,x-y)\,dy\Big)^{p-1}\cdot\varrho(x)\,dx\\
&\leq\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}G(t,x-y)\hat\varrho^{-1}(y)G(t,x-y)\,dy\Big)\cdot\varrho(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G^2(t,x-y)\hat{\varrho}^{-1}(y)\varrho(x)\,dx\,dy\\
&\le (4\pi t)^{-d/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t/2,x-y)\hat{\varrho}^{-1}(y)\varrho(x)\,dx\,dy\\
&\le (4\pi t)^{-d/2}C_\varrho(t/2)\int_{\mathbb{R}^d}\hat{\varrho}^{-1}(x)\varrho(x)\,dx. \qquad (32)
\end{aligned}
$$

From condition (30), it immediately follows that (31) holds and consequently $S(t)$ is p-summing and compact from $L^p_{\hat{\varrho}}$ to L^p_{ϱ}. This completes the proof. ∎

Acknowledgments. Part of the research was carried out during the visit of the first named author to the Department of Mathematical Sciences, Florida Atlantic University. Long is partially supported by FAU start-up funding at the C. E. Schmidt College of Science. Final touches to this paper are due to the visit of the second named author to the Department of Mathematics, University of York, supported by the LMS grant Ref 21029. Both authors would like to thank the respective host Departments for their hospitality.

References

[1] Z. Brzeźniak, *Stochastic partial differential equations in M-type* 2 *Banach spaces*, Potential Anal. 4 (1995), 1–45.

[2] Z. Brzeźniak, *On stochastic convolution in Banach spaces and applications*, Stochastics Stochastics Rep. 61 (1997), 245–295.

[3] Z. Brzeźniak, K. D. Elworthy, *Stochastic differential equations on Banach manifolds*, Methods Funct. Anal. Topology 6 (2000), 43–84.

[4] Z. Brzeźniak, J. van Neerven, *Space-time regularity for linear stochastic evolution equations driven by spatially homogeneous noise*, J. Math. Kyoto Univ. 43 (2003), 261–303.

[5] Z. Brzeźniak, S. Peszat, *Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process*, Studia Math. 137 (1999), 261–299.

[6] G. Da Prato, D. Gątarek, J. Zabczyk, *Invariant measures for semilinear stochastic equations*, Stochastic Anal. Appl. 10 (1992), 387–408.

[7] G. Da Prato, S. Kwapień, J. Zabczyk, *Regularity of solutions of linear stochastic equations in Hilbert spaces*, Stochastics 23 (1987), 1–23.

[8] G. Da Prato, J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge 1996.

[9] E. Dettweiler, *Banach space valued processes with independent increments and stochastic integration*, in: Probability in Banach Spaces IV (Oberwolfach, 1982), Lecture Notes in Math. 990, Springer, Berlin 1983, 54–83.

[10] L. Gross, *Abstract Wiener spaces*, in: 1967 Proc. Fifth Berkeley Sympos. Math. Statist. and Probability, vol. 2, Univ. California Press, Berkeley 1967, 31–42.

[11] N. Kalton, J. M. A. M. van Neerven, M. Veraar, L. Weis, *Embedding vector-valued Besov spaces into spaces of γ-radonifying operators*, Math. Nachr. 281 (2008), 238–252.

[12] H. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Math. 463, Springer, Berlin 1975.

[13] S. Kwapień, W. A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple*, Probab. Appl., Birkhäuser, Boston 1992.

[14] J. M. A. M. van Neerven, *γ-Radonifying operators: a survey*, in: The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, Proc. Centre Math. Appl. Austral. Nat. Univ. 44, Austral. Nat. Univ., Canberra 2010, 1–61.

[15] J. M. A. M. van Neerven, L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. 166 (2005), 131–170.

[16] J. M. A. M. van Neerven, L. Weis, *Weak limits and integrals of Gaussian covariances in Banach spaces*, Probab. Math. Statist. 25 (2005), Acta Univ. Wratislav. No. 2784, 55–74.

[17] A. L. Neidhardt, *Stochastic integrals in 2-uniformly smooth Banach spaces*, Ph.D. thesis, University of Wisconsin, 1978.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer, New York 1983.

[19] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. 28 (1967), 333–353.

[20] L. Schwartz, *Geometry and Probability in Banach Spaces*, Lecture Notes in Math. 852, Springer, Berlin 1981.

[21] G. Tessitore, J. Zabczyk, *Invariant measures for stochastic heat equations*, Probab. Math. Statist. 18 (1998), Acta Univ. Wratislav. No. 2111, 271–287.

[22] K. Yosida, *Functional Analysis*, 6th ed., Grundlehren Math. Wiss. 123, Springer, Berlin 1980.