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PATHWISE UNIQUENESS FOR STOCHASTIC PDEs

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Dedicated to Jerzy Zabczyk

Abstract. We consider a stochastic evolution equation in a separable Hilbert spaces H or in a separable Banach space E with a Hölder continuous perturbation on the drift. We review some recent result about pathwise uniqueness for this equation.

1. Introduction. Let us start with a stochastic differential equation on a separable Hilbert space H (norm $|\cdot|$, scalar product $\langle\cdot,\cdot\rangle$),

$$\begin{cases} dX = (AX + b(X))\,dt + dW(t), \\ X(0) = x, \end{cases} \tag{1}$$

where $A : D(A) \subset H \to H$ is linear, $B : H \to H$ is continuous and $W(\cdot)$ is a cylindrical Wiener process in H. In the first part of the paper we shall assume that

Hypothesis 1.

(i) *A is symmetric negative and there exists $\delta \in (0,1)$ such that $A^{-1+\delta}$ is of trace class.*
(ii) *$b : H \to H$ is α-Hölder continuous and bounded for some $\alpha \in (0,1)$.*
(iii) *W is an H-valued cylindrical process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.*

By Hypothesis 1 (i) A^{-1} is compact, so there exist an orthonormal basis (e_k) in H and a sequence (α_k) of positive numbers such that

$$Ae_k = -\alpha_k e_k, \quad k \in \mathbb{N}. \tag{2}$$

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We assume that W is formally given by

$$W(t) = \sum_{k=1}^{\infty} e_k W_k(t),$$

where (W_k) is a sequence of standard real Brownian motions in $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ mutually independent.

We say that an adapted process X is a *mild* solution of problem (1) if

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} b(X(s))\, ds + W_A(t), \quad \mathbb{P}\text{-a.s.}, \tag{3}$$

where $W_A(t)$ is the *stochastic convolution*,

$$W_A(t) = \int_0^t e^{(t-s)A}\, dW(s), \qquad t \geq 0.$$

Recall that, under Hypothesis 1, the process $W_A(\cdot)$ is well defined and continuous, see [DaZa92].

If b is Lipschitz continuous, one can prove by a straightforward fixed point argument that the mild equation (3) has a unique solution. But if b is only α-Hölder continuous this method fails.

In Section 2 we discuss *pathwise* uniqueness for problem (1) in its mild form (3) under Hypothesis 1.

It is well known that pathwise uniqueness together with existence of a weak solution, yields existence via the infinite dimensional Yamada–Watanabe theory, see [On04], [RoScZh08].

In Section 3 we consider pathwise uniqueness for a more general equation, namely a stochastic reaction-diffusion equation with an α-Hölder continuous perturbation.

When H is finite dimensional, several papers have been devoted to pathwise uniqueness. We recall, besides the seminal paper by Veretennikov, important contributions by Krylov and Röckner [KrRö05] and Flandoli, Gubinelli and Priola [FlGuPr10].

As regard to the infinite dimensional case we quote Gyöngy and Krylov [GyKr96], Da Prato and Flandoli [DaFl10], Da Prato, Flandoli, Priola and Röckner [DFPR13] and Cerrai, Da Prato and Flandoli [CeDaFl13].

We end this section with some notation used in what follows. Let E and F be Banach spaces. We denote by $C_b(E;F)$ the Banach space of all uniformly continuous and bounded mappings $\varphi : E \to F$. Note that $C_b(E;F)$, endowed with the norm

$$\|\varphi\|_0 := \sup_{x\in E} |\varphi(x)|, \quad x \in E,$$

is a Banach space.

Moreover, for any $k \in \mathbb{N}$, $C_b^k(E;F)$ will represent the subspace of $C_b(E,F)$ of those functions which are uniformly continuous and bounded together with their derivatives of order less than k. It is a Banach space with the norm

$$\|\varphi\|_k := \|\varphi\|_0 + \sum_{j=1}^{k} \|D^j\varphi\|_0.$$

Finally, for any $\alpha \in (0,1)$, $C_b^\alpha(E;F)$ is the space of all functions $\varphi \in C_b(E;F)$ such that

$$[\varphi]_\alpha := \sup_{x,y\in E, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} < \infty.$$

We set

$$\|\varphi\|_\alpha := \|\varphi\|_0 + [\varphi]_\alpha.$$

When $F = \mathbb{R}$ we shall write $C_b(E;\mathbb{R}) = C_b(E)$ and so on, for short.

2. Transforming the SPDE. Let us recall some results on pathwise uniqueness when H is finite dimensional, say $H = \mathbb{R}^n$, and $A = 0$. Assume that X is a strong solution to

$$\begin{cases} dX = b(X)\,dt + dW(t), \\ X(0) = x, \end{cases} \tag{4}$$

with $b \in C_b^\alpha(\mathbb{R}^n)$. To prove pathwise uniqueness, the main idea is to transform (4) by the Zvonkin transformation ([Zv74]) into an integral equation with Lipschitz nonlinear coefficients for which uniqueness will follow by Gronwall's lemma, see [FlGuPr10].

Let us fix an orthonormal basis $e_1, \dots, e_n$ in $\mathbb{R}^n$ and for $k = 1, \dots, n$ set

$$\begin{aligned} &X_k(t) = \langle X(t), e_k\rangle, \quad W_k(t) = \langle W(t), e_k\rangle, \quad t \geq 0, \\ &b_k(x) = \langle b(x), e_k\rangle, \quad x \in H. \end{aligned}$$

Then equation (4) is equivalent to the system

$$\begin{cases} dX_k = b_k(X)\,dt + dW_k(t), \quad k = 1, \dots, n, \\ X_k(0) = x_k. \end{cases}$$

Now for any $k \in \mathbb{N}$ consider the elliptic equation

$$\lambda u_k(x) - \frac{1}{2}\,\Delta u_k(x) - \langle b(x), Du_k(x)\rangle = b_k(x), \quad x \in \mathbb{R}^n, \tag{5}$$

where $\lambda > 0$ will be chosen later.

By the classical Schauder estimates, equation (5) has a unique solution $u_k \in C_b^{2+\alpha}(\mathbb{R}^n)$. Moreover, the following estimates hold

$$\begin{aligned} &\|u_k\|_1 \leq C_\alpha \lambda^{-(1+\alpha)/2}\|b_k\|_\alpha, \qquad \forall\, \lambda > 0,\ \forall\, k \in \mathbb{N}, \\ &\|u_k\|_2 \leq C_\alpha \lambda^{-\alpha/2}\|b_k\|_\alpha, \qquad \forall\, \lambda > 0,\ \forall\, k \in \mathbb{N}, \end{aligned} \tag{6}$$

where C_α is a positive constant independent of λ.

Since u_k is C^2 we can apply Itô's formula to $u_k(X(t))$ and write

$$\begin{aligned} du_k(X(t)) = \frac{1}{2}\,\Delta u_k(X(t))\,dt &+ \langle b(X(t), Du_k(X(t))\rangle\,dt \\ &+ \langle Du_k(X(t)), dW(t)\rangle. \end{aligned}$$

Now, taking into account (5), we find

$$du_k(X(t)) = \lambda u_k(X(t))\,dt - b_k(X(t))\,dt + \langle Du_k(X(t)), dW(t)\rangle.$$

Therefore integrating with respect to t we obtain

$$\int_0^t b_k(X(s))\,ds = u_k(x) - u_k(X(t)) + \lambda \int_0^t u_k(X(s))\,ds + \int_0^t \langle Du_k(X(s)), dW(s)\rangle.$$

Now by substituting this into

$$X_k(t) = x_k + \int_0^t b_k(X(s))\,ds + W_k(t), \quad t \geq 0,$$

we get finally

$$X_k(t) = x_k + u_k(x) - u_k(X(t)) + \lambda \int_0^t u_k(X(s))\,ds + \int_0^t \langle Du_k(X(s)), dW(s)\rangle. \tag{7}$$

It is convenient to write (7) in a vector form. Namely let

$$u(x) = \sum_{k=1}^n u_k e_k \quad \text{and} \quad X(t) = \sum_{k=1}^n X_k(t) e_k.$$

By (7), summing up on k, we find

$$X(t) = x + u(X(t)) - u(x) + \lambda \int_0^t u(X(s))\,ds + \sum_{k=1}^n \int_0^t \big\langle Du_k(X(s)), dW(s)\big\rangle. \tag{8}$$

This is an equation with (at least) Lipschitz coefficients because $Du_k \in C^{1+\alpha}(\mathbb{R}^n)$. So, uniqueness will follow from Gronwall's lemma. Let in fact X and Y be two solutions of (8). Then we have

$$\begin{aligned} X(t) - Y(t) = u(X(t)) - u(Y(t)) + \lambda \int_0^t \big(u(X(s)) - u(Y(s))\big)\,ds \\ + \sum_{k=1}^n \int_0^t \big\langle Du_k(X(s)) - Du_k(Y(s)), dW(s)\big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} |X(t) - Y(t)| \leq \|u\|_1\, |X(t) - Y(t)| + \lambda \|u\|_1 \int_0^t |X(s)) - Y(s)|\,ds \\ + \sum_{k=1}^n \Big| \int_0^t \big\langle Du_k(X(s)) - Du_k(Y(s)), dW(s)\big\rangle \Big|, \end{aligned}$$

from which, taking the square and the expectation of both sides, we obtain

$$\begin{aligned} \mathbb{E}|X(t) - Y(t)|^2 &\leq 3\|u\|_1^2\, \mathbb{E}|X(t) - Y(t)|^2 \\ &+ 3\lambda^2 \|u\|_1^2 \int_0^t \mathbb{E}|X(s) - Y(s)|^2\,ds + 3\|u\|_2^2 \sum_{k=1}^n \int_0^t \mathbb{E}|X(s) - Y(s)|^2\,ds. \end{aligned}$$

By taking into account (6) it follows that

$$\begin{aligned} \mathbb{E}|X(t) - Y(t)|^2 &\leq 3C_\alpha^2 \lambda^{-1-\alpha}\, \mathbb{E}|X(t) - Y(t)|^2 \\ &+ 3C_\alpha^2 \lambda^{1-\alpha} \int_0^t |\mathbb{E}X(s) - Y(s)|^2\,ds + 3C_\alpha^2 \lambda^{-\alpha} n \int_0^t \mathbb{E}|X(s) - Y(s)|^2\,ds. \end{aligned} \tag{9}$$

Now choose $\lambda = \lambda_\alpha$ such that $3C_\alpha^2 \lambda_\alpha^{-1-\alpha} = 1/2$. Then by (9) we obtain

$$\mathbb{E}|X(t) - Y(t)|^2 \leq 6C_\alpha^2 (\lambda_\alpha^{1-\alpha} + \lambda_\alpha^{-\alpha} n) \int_0^t \mathbb{E}|X(s) - Y(s)|^2\,ds, \tag{10}$$

which by Gronwall's lemma implies $X = Y$.

Let now go to the infinite dimensional problem (1). One will try to repeat the previous argument based on the Zvonkin transform. Let us list the main problems one is faced with.

First, since A is unbounded one has to write (1) in the mild form (3). Then we recall that there exists an orthonormal basis (e_k) in H and a sequence (α_k) of positive numbers such that (2) is fulfilled and we reduce (3) to a system of infinitely many stochastic differential equations setting

$$X_k(t) = \langle X(t), e_k\rangle, \quad W_k(t) = \langle W(t), e_k\rangle, \quad t \geq 0,\ k \in \mathbb{N},$$
$$b_k(x) = \langle b(x), e_k\rangle, \quad x \in H,\ k \in \mathbb{N}.$$

Then equation (3) is equivalent to the system

$$\begin{cases} dX_k = -\alpha_k X_k + b_k(X)\,dt + dW_k(t), & k \in \mathbb{N}, \\ X_k(0) = x_k. \end{cases}$$

Second, the elliptic equation (5) is replaced by an equation with infinitely many variables, namely,

$$\lambda u_k - \mathscr{L} u_k - \langle b, Du_k\rangle = b_k, \tag{11}$$

where $\mathscr{L}$ is the infinitesimal generator of the Ornstein–Uhlenbeck semigroup R_t, $t \geq 0$, in $C_b(H)$, see [DaZa02]. Let us recall its definition.

$$R_t\varphi(x) = \int_H \varphi(y)\, N_{e^{tA}x, Q_t}(dy), \quad \varphi \in C_b(H), \quad t \geq 0,$$

where $N_{e^{tA}x,Q_t}$ is the Gaussian measure in H with mean $e^{tA}x$ and covariance

$$Q_t = -\tfrac{1}{2}\, A^{-1}(I - e^{2tA}), \quad t \geq 0.$$

The infinitesimal generator $\mathscr{L}$ can be defined through the Laplace transform of R_t, see [Ce94]. Namely we have

$$(\mu - \mathscr{L})^{-1} f(x) = \int_0^{+\infty} e^{-\mu t} R_t f(x)\, dt, \quad \mu > 0,\ f \in C_b(H).$$

However, one can still prove the following Schauder estimates, see [DaZa02].

Proposition 2. *There is $C_\alpha > 0$ independent of λ such that*

$$\|u_k\|_1 \leq C_\alpha \lambda^{-(1+\alpha)/2} \|b_k\|_\alpha, \quad \forall\,\lambda > 0,\ \forall\,k \in \mathbb{N},$$
$$\|u_k\|_2 \leq C_\alpha \lambda^{-\alpha/2} \|b_k\|_\alpha, \quad \forall\,\lambda > 0,\ \forall\,k \in \mathbb{N}.$$

Third, the transformed equation reads now as follows

$$X(t) = x + \int_0^t e^{A(t-s)} X(s)\, ds + u(X(t)) - e^{At}u(x) + A\int_0^t e^{A(t-s)} u(X(s))\, ds$$
$$+ \lambda \int_0^t e^{A(t-s)} u(X(s))\, ds + \sum_{k=1}^\infty \int_0^t e^{A(t-s)} \big\langle Du_k(X(s)), dW(s)\big\rangle.$$

The more difficult terms to handle are

$$A\int_0^t e^{A(t-s)} u(X(s))\, ds$$

and

$$\sum_{k=1}^{\infty} \int_0^t e^{A(t-s)} \big\langle Du_k(X(s)), dW(s) \big\rangle. \tag{12}$$

The first one requires a well known maximal regularity result for the problem

$$y'(s) = Ay(s) + f(s), \ s \in [0,t], \quad y(0) = 0. \tag{13}$$

It says that for the solution of (13),

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)\, ds, \quad t \geq 0,$$

the following estimate holds

$$|Ay|_{L^2(0,t;H)} \leq c|f|_{L^2(0,t;H)}.$$

See e.g. Lions and Magenes [LiMa67].

To estimate the second term (12), one exploits the fact that A^{-1} is of trace class.

Finally, after some manipulation, one can prove the result.

Theorem 3 ([DaFl10]). *Assume that Hypothesis 1 is fulfilled. Then problem* (1) *has at most one solution.*

3. Reaction-diffusion equations. We now consider the following reaction-diffusion equation with Dirichlet boundary conditions, in the Banach space $E := C([0,1])$,

$$\begin{gathered} dX(t,\xi) = \big[D_\xi^2 X(t,\xi) + p(X(t,\xi)) + g(X(t,\xi))\big]\, dt + dW(t,\xi), \quad \xi \in [0,1], \ t > 0, \\ X(t,0) = X(t,1) = 0, \quad t > 0, \\ X(0) = x \in E, \end{gathered} \tag{14}$$

where p is a polynomial of odd degree greater than 1 with negative leading coefficient (for instance $p(x) = -x^3$), $g \in C_b^\alpha(E;E)$ with $\alpha \in (0,1)$ and W is the Brownian sheet in E.

When $g = 0$, equation (14) becomes

$$\begin{gathered} dX(t,\xi) = \big[D_\xi^2 X(t,\xi) + p(X(t,\xi))\big]\, dt + dW(t,\xi), \quad \xi \in [0,1], \ t > 0, \\ X(t,0) = X(t,1) = 0, \quad t > 0, \\ X(0) = x \in E. \end{gathered} \tag{15}$$

Equation (15) has been extensively studied both in $H = L^2(0,1)$ and in $E = C([0,1])$, see in particular the monograph by Da Prato and Zabczyk [DaZa92] and the paper by Cerrai [Ce03].

Let us recall how to solve problem (15) in E in its mild form,

$$X(t) = x + \int_0^t e^{(t-s)A} p(X(s))\, ds + \int_0^t e^{(t-s)A}\, dW(s). \tag{16}$$

Since p is locally Lipschitz in E, local existence for (16) follows by the usual fixed point argument. Moreover, exploiting the fact that p is monotone decreasing, one is able to find an a priori estimate and so, to show that the solution is global.

Finally, we quote a regularity result proved in [Ce03], needed in what follows. For any $\epsilon > 0$ we shall denote by E_ϵ the subspace of E consisting of ϵ-Hölder continuous functions, endowed with the norm

$$|x|_{E_\epsilon} := |x|_E + \sup_{\substack{\xi,\eta \in [0,1] \\ \xi \neq \eta}} \frac{|x(\xi) - x(\eta)|}{|\xi - \eta|^\epsilon}.$$

Moreover, we shall denote by $B_b(E_\epsilon, E_\epsilon)$ the space of all bounded and Borel mappings from E_ϵ into E_ϵ. Then there exists an $\epsilon_0 > 0$ such that for any $x \in E$,

$$X(t,x) \in E_{\epsilon_0}, \quad t > 0, \quad \mathbb{P}\text{-a.s.} \tag{17}$$

and the mapping $E \ni x \to X(t,x) \in E_{\epsilon_0}$ is continuous, $\mathbb{P}$-a.s.

If $g \neq 0$ all these arguments do not work. However, pathwise uniqueness can be proved as we are going to show.

REMARK 4. As we said, problem (14) can also be studied on $H = L^2(0,1)$ in this case, however, b is no more locally Lipschitz (it is in fact not continuous). Consequently in this case we are not able to show pathwise uniqueness for equation (15). So, we have to work on the Banach space $E = C([0,1])$ even if this will cause some extra difficulties which we shall explain later.

3.1. Pathwise uniqueness for problem (14). Repeating the previous argument, we reduce (14) to a system setting,

$$X_k(t) = \langle X(t), e_k\rangle, \quad W_k(t) = \langle W(t), e_k\rangle, \quad t \ge 0,\ k \in \mathbb{N},$$
$$p_k(x) = \langle b(x), e_k\rangle, \quad g_k(x) = \langle g(x), e_k\rangle, \quad x \in H,\ k \in \mathbb{N}.$$

Then equation (15) can be written as

$$\begin{gathered} dX_k = -\alpha_k X_k + p_k(X)\,dt + g_k(X)\,dt + dW_k(t), \quad k \in \mathbb{N}, \\ X_k(0) = x_k. \end{gathered} \tag{18}$$

Now we introduce an auxiliary elliptic equation as in the previous section (recall (11)). To this purpose we consider the transition semigroup associated with problem (15),

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x)], \quad \varphi \in C_b(E),\ t \ge 0,\ x \in E.$$

One can show, see [Ce01], that there exists a unique m-dissipative operator $\mathscr{K}$ in $C_b(E)$, the infinitesimal generator of P_t, such that

$$(\lambda - \mathscr{K})^{-1} f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\,dt, \quad \lambda > 0,\ x \in E,\ f \in C_b(E).$$

Now we consider the elliptic equation

$$\lambda u_k - \mathscr{K} u_k - \langle b, Du_k\rangle_{E,E^*} = b_k. \tag{19}$$

Here the notation $\langle \cdot,\cdot\rangle_{E,E^*}$ means the duality between E and its topological dual E^*.

Equation (19) can be solved by a fixed point argument. However, in this case the Schauder estimates require more work and some new tools in interpolation theory, see Cerrai and Da Prato [CeDa11].

REMARK 5. Notice that the Schauder estimates for reaction-diffusion equations seem not to hold in the Hilbert space $H = L^2(0,1)$, see [Ce01].

Let us recall the result:

PROPOSITION 6. *Let* $\lambda > 0$ *and* $f \in C_b^\alpha(E)$*. Then there exists a unique solution* $\varphi \in D(\mathscr{K}) \cap C_b^{2+\alpha}(E)$ *of* (19)*. Moreover, there is* $C_\alpha > 0$*, independent of* λ *such that*

$$\|u_k\|_1 \le C_\alpha \lambda^{-(1+\alpha)/2} \|b_k\|_\alpha, \quad \forall\, \lambda > 0,\ \forall\, k \in \mathbb{N},$$

$$\|u_k\|_2 \le C_\alpha \lambda^{-\alpha/2} \|b_k\|_\alpha, \quad \forall\, \lambda > 0,\ \forall\, k \in \mathbb{N},$$

where C_α *is a positive constant independent of* λ*.*

Now we can proceed as in Section 1 using the Zvorkin transformation. Several technicalities arise, however, when working in the Banach space E and the proof is much longer than in the case of the Hilbert space.

We end this section giving some insight on the Kolmogorov equations in the Banach space E.

3.1.1. *Kolmogorov equation on E.* If $\varphi \in C_b^2(E)$, $\mathscr{K}$ looks like

$$\mathscr{K}\varphi = \frac{1}{2}\sum_{k=1}^{\infty} D^2\varphi(e_k, e_k) + \langle Ax + p(x), D\varphi(x)\rangle_{E,E^*},$$

where E^* is the topological dual of E.

Moreover, the *identité du carré des champs* takes the form

$$\mathscr{K}(\varphi^2) = 2\varphi\mathscr{K}\varphi + \sum_{k=1}^{\infty} \left|\langle e_k, D\varphi\rangle_{E,E^*}\right|^2, \tag{20}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal system in $L^2(0,1)$ given by the eigenfunctions of the Laplacian endowed with Dirichlet boundary conditions.

Notice that, in order to give a meaning to equation (20) for $\varphi \in D(\mathscr{K})$, we have to prove that, see [CeDa11]:

(i) $D(\mathscr{K})$ is included in $C_b^1(E)$,
(ii) the series in (20) is convergent for any $\varphi \in D(\mathscr{K})$,
(iii) $\varphi^2 \in D(\mathscr{K})$, for any $\varphi \in D(\mathscr{K})$, and (20) is fulfilled.

In conclusion, the following result can be proved.

THEOREM 7 ([CeDaFl13]). *Assume that there exist* $\alpha, \epsilon > 0$ *such that* $B \in C_b^\alpha(E,E) \cap B_b(E_{\epsilon_0}, E_\epsilon)$*, where* ϵ_0 *is defined by* (17)*. Then equation* (14) *has at most one mild solution.*

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