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EXISTENCE OF EXPLOSIVE SOLUTIONS TO SOME NONLINEAR PARABOLIC ITÔ EQUATIONS

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Abstract. The paper is concerned with the problem of existence of explosive solutions for a class of nonlinear parabolic Itô equations. Under some sufficient conditions on the initial state and the coefficients, it is proven by the method of auxiliary functionals that there exist explosive solutions with positive probability. The main results are presented in Theorems 3.1 and 3.2 under different sets of conditions. An example is given to illustrate some application of the second theorem.

1. Introduction. Consider the initial-boundary problem for a reaction-diffusion equation in domain $\mathcal{D} \subset \mathbb{R}^d$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \triangle\, u + f(u), && t > 0, \\ u(x,0) &= g(x), && x \in \mathcal{D}, \\ u(x,t) &= 0, && x \in \partial\mathcal{D}, \end{aligned} \tag{1.1}$$

where $\triangle$ is the Laplacian operator, $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$, and the functions f and g are given such that the problem (1.1) has a unique local solution. In 1963 it was first shown by S. Kaplan [10] that, for a certain class of nonlinear functions $f(u)$, the solution of equation (1.1) becomes infinite or explodes at a finite time, provided that the initial state $g(x)$ and the nonlinear function $f(u)$ satisfy appropriate conditions. His result was later extended by Fujita [6] and many others. Since then it has become known that solutions to more general nonlinear parabolic equations may develop singularities in finite time, see, e.g., the review article [7] and the book [13], where an extensive references can be found. Physically this phenomenon is manifested as the explosion in combustion,

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reaction diffusion and branching diffusion problems. It is therefore of interest to examine the effect of a random perturbation to equation (1.1) on the existence of an explosive solution. This consideration has led us to investigate the question of nonexistence of a global solution to the following type of parabolic Itô equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \triangle u + f(u) + \sigma(u)\partial_t W(x,t), && t > 0, \\ &u(x,0) = g(x), && x \in \mathcal{D}, \\ &u(x,t) = 0, && x \in \partial\mathcal{D}, \end{aligned} \tag{1.2}$$

with a multiplicative noise, where σ is a given function and $W(x,t)$ is a Wiener random field. So far, little is known about the existence of explosive solutions for SPDEs. Recently we studied the existence of explosive solutions for a class of nonlinear stochastic wave equations. Based on a stochastic energy method, we were able to obtain some sufficient conditions for the blow-up of the second moments of solutions, or in the L^2-norm [2]. In the case of nonlinear parabolic Itô equations, we studied the explosive positive solutions in the papers [3] and [4]. The results are restricted to the explosion of positive solutions in the L^p-norm, or the p-th moment of the solution. In contrast, for stochastic ordinary differential equations, general results on the explosion and non-explosion of solutions, in probability or with probability one, have been well established (see, e.g., [9], [11] among others). Technically the method of auxiliary (Lyapunov) function has been employed extensively to find sufficient conditions for stability and explosion of solutions. In this paper this method will be adopted to establish certain sufficient conditions on the coefficients of some parabolic Itô equations for explosion with positive probability. This approach does not require the positivity of the solution and the coefficients may also depend on the gradients of the solution.

In this paper we shall give a brief review of some basic results for nonlinear parabolic Itô equations in Section 2. The main results are presented in Section 3 as two theorems. In Theorem 3.1, by means of a globally defined auxiliary functional, it is proven that, under certain sufficient conditions on the coefficients, there exists an explosive solution with positive probability. By relaxing the global condition on the auxiliary functional, it is only required to be defined outside a large ball. A similar explosion result is proved in Theorem 3.2. Finally an example is given in Section 4 to show a possible application of Theorem 3.2. However, the problem of proving an almost sure explosion result remains open.

2. Preliminaries. Let $\mathcal{D}$ be a domain in $\mathbb{R}^d$, which has a smooth boundary $\partial\mathcal{D}$ if it is bounded. We set $H = L^2(\mathcal{D})$ with the inner product and norm denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $H^1 = H^1(\mathcal{D})$ be the L^2-Sobolev space of first order and let $V = H_0^1$ denote the closure in H^1 of the space of C^1-functions with compact support in $\mathcal{D}$. Let $\|\cdot\|_1$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the norm in V and the duality pairing between V and $V' = H^{-1}$.

Let $W(x,t)$, for $x \in \mathbb{R}^d$, $t \geq 0$, be a continuous Wiener random field defined in a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$ ([1, p. 38]). It has mean

$EW(x,t)=0$ and covariance function $q(x,y)$ defined by

$$EW(x,t)W(y,s) = (t \wedge s)q(x,y), \quad x, y \in \mathbb{R}^d,$$

where $(t \wedge s) = \min\{t,s\}$ for $0 \le t, s \le T$.

Consider the initial-boundary value problem for the parabolic Itô equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = Au + f(u, \nabla u, x, t) + \sigma(u, \nabla u, x, t)\partial_t W(x,t), \\ &u(x,0) = h(x), \qquad x \in \mathcal{D}, \\ &u(x,t)|_{\partial\mathcal{D}} = 0, \qquad t \in (0,T), \end{aligned} \tag{2.1}$$

where $A = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}[a_{ij}(x)\frac{\partial}{\partial x_j}]$ is a symmetric, uniformly elliptic operator with smooth coefficients (say, in $C^3(\overline{\mathcal{D}})$), that is, there exists a constant $a_0 > 0$ such that

$$b(x,\xi) := \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge a_0|\xi|^2, \tag{2.2}$$

for all $x \in \overline{\mathcal{D}}$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Certain conditions will be imposed on the functions f, σ, h later.

Now, to regard the equation (2.1) with a homogeneous boundary condition as an Itô equation in the Hilbert space H, we set $u_t = u(\cdot,t), F_t(u) = f(u, \nabla u, \cdot, t)$, $\Sigma_t(u) = \sigma(u, \nabla u, \cdot, t)$ and so on, and rewrite it as

$$\begin{aligned} du_t &= [Au_t + F_t(u_t)]\,dt + \Sigma_t(u_t)\,dW_t, \quad 0 < t < T, \\ u_0 &= h, \end{aligned} \tag{2.3}$$

where A is now regarded as a continuous linear operator from V to V'. The nonlinear function $F_t : V \to H$ and, with $v \in V$, the linear operator $\Sigma_t(v) : V \to H$ are all continuous for $t > 0$. We assume that the covariance function $q(x,y)$ satisfies the condition

$$\int_{\mathcal{D}} q(x,x)\,dx < \infty.$$

Then we can rewrite equation (2.3) as

$$u_t = g + \int_0^t [Au_s + F_s(u_s)]\,ds + \int_0^t \Sigma_s(u_s)\,dW_s, \tag{2.4}$$

where the stochastic integral is well defined (see [1, Theorem 2.4]). By the semigroup approach [5], it is well known that there exists a mild solution of equation (2.4) under suitable conditions. However, for the problem under consideration, it requires the use of Itô calculus. Therefore we need to consider the strong or variational solution [12].

Under the usual conditions, such as the stochastic coercivity, Lipschitz continuity and monotonicity conditions, the equation (2.4) is known to have a unique global strong solution $u \in C\big([0,T]; H\big) \cap L^2\big((0,T); V\big)$ for any $T > 0$ ([1, Theorem 7.4]). Moreover, let $\Phi : [0,T] \times H \to \mathbb{R}$. A functional $\Phi(t,v)$ will be called an Itô functional if it is once differentiable in t and twice Fréchet-differentiable in v such that the first Fréchet derivative $\Phi'(t,v) \in V$ for $v \in V$. Then the following Itô formula holds [12], [8]

$$\begin{aligned}\Phi(t,u_t) = \Phi(0,u_0) + \int_0^t \Big\{\frac{\partial}{\partial s}\Phi(s,u_s) + \mathcal{L}\,\Phi(s,u_s)\Big\}\,ds \\ + \int_0^t \big(\Phi'(s,u_s), \Sigma_s(u_s)\,dW_s\big)\end{aligned} \tag{2.5}$$

and

$$\mathcal{L}\,\Phi = \frac{1}{2}\operatorname{Tr}\Phi''\Sigma Q\Sigma^\star + \langle Au_s, \Phi'\rangle + \big(F_s(u_s), \Phi'\big) \tag{2.6}$$

where Φ'' denotes the second Fréchet derivative of Φ, Q is the covariance operator with kernel q, the star means the adjoint and Tr is the trace of an operator.

On the other hand, if the nonlinear terms are only locally Lipschitz continuous and the monotonicity condition is dropped, one can only assert the existence of a unique local solution. In this case, for $h \in H$, the solution u_t^h in H is said to explode or blow up with positive probability if the probability $\mathbb{P}\{\tau^h < \infty\} > 0$, where τ^h is the explosion time defined by $\tau^h = \inf\{t > 0 : \|u_t^h\| = \infty\}$ [9].

3. Existence of explosive solutions. To simplify the notation, we consider the autonomous parabolic Itô equation (2.1), where the functions f and σ are independent of t. Then the corresponding stochastic evolution equation (2.3) yields

$$\begin{aligned} du_t &= [Au_t + F(u_t)]\,dt + \Sigma(u_t)\,dW_t, \quad 0 < t < T,\\ u_0 &= h, \end{aligned} \tag{3.1}$$

THEOREM 3.1. *Suppose that, given $h \in H$, the parabolic Itô equation* (3.1) *has a unique strong local solution u_t^h. Assume that there exist a bounded nonnegative Itô functional U on H and a constant $C > 0$ such that*

$$\mathcal{L}\,U(v) \geq CU(v), \quad \text{for any } v \in H^1. \tag{3.2}$$

Then, for any $h \in H$ with $U(h) > 0$, the solution u_t^h will explode with a positive probability. More precisely, for any $\varepsilon > 0$, the following inequality holds

$$\mathbb{P}\Big(\tau^h < \frac{1}{C}\ln\Big[\frac{M}{U(h)}\Big] + \varepsilon\Big) > 0, \tag{3.3}$$

where we set $M = \sup_{\eta \in H} U(\eta)$.

Proof. Let $\Phi(t,v) = U(v)e^{-Ct}$. Then Φ is also an Itô functional. Since a strong solution u_t^h is known to be a strong Markov process, for any stopping time τ, we can apply the Itô–Dynkin formula to obtain

$$\Phi(t\wedge\tau, u_{t\wedge\tau}) = U(h) + \int_0^{t\wedge\tau} e^{-Cs}(\mathcal{L} - C)U(u_s)\,ds + \int_0^{t\wedge\tau} e^{-Cs}\big(U'(s,u_s), \Sigma(u_s)\,dW_s\big).$$

Define $\tau_n^h = \inf\{t > 0 : \|u_t^h\| > n\}$. By setting $\tau = \tau_n^h$, taking an expectation and making use of condition (3.2), the above inequality yields

$$\mathbb{E}\Phi(t\wedge\tau_n^h, u^h_{t\wedge\tau_n^h}) = \mathbb{E}\big\{U(u_{t\wedge\tau_n^h})e^{\{-C(t\wedge\tau_n^h)\}}\big\} \geq U(h)$$

or

$$M\mathbb{E}e^{\{-C(t\wedge\tau_n^h)\}} \geq U(h)$$

with $M = \sup_{\eta \in H} U(\eta)$. As $n \to \infty$, we have $\tau_n^h \to \tau^h$ a.s., where τ^h is an explosion time. Then the above inequality gives

$$\mathbb{E}e^{\{-C(t\wedge\tau^h)\}} \geq U(h)/M. \tag{3.4}$$

We will show that $\mathbb{P}(\tau^h < \infty) > 0$. Suppose otherwise and assume $\mathbb{P}(\tau^h = \infty) = 1$. Then the above inequality implies that

$$e^{\{-Ct\}} \geq U(h)/M,$$

or

$$t \leq \frac{1}{C}\ln\Big[\frac{M}{U(h)}\Big], \quad \forall\, t > 0,$$

which is impossible. Hence we have $\mathbb{P}(\tau^h < \infty) > 0$ by contradiction. Moreover, by taking $t \to \infty$ in equation (3.4), it can be rewritten as

$$\mathbb{E}\exp\Big\{-C\Big[\tau^h - \frac{1}{C}\ln\frac{M}{U(h)}\Big]\Big\} \geq 1,$$

which implies

$$\mathbb{P}\Big(\tau^h < \frac{1}{C}\ln\Big[\frac{M}{U(h)}\Big] + \varepsilon\Big) > 0,$$

for any $\varepsilon > 0$. ∎

In the above theorem, the auxiliary function $U(v)$ must be defined for each $v \in H$. It seems plausible to speculate that the explosive behavior of solutions u_t^h depends mostly on the properties of the coefficients far away from the origin. This consideration leads to the following theorem under some milder conditions on the coefficients. For $\rho > 0$, define $B_\rho = \{v \in H : \|v\| < \rho\}$ with complement $B_\rho^c = \{v \in H : \|v\| \geq \rho\}$, $S_\rho = \{v \in H : \|v\| = \rho\}$. Also let $\tau_\rho = \inf\{t > 0 : \|u_t^h\| \geq \rho\}$ and $\tau_{S_\rho} = \{t > 0 : u_t^h \in S_\rho\}$.

Theorem 3.2. *Suppose that, given $h \in H$, the parabolic Itô equation* (3.1) *has a unique local strong solution u_t. Assume there exist a positive Itô functional $\Phi(t, v)$ defined for $t > 0$ and $v \in B_r^c$ for some $r > 0$, and positive constants K_1, K_2, K_3 and C such that the following conditions hold*:

(C1)
$$\sup_{t\geq 0, v\in B_r^c} \Phi(t, v) = K_1,$$

(C2)
$$\sup_{t\geq 0, v\in S_r} \Phi(t, v) = K_2 < \inf_{t\geq 0, v\in S_\rho} \Phi(t, v) = K_3, \quad \textit{for some } \rho > r,$$

(C3)
$$\mathcal{L}_t\,\Phi(t, v) := \Big(\frac{\partial}{\partial t} + \mathcal{L}\Big)\Phi(t, v) \geq C\Phi(t, v), \quad \textit{for } t \geq 0,\ v \in H^1 \cap B_r^c.$$

If $u_{t_1}^h = v_1 \in S_\rho$, for some $t_1 \geq 0$, then the solution will explode with positive probability.

Proof. Let $\tau_R(t) = \min\{\tau_R, t, \tau_{S_r}\}$. It follows from condition (C3) that

$$\mathcal{L}_t\,\{\Phi(t, v)\exp[-C(t - t_1)]\} \geq 0$$

for $v \in B_r^c$, $t > t_1$. This inequality and Itô–Dynkin's formula imply, for $t > t_1$, the inequality

$$\mathbb{E}^{t_1, v_1}\big\{\Phi(\tau_R(t), u_{\tau_R(t)})\exp[-C(\tau_R(t) - t_1)]\big\} \geq \Phi(t_1, v), \tag{3.5}$$

where $\mathbb{E}^{t_1,v_1}$ denotes the conditional expectation given $u_{t_1} = v_1$. Note that the inequality (3.5) can be expressed as follows:

$$\begin{aligned} &\mathbb{E}^{t_1,v_1}\big\{\Phi(\tau_R, u_{\tau_R})\exp[-C(\tau_R - t_1)]\mathbb{I}_{\{\tau_R<t\wedge\tau_{S_r}\}}\big\} \\ &+ \mathbb{E}^{t_1,v_1}\big\{\Phi(\tau_{S_r}, u_{\tau_{S_r}})\exp[-C(\tau_{S_r} - t_1)]\mathbb{I}_{\{\tau_{S_r}<t\wedge\tau_R\}}\big\} \\ &+ \mathbb{E}^{t_1,v_1}\{\Phi(t, u_t)\exp[-C(t-t_1)]\mathbb{I}_{\{t<\tau_R\wedge\tau_{S_r}\}}\} \geq \Phi(t_1, v_1), \end{aligned}$$

where $t \wedge s = \min\{t, s\}$ and $\mathbb{I}_D$ denotes the indicator function of D. It follows from this inequality and conditions (C2) and (C3) that, for $v_1 \in \Gamma$, $t_1 \geq 0$,

$$\begin{aligned} &\mathbb{E}^{t_1,v_1}\big\{\Phi(\tau_R, u_{\tau_R})\exp[-C(\tau_R - t_1)]\mathbb{I}_{\{\tau_R<t\wedge\tau_{S_r}\}}\big\} \\ &\geq \Phi(t_1, v_1) - K_2 - K_1\exp[-C(t-t_1)] \geq K_3 - K_2 - K_1\exp[-C(t-t_1)]. \end{aligned} \tag{3.6}$$

Letting $t \to \infty$, $R \to \infty$ and making use of condition (C1) again, for $v_1 \in S_\rho$ and $t_1 \geq 0$, we obtain

$$K_1\mathbb{E}^{t_1,v_1}\exp[-C(\tau_\infty - t_1)] \geq K_3 - K_2 > 0,$$

which implies $\mathbb{P}^{t_1,v_1}\{\tau_\infty < \infty\} > 0$. This proves the theorem. ∎

Remark. Even though we considered the autonomous equation (3.1), with an obvious modification, Theorem 3.2 is also applicable to the equation when the noise term Σ depends on t continuously.

4. Example. As mentioned in Introduction, the existence of positive explosive solutions to some nonlinear parabolic Itô equations was treated in the sense of L^p-norm. Here we shall give an example of using Theorem 3.2 to show the non-existence of a global solution with positive probability. To be specific, consider the following problem in one dimension:

$$\begin{gathered} \partial_t u = \partial_x^2 u + f(u, \partial_x u) + \sigma(t)(\partial_x u)\partial_t W(x,t), \quad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0;\ u(x,0) = h(x), \end{gathered} \tag{4.1}$$

where $W(x,t)$ is a continuous Wiener random field with the covariance function $q(x,y)$, $f(u, \partial_x u)$ is a nonlinear function to be specified, and $\sigma(t)$ is a strictly increasing, continuous function with $\sigma(0) = 0$ and bounded by a sufficiently large constant σ_0. Assume that the function $f(\xi, \eta)$ is of the form

$$f(\xi,\eta) = g(\xi) + \xi\eta^2 \quad \text{for} \quad \xi, \eta \in \mathbb{R}, \tag{4.2}$$

where g is a continuous function as yet to be specified. Define

$$(Qv, v) := \int_0^1\int_0^1 q(x,y)v(x)v(y)\,dx\,dy \geq q_1\|v\|^2.$$

Assume that the correlation function $q(x,y)$ is bounded and positive definite such that

$$q_1\|v\|^2 \leq (Qv, v) \leq q_2\|v\|^2, \tag{4.3}$$

for some constants $q_2 > q_1 > 0$ and $v \in H = L^2(0,1)$. Introduce the auxiliary functional $\Phi(v)$, independent of t, as follows

$$\Phi(v) = K_1 - \frac{1}{(\ln\|v\|^2)^\alpha},$$

for some $\alpha > 0$ and $\|v\| > r > 1$, where K_1 is chosen such that

$$K_1 > \frac{1}{(\ln r^2)^\alpha}\,.$$

Then it is easy to check that the conditions (C1) and (C2) of Theorem 3.2 are satisfied. To verify condition (C3), since $\Phi(v)$ is bounded for $\|v\| \geq r$, it suffices to show that $\mathcal{L}_t\Phi(v) \geq C_0$ with $t \geq t_1$ for some positive constants C_0 and t_1. By some simple computations, we can obtain

$$\begin{aligned}\mathcal{L}_t\Phi(v) = {} & \frac{\alpha}{\|v\|^2(\ln\|v\|^2)^{(1+\alpha)/2}}\Big\{\Big[\frac{\sigma^2(t)}{2}(Q\partial_x v, \partial_x v) \\ & - \|\partial_x v\|^2 + (g(v), v) + \|v\partial_x v\|^2\Big] \\ & - \frac{2}{\|v\|^2}\Big(1 + \frac{1+\alpha}{2\ln\|v\|^2}\Big)\sigma^2(Qv\partial_x v, v\partial_x v)\Big\} \\ \geq {} & \frac{\alpha}{\|v\|^2(\ln\|v\|^2)^{(1+\alpha)/2}}\Big\{\Big[\Big(\frac{1}{2}\, q_1\sigma^2(t) - 1\Big)\|\partial_x v\|^2 + (g(v), v) + \|v\partial_x v\|^2\Big] \\ & - \frac{2\sigma_0^2 q_2}{\|v\|^2}\Big(1 + \frac{1+\alpha}{2\ln\|v\|^2}\Big)\|v\partial_x v\|^2\Big\},\end{aligned} \tag{4.4}$$

where use was made of condition (4.3). Let $t_1 > 0$ such that

$$\frac{1}{2}\, q_1\sigma^2(t_1) > 1.$$

Then we make use of condition (4.3) and take r to be so large that

$$\frac{2\sigma_0^2 q_2}{\|v\|^2}\Big(1 + \frac{1+\alpha}{2\ln\|v\|^2}\Big) \leq 1,$$

for $\|v\| > r$. By using the preceding two inequalities in equation (4.4), we obtain

$$\mathcal{L}_t\,\Phi(v) \geq \frac{\alpha(g(v), v)}{\|v\|^2(\ln\|v\|^2)^{(1+\alpha)/2}}\,,$$

for $t \geq t_1$ and $\|v\| \geq r$. If, in addition,

$$(g(v), v) \geq C_1\|v\|^2(\ln\|v\|^2)^{(1+\alpha)/2}$$

for $\|v\| \geq r$, we can conclude that there is $C_0 > 0$ such that $\mathcal{L}_t\,\Phi(v) \geq C_0$. By Theorem 3.2, for $\|u_{t_1}\| \geq r$, the solution u_t will explode with positive probability.

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