

DYNAMIC PROGRAMMING FOR AN INVESTMENT/CONSUMPTION PROBLEM IN ILLIQUID MARKETS WITH REGIME-SWITCHING

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Abstract. We consider an illiquid financial market with different regimes modeled by a continuous time finite-state Markov chain. The investor can trade a stock only at the discrete arrival times of a Cox process with intensity depending on the market regime. Moreover, the risky asset price is subject to liquidity shocks, which change its rate of return and volatility, and induce jumps on its dynamics. In this setting, we study the problem of an economic agent optimizing her expected utility from consumption under a non-bankruptcy constraint. In this paper we perform the first step needed to treat this model: the proof of the dynamic programming principle (DPP) and the characterization of the value function as the unique viscosity solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. This puts the basis for the analysis of the optimal solution of the model which is done in the companion paper of the authors (SIAM J. Control. Optim. 52 (2014)). The proof of the dynamic programming principle is not standard as in this case we do not know a priori if the value function is continuous up to the boundary.

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1. Introduction. A classical assumption in the theory of optimal portfolio/consumption choice as in Merton [11] is that assets are continuously tradable by agents. This is not always realistic in practice, and illiquid markets provide a prime example. Indeed, an important aspect of market liquidity is the time restriction on assets trading: investors cannot buy and sell them immediately, and have to wait some time before being able to unwind a position in some financial assets. In the past years, there was a significant strand of literature addressing these liquidity constraints. In [13], [10], the price process is observed continuously but the trades succeed only at the jump times of a Poisson process. Recently, the papers [12], [3], [7] relax the continuous-time price observation by considering that asset is observed only at the random trading times. In all these cited papers, the intensity of trading times is constant or deterministic. However, the market liquidity is also affected by long-term macroeconomic conditions, for example by financial crisis or political turmoil, and so the level of trading activity measured by its intensity should vary randomly over time. Moreover, liquidity breakdowns would typically induce drops on the stock price in addition to changes in its rate of return and volatility.

In this paper, we investigate the effects of such liquidity features on the optimal portfolio choice. We model the index of market liquidity as an observable continuous-time Markov chain with finite-state regimes, which is consistent with some cyclicity observed in financial markets. The economic agent can trade only at the discrete arrival times of a Cox process with intensity depending on the market regimes. Moreover, the risky asset price is subject to liquidity shocks, which switch its rate of return and volatility, while inducing jumps on its dynamics. In this hybrid jump-diffusion setting with regime switching, we study the optimal investment/consumption problem over an infinite horizon under a non-bankruptcy state constraint.

Our main results are the following.

- We first prove carefully that the dynamic programming principle (DPP) holds in our framework. Due to the state constraints in two dimensions, we have to slightly weaken the standard continuity assumption, see Remark 3.3.
- Then, using the DPP, we characterize the value function of this stochastic control problem as the unique constrained viscosity solution to a system of integro-partial differential equations.

These two results set up the basis for the deep study (analytical and numerical of the optimal solution of the model), which is done in the companion paper [6].

The rest of the paper is structured as follows. Section 2 describes our continuous-time market model with regime-switching liquidity, and formulates the optimization problem for the investor. In Section 3 we state and prove the DPP. Section 4 is devoted to the analytic characterization of the value function as the unique viscosity solution to the dynamic programming equation.

2. A market model with regime-switching liquidity. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Let I be a continuous-time Markov chain valued in the finite state space $\mathbb{I}_d = \{1, \dots, d\}$, with intensity matrix $Q = (q_{ij})$. For $i \neq j$ in \mathbb{I}_d , we can associate to the jump process I , a Poisson process N^{ij} with intensity rate $q_{ij} \geq 0$, such that a switch from state i to j corresponds to a jump of N^{ij} when I is in state i . We interpret the process I as a proxy for market liquidity with states (or regimes) representing the level of liquidity activity, in the sense that the intensity of trading times varies with the regime value. This is modeled through a Cox process $(N_t)_{t \geq 0}$ with intensity $(\lambda_{I_t})_{t \geq 0}$, where $\lambda_i > 0$ for each $i \in \mathbb{I}_d$. For example, if $\lambda_i < \lambda_j$, this means that trading times occur more often in regime j than in regime i . The increasing sequence of jump times $(\tau_n)_{n \geq 0}$, $\tau_0 = 0$, associated to the counting process N represents the random times when an investor can trade a risky asset of price process S . Note that under these assumptions the jumps of I and N are a.s. disjoint.

In the liquidity regime $I_t = i$, the stock price follows the dynamics

$$dS_t = S_t(b_i dt + \sigma_i dW_t),$$

where W is a standard Brownian motion independent of (I, N) , and $b_i \in \mathbb{R}$, $\sigma_i \geq 0$, for $i \in \mathbb{I}_d$. Moreover, at the times of transition from $I_{t^-} = i$ to $I_t = j$, the stock changes as follows:

$$S_t = S_{t^-}(1 - \gamma_{ij})$$

for a given $\gamma_{ij} \in (-\infty, 1)$, so the stock price remains strictly positive, and we may have a relative loss (if $\gamma_{ij} > 0$), or gain (if $\gamma_{ij} \leq 0$). Typically, there is a drop of the stock price after a liquidity breakdown, i.e. $\gamma_{ij} > 0$ for $\lambda_j < \lambda_i$. Overall, the risky asset is governed by a regime-switching jump-diffusion model:

$$dS_t = S_{t^-}(b_{I_{t^-}} dt + \sigma_{I_{t^-}} dW_t - \gamma_{I_{t^-}, I_t} dN_t^{I_{t^-}, I_t}).$$

Portfolio dynamics under liquidity constraint. We consider an agent investing and consuming in this regime-switching market. We denote by (Y_t) the total amount invested in the stock, and by (c_t) the consumption rate per unit of time, which is a nonnegative adapted process. Since the number of shares Y_t/S_t in the stock held by the investor has to be kept constant between two trading dates τ_n and τ_{n+1} , then between such trading times, the process Y follows the dynamics:

$$dY_t = Y_{t^-} \frac{dS_t}{S_{t^-}}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0.$$

The trading strategy is represented by a predictable process (ζ_t) such that at a trading time $t = \tau_{n+1}$, the rebalancing on the number of shares induces a jump ζ_t in the amount invested in the stock:

$$\Delta Y_t = \zeta_t.$$

Overall, the càdlàg process Y is governed by the hybrid controlled jump-diffusion process

$$dY_t = Y_{t^-}(b_{I_{t^-}} dt + \sigma_{I_{t^-}} dW_t - \gamma_{I_{t^-}, I_t} dN_t^{I_{t^-}, I_t}) + \zeta_t dN_t. \tag{2.1}$$

If we assume for simplicity a constant savings account, the amount (X_t) invested in cash then follows

$$dX_t = -c_t dt - \zeta_t dN_t. \tag{2.2}$$

The total wealth is defined at any time $t \geq 0$, by $R_t = X_t + Y_t$, and we shall require the non-bankruptcy constraint at any trading time:

$$R_{\tau_n} \geq 0, \quad \text{a.s.} \quad \forall n \geq 0. \tag{2.3}$$

Actually since the asset price may become arbitrarily large or small between two trading dates, this non-bankruptcy constraint is equivalent to a no-short sale constraint on both the stock and savings account (see Lemma 2.1 in [6]):

$$X_t \geq 0 \text{ and } Y_t \geq 0, \quad \forall t \geq 0. \tag{2.4}$$

Given an initial state $(i, x, y) \in \mathbb{I}_d \times \mathbb{R}_+ \times \mathbb{R}_+$, we shall denote by $\mathcal{A}_i(x, y)$ the set of investment/consumption control process (ζ, c) such that the corresponding process (X, Y) solution to (2.1)–(2.2) with a liquidity regime I , and starting from $(I_{0-}, X_{0-}, Y_{0-}) = (i, x, y)$, satisfy the non-bankruptcy constraint (2.3).

Optimal investment/consumption problem. The preferences of the agent are described by a utility function U which is increasing, concave, C^1 on $(0, \infty)$ with $U(0) = 0$, and satisfies the usual Inada conditions: $U'(0) = \infty$, $U'(\infty) = 0$. We assume the following growth condition on U : there exist some positive constant K , and $p \in (0, 1)$ such that

$$U(x) \leq Kx^p, \quad x \geq 0. \tag{2.5}$$

We denote by \tilde{U} the convex conjugate of U , defined from \mathbb{R} into $[0, \infty]$ by

$$\tilde{U}(\ell) = \sup_{x \geq 0} [U(x) - x\ell].$$

The agent’s objective is to maximize over portfolio/consumption strategies in the above illiquid market model the expected utility from consumption rate over an infinite horizon. We then consider, for each $i \in \mathbb{I}_d$, the value function

$$v_i(x, y) = \sup_{(\zeta, c) \in \mathcal{A}_i(x, y)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(c_t) dt \right], \quad (x, y) \in \mathbb{R}_+^2, \tag{2.6}$$

where ρ is a discount factor. We also introduce, for $i \in \mathbb{I}_d$, the function

$$\hat{v}_i(r) = \sup_{x \in [0, r]} v_i(x, r - x), \quad r \geq 0, \tag{2.7}$$

which represents the maximal utility performance that the agent can achieve starting from an initial nonnegative wealth r and from the regime i . More generally, for any locally bounded function w_i on \mathbb{R}_+^2 , we associate the function \hat{w}_i defined on \mathbb{R}_+ by $\hat{w}_i(r) = \sup_{x \in [0, r]} w_i(x, y)$, so that

$$\hat{w}_i(x + y) = \sup_{e \in [-y, x]} w_i(x - e, y + e), \quad (x, y) \in \mathbb{R}_+^2.$$

In the sequel, we shall often identify a d -tuple function $(w_i)_{i \in \mathbb{I}_d}$ defined on \mathbb{R}_+^2 with the function w defined on $\mathbb{R}_+^2 \times \mathbb{I}_d$ by $w(x, y, i) = w_i(x, y)$.

In this paper, we focus on the characterization of the value functions v_i (and so \hat{v}_i), $i \in \mathbb{I}_d$, as unique solutions of the associated HJB system.

We first need to check that the value functions are well-defined and finite. Let us consider for $p > 0$, the positive constant

$$k(p) := \max_{i \in \mathbb{I}_d, z \in [0, 1]} \left[pb_i z - \frac{\sigma_i^2}{2} p(1-p)z^2 + \sum_{j \neq i} q_{ij} ((1 - z\gamma_{ij})^p - 1) \right] < \infty.$$

Under (2.5), we make the standing assumption that

$$\rho > k(p),$$

which shall ensure the finiteness of the value function. More precisely, the following growth condition is showed in [6]: there exists some positive constant C such that

$$v_i(x, y) \leq C(x + y)^p, \quad \forall (i, x, y) \in \mathbb{I}_d \times \mathbb{R}_+^2. \tag{2.8}$$

We next state continuity properties of the value functions.

PROPOSITION 2.1. *The value functions v_i , $i \in \mathbb{I}_d$, are concave in \mathbb{R}_+^2 , and so continuous on the interior of \mathbb{R}_+^2 . Moreover, for $i \in \mathbb{I}_d$ the function $x \mapsto v_i(x, 0)$ and $y \mapsto v_i(0, y)$ are continuous on \mathbb{R}_+ .*

Proof. The concavity of v_i in (x, y) follows from the linearity of the admissibility constraints in X, Y, ζ, c , and the concavity of U . Since v_i is concave, it is continuous on the interior of its domain \mathbb{R}_+^2 and on the axes. ■

3. Dynamic programming principle. First of all we formulate our control problem in a suitable weak sense which is the good one to prove the dynamic programming principle.

DEFINITION 3.1. Given $(i, x, y) \in \mathbb{I}_d \times \mathbb{R}_+ \times \mathbb{R}_+$, a control \mathcal{U} is a 9-tuple $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, W, I, N, c, \zeta)$, where:

1. $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a filtered probability space satisfying the usual conditions.
2. I is a Markov chain with space state \mathbb{I}_d and generator Q , $I_0 = i$ a.s., N is a Cox process with intensity (λ_{I_t}) , and W is an \mathbb{F} -Brownian motion independent of (I, N) .
3. $\mathcal{F}_t = \sigma(W_s, I_s, N_s; s \leq t) \vee \mathcal{N}$, where \mathcal{N} is the collection of all \mathbb{P} -null sets of \mathcal{F} .
4. (c_t) is \mathbb{F} -progressively measurable, (ζ_t) is \mathbb{F} -predictable.

We say that \mathcal{U} is *admissible* (writing $\mathcal{U} \in \mathcal{A}_i^w(x, y)$), if the solution (X, Y) to (2.2)–(2.1) with $X_0 = x, Y_0 = y$, satisfies $X_t \geq 0, Y_t \geq 0$ a.s.

Given $\mathcal{U} \in \mathcal{A}_i^w(x, y)$, define $J(\mathcal{U}) = \mathbb{E}[\int_0^\infty e^{-\rho s} U(c_s) ds]$, and the value function

$$v_i(x, y) = \sup_{\mathcal{U} \in \mathcal{A}_i^w(x, y)} J(\mathcal{U}).$$

Note that in the above formula the supremum is taken in a set which is larger than the one used in (2.6). Indeed it can be proved that the resulting value function is the same. This fact can be proved exactly as e.g in [5], Chapter 2. We skip the proof here for brevity giving a hint in Remark 3.2. For this reason we will use the name v_i for the value functions also in this weak setting.

The main result of this section is the following

THEOREM 3.1. *For every initial conditions i, x, y ,*

$$\begin{aligned} v_i(x, y) &= \sup_{\mathcal{U} \in \mathcal{A}_i^w(x, y)} \sup_{\tau \in \mathcal{T}_{\mathbb{F}}} \mathbb{E} \left[\int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} v_{I_\tau}(X_\tau, Y_\tau) \right] \\ &= \sup_{\mathcal{U} \in \mathcal{A}_i^w(x, y)} \inf_{\tau \in \mathcal{T}_{\mathbb{F}}} \mathbb{E} \left[\int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} v_{I_\tau}(X_\tau, Y_\tau) \right], \end{aligned} \tag{3.1}$$

where by $\mathcal{T}_{\mathbb{F}}$ we denote the set of finite stopping times with respect to the filtration \mathbb{F} (here, given with the admissible control \mathcal{U}).

Before proving this theorem we state some technical lemmas.

LEMMA 3.1. *Given $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t), W, I, N)$ satisfying the conditions of Definition 3.1, define $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t \geq 0}$, where $\mathcal{F}_t^0 = \sigma(W_s, I_s, N_s; s \leq t)$. Then if (c_t) is \mathbb{F} -progressively measurable (resp. predictable), there exists $c_1 \mathbb{F}^0$ -progressively measurable (resp. predictable) such that $c = c_1 d\mathbb{P} \otimes dt$ a.e.*

Proof. The proof can be found in [5], Chapter 2, from which this Lemma is taken. Here we give a sketch of the proof for the reader convenience. We first use Lemma 3.2.4, page 133 in [9], to find, for each $n \in \mathbb{N}$, an approximating \mathcal{F}_t -simple process c^n converging to c in the $L^2(dt \otimes d\mathbb{P})$ norm. Then, using Lemma 1.25, page 13 in [8], we can change every c^n on a null-set and find a sequence of $\mathcal{F}_s^{t,0}$ -simple process $c_1^n(t)$ that again converges to c in the $L^2(dt \otimes d\mathbb{P})$ norm. We now extract a subsequence (denoted again by c_1^n) such that $c_1^n \rightarrow c$ a.e. and we define $c_1 := \liminf_{n \rightarrow +\infty} c_1^n$. This is $\mathcal{F}_s^{t,0}$ -progressively measurable and $c = c_1, dt \otimes d\mathbb{P}$ a.e. on $[0, +\infty) \times \Omega$. This concludes the proof. ■

REMARK 3.1. With the notation of the previous lemma, it is easy to check that $(X^{c', \zeta'}, Y^{c', \zeta'}) \sim (X^{c, \zeta}, Y^{c, \zeta})$ in law. Hence without loss of generality we can assume that c is \mathbb{F}^0 -progressively measurable and ζ is \mathbb{F}^0 -predictable.

Define \mathcal{W} as the space of continuous functions on $\mathbb{R}_+, \mathcal{I}$ the space of càdlàg \mathbb{I}_d -valued functions, \mathcal{N} the space of nondecreasing càdlàg \mathbb{N} -valued functions. On $\mathcal{W} \times \mathcal{I} \times \mathcal{N}$, define the filtration $(\mathcal{B}_t^0)_{t \geq 0}$, where \mathcal{B}_t^0 is the smallest σ -algebra making the coordinate mappings for $s \leq t$ measurable, and define $\mathcal{B}_{t+}^0 = \bigcap_{s > t} \mathcal{B}_s^0$.

LEMMA 3.2. *If c is \mathbb{F}^0 -progressively measurable (resp. \mathbb{F}^0 -predictable), there exists a \mathcal{B}_{t+}^0 -progressively measurable (resp. \mathcal{B}_t^0 -predictable) process $f_c : \mathbb{R}_+ \times \mathcal{W} \times \mathcal{I} \times \mathcal{N} \rightarrow \mathbb{R}$, such that*

$$c_t = f_c(t, W_{\cdot \wedge t}, I_{\cdot \wedge t}, N_{\cdot \wedge t}), \quad \text{for } \mathbb{P}\text{-a.e. } \omega, \quad \text{for all } t \in \mathbb{R}_+.$$

Proof. For the progressively measurable part one can see e.g. Theorem 2.10 in [14]. For the predictable part the proof can be found in [5], Chapter 2. Here we give a sketch of the proof for the reader convenience. For c predictable we notice that the claim is true if $c = X \mathbf{1}_{(t, s]}$, where X is \mathcal{F}_t^0 -measurable; then a monotone class argument allows us to conclude. ■

Proof of Theorem 3.1. Let $V_i^+(x, y), V_i^-(x, y)$ be respectively the right hand side in the first and second lines of (3.1). Clearly $V_i^+(x, y) \geq V_i^-(x, y)$, so that it will be enough to prove $V_i^+(x, y) \leq v_i(x, y) \leq V_i^-(x, y)$.

Step 1. $v_i(x, y) \leq V_i^-(x, y)$: Take $\mathcal{U} \in \mathcal{A}_i^w(x, y)$ and $\tau \in \mathcal{T}_{\mathbb{F}}$. Then

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} U(c_t) dt \mid \mathcal{F}_\tau \right] = \int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} \mathbb{E} \left[\int_0^\infty e^{-\rho s} U(c_{\tau+s}) ds \mid \mathcal{F}_\tau \right]. \quad (3.2)$$

By Remark 3.1, without loss of generality we can assume that c is \mathbb{F}^0 -progressively measurable (resp. $\zeta \mathbb{F}^0$ -predictable). For $\omega_0 \in \Omega$, define the shifted control $\tilde{U}^{\omega_0} = (\Omega, \tilde{\mathcal{F}}^\tau, \mathbb{P}_{\omega_0}, \tilde{\mathcal{F}}_\tau, \tilde{W}, \tilde{I}, \tilde{N}, \tilde{c}, \tilde{\zeta})$, where

- $\mathbb{P}_{\omega_0} = \mathbb{P}(\cdot \mid \mathcal{F}_\tau)(\omega_0)$,
- $\tilde{W}_t = W_{\tau+t} - W_\tau$,
- $\tilde{I}_t = I_{\tau+t}$,

- $N'_t = N_{\tau+t} - N_\tau$,
- \tilde{F}^τ is the augmentation of \mathcal{F} by the \mathbb{P}_{ω_0} -null sets, and $\tilde{\mathcal{F}}^\tau_t$ is the augmented filtration generated by $(\tilde{W}, \tilde{I}, \tilde{N})$,
- $\tilde{c}_t = c_{t+\tau}$, $\tilde{\zeta}_t = \zeta_{t+\tau}$.

Then we can check that for almost all ω_0 , \tilde{U}^{ω_0} satisfies the conditions of Definition 3.1 (with initial conditions $(I_\tau(\omega_0), X_\tau(\omega_0), Y_\tau(\omega_0))$): 2. comes from the independence of W and (I, N) and the strong Markov property, and 4. is fulfilled because for almost all ω_0 $\mathcal{F}^0_{t+\tau} \subset \tilde{\mathcal{F}}^\tau_t$.

Moreover, there is a modification (X', Y') of (X, Y) such that $(X'_{\tau+t}, Y'_{\tau+t})$ is \tilde{F}^τ -adapted, and a solution of (2.1)–(2.2) for $(\tilde{W}, \tilde{I}, \tilde{N})$. Hence

$$\tilde{U}^{\omega_0} \in \mathcal{A}^w_{I_\tau(\omega_0)}(X_\tau(\omega_0), Y_\tau(\omega_0)),$$

and

$$\mathbb{E} \left[\int_0^\infty e^{-\rho s} U(c_{\tau+s}) ds \mid \mathcal{F}_\tau \right] (\omega_0) = J(\tilde{U}^{\omega_0}) \leq v_{I_\tau}(X_\tau, Y_\tau)(\omega_0).$$

Hence taking the expectation over ω_0 in (3.2),

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} U(c_t) dt \right] \leq \mathbb{E} \left[\int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} v_{I_\tau}(X_\tau, Y_\tau) \right],$$

and taking first the infimum over τ in the right hand side, then the supremum over \mathcal{U} on both sides, we obtain $v_i(x, y) \leq V_i^-(x, y)$.

Step 2. $v_i(x, y) \geq V_i^+(x, y)$: We already know from Proposition 2.1 that v_i is continuous on $\text{Int}(\mathbb{R}_+^2)$, and that the restriction of v_i to the boundary is continuous. One can then find a countable sequence $(U_k)_{k \geq 0}$ such that

- (i) $(U_k)_k$ is a partition of \mathbb{R}_+^2 ,
- (ii) $\forall (x, y) (x', y') \in U_k, \forall i |v_i(x, y) - v_i(x', y')| \leq \varepsilon$,
- (iii) U_k contains its bottom-left corner $(x_k, y_k) = (\min_{(x,y) \in U_k} x, \min_{(x,y) \in U_k} y)$.

Indeed, we can construct such a partition in the following way: v_i is continuous on the boundary so we can partition each of the boundary lines into a countable number of segments satisfying (ii) and (iii). Then in the interior we have first a partition in “squared rings”: $\text{Int}(\mathbb{R}_+^2) = \bigcup_{n \geq 1} K_n$, where $K_n = [1/(n+1), n+1]^2 \setminus [1/n, n]^2$. Since v_i is continuous on the interior, we can partition each K_n into a finite number of squares satisfying (ii) and (iii). By taking the union of the line segments and the squares for each K_n , we obtain a sequence (U_k) satisfying (i)–(iii).

Notice that (iii) implies the inclusion $\mathcal{A}_i(x_k, y_k) \subset \mathcal{A}_i(x, y)$, for all $(x, y) \in U_k$. For each k , take $\mathcal{U}^{i,k} = (\Omega^{i,k}, \mathcal{F}^{i,k}, \mathbb{P}^{i,k}, \mathbb{F}^{i,k}, W^{i,k}, I^{i,k}, N^{i,k}, c^{i,k}, \zeta^{i,k})_\varepsilon$ -optimal for (i, x_k, y_k) , and $f_c^{i,k}, f_\zeta^{i,k}$ associated to $(c^{i,k}, \zeta^{i,k})$ by Lemma 3.2. Then for each $\mathcal{U} \in \mathcal{A}_i(x, y)$ and $\tau \in \mathcal{T}_\mathbb{F}$, let us define a control $\tilde{c}, \tilde{\zeta}$ (on the same probability space as \mathcal{U}) by

$$\tilde{c}_t = \begin{cases} c_t & \text{when } t < \tau \\ f_c^{i,k}(t - \tau, \tilde{W}(\cdot \wedge (t - \tau)), \tilde{I}(\cdot \wedge (t - \tau)), \tilde{N}(\cdot \wedge (t - \tau))) & \text{when } t \geq \tau, \\ I_\tau = i, (X_\tau, Y_\tau) \in U_k. \end{cases}$$

Then \tilde{c} (resp. $\tilde{\zeta}$) is \mathbb{F} -progressively measurable (resp. predictable). Furthermore, for almost all ω_0 , with $i = I_\tau(\omega_0)$ and $(X_\tau, Y_\tau)(\omega_0) \in U_k$,

$$\mathcal{L}^{\mathbb{P}^{\omega_0}}(\tilde{W}, \tilde{I}, \tilde{N}, (\tilde{c}_{t+\tau}), (\tilde{\zeta}_{t+\tau})) = \mathcal{L}^{\mathbb{P}^{i,k}}(W^{i,k}, I^{i,k}, N^{i,k}, c^{i,k}, \zeta^{i,k}),$$

and since $\mathcal{A}_i(x_k, y_k) \subset \mathcal{A}_{I_\tau(\omega_0)}(X_\tau(\omega_0), Y_\tau(\omega_0))$, this implies $X_t^{\tilde{c}, \tilde{\zeta}}, Y_t^{\tilde{c}, \tilde{\zeta}} \geq 0$ a.s., and $(\tilde{c}, \tilde{\zeta}) \in \mathcal{A}_i(x, y)$. We also have

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-\rho s} U(\tilde{c}_{\tau+s}) ds \mid \mathcal{F}_\tau\right](\omega_0) &= \mathbb{E}^{i,k}\left[\int_0^\infty e^{-\rho s} U(c_s^{i,k}) ds\right] \\ &\geq v_i(x_k, y_k) - \varepsilon \geq v_{I_\tau}(X_\tau, Y_\tau)(\omega_0) - 2\varepsilon. \end{aligned}$$

By taking expectation in (3.2), we have

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\tilde{c}_t) dt\right] \geq \mathbb{E}\left[\int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho\tau} v_{I_\tau}(X_\tau, Y_\tau)\right] - 2\varepsilon.$$

Finally, by taking the supremum on both sides, and letting ε go to 0, we obtain $v_i(x, y) \geq V_i^+(x, y)$. ■

REMARK 3.2. Actually the weak value function is equal to the value function defined in (2.6) for any $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W, I, N)$ satisfying 1–3. in Definition 3.1. Indeed, given any $\mathcal{U}' = (\Omega', \mathcal{F}', \mathbb{P}', \mathbb{F}', W', I', N') \in \mathcal{A}_i^w(x, y)$, letting $f_{c'}$ and $f_{\zeta'}$ being associated to c' and ζ' by Lemmas 3.1 and 3.2, and defining (almost surely) $c_t = f_{c'}(t, W, I, N)$, $\zeta_t = f_{\zeta'}(t, W, I, N)$, by the same arguments as in the proof of Theorem 3.1, we deduce that $\mathcal{U} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W, I, N, c, \zeta) \in \mathcal{A}_i^w(x, y)$, and $J(\mathcal{U}) = J(\mathcal{U}')$. Hence

$$\sup_{\mathcal{U}' \in \mathcal{A}_i^w(x, y)} J(\mathcal{U}') = \sup_{(c, \zeta) \in \mathcal{A}_i(x, y)} \mathbb{E}\left[\int_0^\infty e^{-\rho s} U(c_s) ds\right].$$

REMARK 3.3. Usually the proof of dynamic programming principle is done (or referred to) in two parts: the “easy” one (\leq) which does not require continuity of the value function, and the “difficult” one (\geq) which requires the continuity of the value function up to the boundary. The proof of continuity at the boundary in many cases can be proved using only the “easy” inequality (see e.g. [4] for a similar result). In our case, due to the specific boundary condition of our problem, the “easy” inequality is not enough to prove the continuity at the boundary. We need also the “hard” inequality. For this reason we need to provide a proof of the “hard” inequality without knowing the continuity up to the boundary. Our proof is then a modification of the one given e.g. in [5], Chapter 2, or [14], Chapter 5. Differently from such proofs our proof only uses the continuity of v_i in the interior and the continuity of its restriction to the boundary (which are both implied by the concavity and by the growth condition (2.8)). So, once this proof is given we are able to prove also the continuity of the value function up to boundary (see the companion paper [6]).

4. Viscosity characterization. In this section, we provide an analytic characterization of the value functions v_i , $i \in \mathbb{I}_d$, to our control problem (2.6), by relying on the dynamic programming principle, which has been proved in the previous section, Theorem 3.1.

The associated dynamic programming system (also called Hamilton–Jacobi–Bellman or HJB system) for v_i , $i \in \mathbb{I}_d$, is written as

$$\begin{aligned} \rho v_i - b_i y \frac{\partial v_i}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_i}{\partial y^2} - \tilde{U} \left(\frac{\partial v_i}{\partial x} \right) - \sum_{j \neq i} q_{ij} [v_j(x, y(1 - \gamma_{ij})) - v_i(x, y)] \\ - \lambda_i [\hat{v}_i(x + y) - v_i(x, y)] = 0, \quad (x, y) \in (0, \infty) \times \mathbb{R}_+, \quad i \in \mathbb{I}_d. \end{aligned} \quad (4.1)$$

The appropriate boundary condition follows from the dynamic programming principle. Indeed, from Theorem 3.1 it follows that

$$\begin{aligned} v_i(x, y) &= \sup_{c \in \mathcal{C}(x)} \mathbb{E} \left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} \hat{v}_{I_{\tau_1}^i}(R_{\tau_1}) \right] \\ &= \sup_{c \in \mathcal{C}(x)} \mathbb{E} \left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} \hat{v}_{I_{\tau_1}^i} \left(x - \int_0^{\tau_1} c_t dt + y \frac{S_{\tau_1}}{S_0} \right) \right], \quad \forall (x, y) \in \mathbb{R}_+^2, \end{aligned}$$

where $\mathcal{C}(x)$ denotes the set of nonnegative adapted processes (c_t) such that $\int_0^{\tau_1} c_t dt \leq x$ a.s. This implies that for v_i , $i \in \mathbb{I}_d$, on $\{0\} \times \mathbb{R}_+$ we have

$$v_i(0, y) = \begin{cases} 0, & \text{if } y = 0, \\ \mathbb{E} \left[e^{-\rho \tau_1} \hat{v}_{I_{\tau_1}^i} \left(y \frac{S_{\tau_1}}{S_0} \right) \right], & \text{if } y > 0. \end{cases} \quad (4.2)$$

Here I^i denotes the continuous-time Markov chain I starting from i at time 0.

In our context, the notion of viscosity solution to the non-local second-order system (4.1) is defined as follows.

DEFINITION 4.1.

- (i) A d -tuple $w = (w_i)_{i \in \mathbb{I}_d}$ of continuous functions on \mathbb{R}_+^2 is a *viscosity supersolution* (resp. *subsolution*) to (4.1) if

$$\begin{aligned} \rho \varphi_i(\bar{x}, \bar{y}) - b_i \bar{y} \frac{\partial \varphi_i}{\partial y}(\bar{x}, \bar{y}) - \frac{1}{2} \sigma_i^2 \bar{y}^2 \frac{\partial^2 \varphi_i}{\partial y^2}(\bar{x}, \bar{y}) - \tilde{U} \left(\frac{\partial \varphi_i}{\partial x}(\bar{x}, \bar{y}) \right) \\ - \sum_{j \neq i} q_{ij} [\varphi_j(\bar{x}, \bar{y}(1 - \gamma_{ij})) - \varphi_i(\bar{x}, \bar{y})] - \lambda_i [\hat{\varphi}_i(\bar{x} + \bar{y}) - \varphi_i(\bar{x}, \bar{y})] \geq 0 \quad (\text{resp. } \leq 0), \end{aligned}$$

for every d -tuple $\varphi = (\varphi_i)_{i \in \mathbb{I}_d}$ of C^2 functions on \mathbb{R}_+^2 , and any $(\bar{x}, \bar{y}, i) \in (0, \infty) \times \mathbb{R}_+ \times \mathbb{I}_d$, such that $w_i(\bar{x}, \bar{y}) = \varphi_i(\bar{x}, \bar{y})$, and $w \geq \varphi$ (resp. $w \leq \varphi$) on $\mathbb{R}_+^2 \times \mathbb{I}_d$.

- (ii) A d -tuple $w = (w_i)_{i \in \mathbb{I}_d}$ of continuous functions on \mathbb{R}_+^2 is a *viscosity solution* to (4.1) if it is both a viscosity supersolution and subsolution to (4.1).

The main result of this section is to provide an analytic characterization of the value functions in terms of viscosity solutions to the dynamic programming system.

THEOREM 4.1. *The value function $v = (v_i)_{i \in \mathbb{I}_d}$ is the unique viscosity solution to (4.1) satisfying the boundary condition (4.2) and the growth condition (2.8).*

We first prove the viscosity property of the value function to its dynamic programming system (4.1), written as:

$$F_i(x, y, v_i(x, y), Dv_i(x, y), D^2v_i(x, y)) + G_i(x, y, v) = 0, \quad (x, y) \in (0, \infty) \times \mathbb{R}_+,$$

for any $i \in \mathbb{I}_d$, where F_i is the local operator defined by

$$F_i(x, y, u, p, A) = \rho u - b_i y p_2 - \frac{1}{2} \sigma_i^2 y^2 a_{22} - \tilde{U}(p_1)$$

for $(x, y) \in (0, \infty) \times \mathbb{R}_+$, $u \in \mathbb{R}$, $p = (p_1 \ p_2) \in \mathbb{R}^2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \in \mathcal{S}^2$ (the set of symmetric 2×2 matrices), and G_i is the nonlocal operator defined by

$$G_i(x, y, w) = - \sum_{j \neq i} q_{ij} [w_j(x, y(1 - \gamma_{ij})) - w_i(x, y)] - \lambda_i [\hat{w}_i(x + y) - w_i(x, y)]$$

for $w = (w_i)_{i \in \mathbb{I}_d}$ d -tuple of continuous functions on \mathbb{R}_+^2 .

PROPOSITION 4.1. *The value function $v = (v_i)_{i \in \mathbb{I}_d}$ is a viscosity solution to (4.1).*

Proof. Viscosity supersolution: Let $(i, \bar{x}, \bar{y}) \in \mathbb{I}_d \times (0, \infty) \times \mathbb{R}_+$, $\varphi = (\varphi_i)_{i \in \mathbb{I}_d}$, C^2 test functions such that $v_i(\bar{x}, \bar{y}) = \varphi_i(\bar{x}, \bar{y})$, and $v \geq \varphi$. Take some arbitrary $e \in (-\bar{y}, \bar{x})$, and $c \in \mathbb{R}_+$. Since $\bar{x} > 0$, there exists a strictly positive stopping time $\tau > 0$ a.s. such that the control process $(\bar{\zeta}, \bar{c})$ defined by:

$$\bar{\zeta}_t = e \mathbf{1}_{t \leq \tau}, \quad \bar{c}_t = c \mathbf{1}_{t \leq \tau}, \quad t \geq 0, \quad (4.3)$$

with associated state process (\bar{X}, \bar{Y}, I) starting from (x, y, i) at time 0, satisfies $\bar{X}_t \geq 0$, $\bar{Y}_t \geq 0$, for all t . Thus, $(\bar{\zeta}, \bar{c}) \in \mathcal{A}_i(x, y)$. Let \mathcal{V} be a compact neighbourhood of (x, y, i) in $(0, \infty) \times \mathbb{R}_+ \times \mathbb{I}_d$, and consider the sequence of stopping time: $\theta_n = \theta \wedge h_n$, where $\theta = \inf \{t \geq 0 : (\bar{X}_t, \bar{Y}_t, I_t) \notin \mathcal{V}\}$, and (h_n) is a strictly positive sequence converging to zero. From the dynamic programming principle, Theorem 3.1, and by applying Itô's formula to $e^{-\rho t} \varphi(\bar{X}_t, \bar{Y}_t, I_t)$ between 0 and θ_n , we get

$$\begin{aligned} \varphi(\bar{x}, \bar{y}, i) &= v(x, y, i) \geq \mathbb{E} \left[\int_0^{\theta_n} e^{-\rho t} U(\bar{c}_t) dt + e^{-\rho \theta_n} v(\bar{X}_{\theta_n}, \bar{Y}_{\theta_n}, I_{\theta_n}) \right] \\ &\geq \mathbb{E} \left[\int_0^{\theta_n} e^{-\rho t} U(\bar{c}_t) dt + e^{-\rho \theta_n} \varphi(\bar{X}_{\theta_n}, \bar{Y}_{\theta_n}, I_{\theta_n}) \right] \\ &= \varphi(\bar{x}, \bar{y}, i) + \mathbb{E} \left[\int_0^{\theta_n} e^{-\rho t} \left(U(\bar{c}_t) - \rho \varphi - \bar{c}_t \frac{\partial \varphi}{\partial x} + b_{I_t} \bar{Y}_t - \frac{\partial \varphi}{\partial y} + \frac{1}{2} \sigma_{I_t}^2 \bar{Y}_t^2 \frac{\partial^2 \varphi}{\partial y^2} \right. \right. \\ &\quad \left. \left. + \sum_{j \neq I_t} q_{I_t-j} [\varphi(\bar{X}_{t-}, \bar{Y}_{t-}(1 - \gamma_{I_t-j}), j) - \varphi(\bar{X}_{t-}, \bar{Y}_{t-}, I_{t-})] \right. \right. \\ &\quad \left. \left. + \lambda_{I_t} [\varphi(\bar{X}_{t-} - \bar{\zeta}_t, \bar{Y}_{t-} + \bar{\zeta}_t, I_{t-}) - \varphi(\bar{X}_{t-}, \bar{Y}_{t-}, I_{t-})] \right) dt \right], \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E} \left[\frac{1}{h_n} \int_0^{\theta_n} e^{-\rho t} \left(\rho \varphi - U(\bar{c}_t) + \bar{c}_t \frac{\partial \varphi}{\partial x} - b_{I_t} \bar{Y}_t - \frac{\partial \varphi}{\partial y} - \frac{1}{2} \sigma_{I_t}^2 \bar{Y}_t^2 \frac{\partial^2 \varphi}{\partial y^2} \right. \right. \\ \left. \left. - \sum_{j \neq I_t} q_{I_t-j} [\varphi(\bar{X}_{t-}, \bar{Y}_{t-}(1 - \gamma_{I_t-j}), j) - \varphi(\bar{X}_{t-}, \bar{Y}_{t-}, I_{t-})] \right. \right. \\ \left. \left. - \lambda_{I_t} [\varphi(\bar{X}_{t-} - \bar{\zeta}_t, \bar{Y}_{t-} + \bar{\zeta}_t, I_{t-}) - \varphi(\bar{X}_{t-}, \bar{Y}_{t-}, I_{t-})] \right) dt \right] \geq 0. \quad (4.4) \end{aligned}$$

Now, we have almost surely for n large enough, $\theta \geq h_n$, i.e. $\theta_n = h_n$, so that by using also (4.3),

$$\begin{aligned} & \frac{1}{h_n} \int_0^{\theta_n} e^{-\rho t} \left(\rho \varphi - U(\bar{c}_t) + \bar{c}_t \frac{\partial \varphi}{\partial x} - b_{I_{t-}} \bar{Y}_{t-} \frac{\partial \varphi}{\partial y} - \frac{1}{2} \sigma_{I_{t-}}^2 \bar{Y}_{t-}^2 \frac{\partial^2 \varphi}{\partial y^2} \right. \\ & \quad - \sum_{j \neq I_{t-}} q_{I_{t-}j} [\varphi(\bar{X}_{t-}, \bar{Y}_{t-}(1 - \gamma_{I_{t-}j}), j) - \varphi(\bar{X}_{t-}, \bar{Y}_{t-}, I_{t-})] \\ & \quad \left. - \lambda_{I_{t-}} [\varphi(\bar{X}_{t-} - \bar{\zeta}_t, \bar{Y}_{t-} + \bar{\zeta}_t, I_{t-}) - \varphi(\bar{X}_{t-}, \bar{Y}_{t-}, I_{t-})] \right) dt \\ & \rightarrow \rho \varphi_i(\bar{x}, \bar{y}) - U(c) + c \frac{\partial \varphi_i}{\partial x}(\bar{x}, \bar{y}) - b_i \bar{y} \frac{\partial \varphi_i}{\partial y}(\bar{x}, \bar{y}) - \frac{1}{2} \sigma_i^2 \bar{y}^2 \frac{\partial^2 \varphi_i}{\partial y^2}(\bar{x}, \bar{y}) \\ & \quad - \sum_{j \neq i} q_{ij} [\varphi_j(\bar{x}, \bar{y}(1 - \gamma_{ij})) - \varphi_i(\bar{x}, \bar{y})] - \lambda_i [\varphi_i(\bar{x} - e, \bar{y} + e) - \varphi_i(\bar{x}, \bar{y})], \quad \text{a.s.} \end{aligned}$$

when n goes to infinity. Moreover, since the integrand of the Lebesgue integral term in (4.4) is bounded for $t \leq \theta$, one can apply the dominated convergence theorem in (4.4), which gives

$$\begin{aligned} & \rho \varphi_i(\bar{x}, \bar{y}) - U(c) + c \frac{\partial \varphi_i}{\partial x}(\bar{x}, \bar{y}) - b_i \bar{y} \frac{\partial \varphi_i}{\partial y}(\bar{x}, \bar{y}) - \frac{1}{2} \sigma_i^2 \bar{y}^2 \frac{\partial^2 \varphi_i}{\partial y^2}(\bar{x}, \bar{y}) \\ & \quad - \sum_{j \neq i} q_{ij} [\varphi_j(\bar{x}, \bar{y}(1 - \gamma_{ij})) - \varphi_i(\bar{x}, \bar{y})] - \lambda_i [\varphi_i(\bar{x} - e, \bar{y} + e) - \varphi_i(\bar{x}, \bar{y})] \geq 0. \end{aligned}$$

Since c and e are arbitrary, we obtain the required viscosity supersolution inequality by taking the supremum over $c \in \mathbb{R}_+$ and $e \in (-\bar{y}, \bar{x})$.

Viscosity subsolution: Let $(\bar{i}, \bar{x}, \bar{y}) \in \mathbb{I}_d \times (0, \infty) \times \mathbb{R}_+$, $\varphi = (\varphi_i)_{i \in \mathbb{I}_d}$, C^2 test functions such that $v(\bar{x}, \bar{y}, \bar{i}) = \varphi(\bar{x}, \bar{y}, \bar{i})$, and $v \leq \varphi$. We can also assume without loss of generality that $v < \varphi$ outside $(\bar{x}, \bar{y}, \bar{i})$. We argue by contradiction by assuming that

$$\begin{aligned} & \rho \varphi_{\bar{i}}(\bar{x}, \bar{y}) - b_{\bar{i}} \bar{y} \frac{\partial \varphi_{\bar{i}}}{\partial y}(\bar{x}, \bar{y}) - \frac{1}{2} \sigma_{\bar{i}}^2 \bar{y}^2 \frac{\partial^2 \varphi_{\bar{i}}}{\partial y^2}(\bar{x}, \bar{y}) - \tilde{U} \left(\frac{\partial \varphi_{\bar{i}}}{\partial x}(\bar{x}, \bar{y}) \right) \\ & \quad - \sum_{j \neq \bar{i}} q_{\bar{i}j} [\varphi_j(\bar{x}, \bar{y}(1 - \gamma_{\bar{i}j})) - \varphi_{\bar{i}}(\bar{x}, \bar{y})] - \lambda_{\bar{i}} [\hat{\varphi}_{\bar{i}}(\bar{x} + \bar{y}) - \varphi_{\bar{i}}(\bar{x}, \bar{y})] > 0. \end{aligned}$$

By continuity of φ and of its derivatives, there exist some compact neighbourhood $\bar{\mathcal{V}}$ of $(\bar{x}, \bar{y}, \bar{i})$ in $(0, \infty) \times \mathbb{R}_+ \times \mathbb{I}_d$, and $\varepsilon > 0$, such that for all $(x, y, i) \in \bar{\mathcal{V}}$

$$\begin{aligned} & \rho \varphi_i(x, y) - b_i y \frac{\partial \varphi_i}{\partial y}(x, y) - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 \varphi_i}{\partial y^2}(x, y) - \tilde{U} \left(\frac{\partial \varphi_i}{\partial x}(x, y) \right) \\ & \quad - \sum_{j \neq i} q_{ij} [\varphi_j(x, y(1 - \gamma_{ij})) - \varphi_i(x, y)] - \lambda_i [\hat{\varphi}_i(x + y) - \varphi_i(x, y)] \geq \varepsilon. \quad (4.5) \end{aligned}$$

Since $v < \varphi$ outside $(\bar{x}, \bar{y}, \bar{i})$, there exists some $\delta > 0$ such that $v < \varphi - \delta$ outside of $\bar{\mathcal{V}}$. We can also assume that $\varepsilon \leq \delta \rho$. By Theorem 3.1, there exists $(\zeta, c) \in \mathcal{A}_{\bar{i}}(\bar{x}, \bar{y})$ such that

$$v(\bar{x}, \bar{y}, \bar{i}) - \varepsilon \frac{1 - e^{-\rho}}{2\rho} \leq \mathbb{E} \left[\int_0^{\theta \wedge 1} e^{-\rho t} U(c_t) dt + e^{-\rho(\theta \wedge 1)} v(X_{\theta \wedge 1}, Y_{\theta \wedge 1}, I_{\theta \wedge 1}) \right],$$

where (X, Y, I) is controlled by (ζ, c) , and we take $\theta = \inf\{t \geq 0 : (X_t, Y_t, I_t) \notin \bar{\mathcal{V}}\}$. We then get:

$$\begin{aligned} & \varphi(\bar{x}, \bar{y}, \bar{i}) - \varepsilon \frac{1 - e^{-\rho}}{2\rho} = v(\bar{x}, \bar{y}, \bar{i}) - \varepsilon \frac{1 - e^{-\rho}}{2\rho} \\ & \leq \mathbb{E} \left[\int_0^{\theta \wedge 1} e^{-\rho t} U(c_t) dt + e^{-\rho(\theta \wedge 1)} \varphi(X_{\theta \wedge 1}, Y_{\theta \wedge 1}, I_{\theta \wedge 1}) - e^{-\rho\theta} \delta \mathbf{1}_{\{\theta < 1\}} \right] \\ & = \varphi(\bar{x}, \bar{y}, \bar{i}) + \mathbb{E} \left[\int_0^{\theta \wedge 1} e^{-\rho t} \left(U(c_t) - \rho\varphi - c_t \frac{\partial \varphi}{\partial x} + b_{I_{t-}} Y_{t-} \frac{\partial \varphi}{\partial y} + \frac{1}{2} \sigma_{I_{t-}}^2 Y_{t-}^2 \frac{\partial^2 \varphi}{\partial y^2} \right. \right. \\ & \quad \left. \left. + \sum_{j \neq I_{t-}} q_{I_{t-}j} [\varphi(X_{t-}, Y_{t-}(1 - \gamma_{I_{t-}j}), j) - \varphi(X_{t-}, Y_{t-}, I_{t-})] \right. \right. \\ & \quad \left. \left. + \lambda_{I_{t-}} [\varphi(X_{t-} - \zeta_t, Y_{t-} + \zeta_t, I_{t-}) - \varphi(X_{t-}, Y_{t-}, I_{t-})] \right) dt - e^{-\rho\theta} \delta \mathbf{1}_{\{\theta < 1\}} \right] \\ & \leq \varphi(\bar{x}, \bar{y}, \bar{i}) + \mathbb{E} \left[\int_0^{\theta \wedge 1} -\varepsilon e^{-\rho t} dt - e^{-\rho\theta} \delta \mathbf{1}_{\{\theta < 1\}} \right] \end{aligned}$$

where we applied Itô's formula in the second equality, and used (4.5) in the last inequality. This means that

$$\begin{aligned} -\varepsilon \frac{1 - e^{-\rho}}{2\rho} & \leq \mathbb{E} \left[\int_0^{\theta \wedge 1} -\varepsilon e^{-\rho t} dt - e^{-\rho\theta} \delta \mathbf{1}_{\{\theta < 1\}} \right] \\ & = \mathbb{E} \left[-\frac{\varepsilon}{\rho} + \frac{\varepsilon}{\rho} e^{-\rho(\theta \wedge 1)} - e^{-\rho\theta} \delta \mathbf{1}_{\{\theta < 1\}} \right] \leq -\frac{\varepsilon}{\rho} (1 - e^{-\rho}), \end{aligned}$$

since $\varepsilon/\rho \leq \delta$, and we get the required contradiction. ■

Let us now prove comparison principle for our dynamic programming system. As usual, it is convenient to formulate an equivalent definition for viscosity solutions to (4.1) in terms of semi-jets. We shall use the notation $X = (x, y)$ for $\mathbb{R}_+ \times \mathbb{R}_+$ -valued vectors. Given $w = (w_i)_{i \in \mathbb{I}_d}$ a d -tuple of continuous functions on \mathbb{R}_+^2 , the second-order *superjet* of w_i at $X \in \mathbb{R}_+^2$ is defined by

$$\begin{aligned} \mathcal{P}^{2,+} w_i(X) & = \left\{ (p, A) \in \mathbb{R}^2 \times \mathcal{S}^2 \text{ such that } w_i(X') \leq w_i(X) + \langle p, X' - X \rangle \right. \\ & \quad \left. + \frac{1}{2} \langle A(X' - X), X' - X \rangle + o(\|X' - X\|^2) \text{ as } X' \rightarrow X \right\}, \end{aligned}$$

and its closure $\bar{\mathcal{P}}^{2,+} w_i(X)$ as the set of elements $(p, A) \in \mathbb{R}^2 \times \mathcal{S}^2$ for which there exists a sequence $(X_m, p_m, A_m)_m$ of $\mathbb{R}_+^2 \times \mathcal{P}^{2,+} w_i(X_m)$ satisfying $(X_m, p_m, A_m) \rightarrow (X, p, A)$. We also define the second-order *subjet* $\mathcal{P}^{2,-} w_i(X) = -\mathcal{P}^{2,+}(-w_i)(X)$, and $\bar{\mathcal{P}}^{2,-} w_i(X) = -\bar{\mathcal{P}}^{2,+}(-w_i)(X)$. By standard arguments (see e.g. [1] for equations with nonlocal terms), one has an equivalent definition of viscosity solutions in terms of semi-jets:

A d -tuple $w = (w_i)_{i \in \mathbb{I}_d}$ of continuous functions on \mathbb{R}_+^2 is a *viscosity supersolution* (resp. *subsolution*) of (4.1) if and only if for all $(i, x, y) \in \mathbb{I}_d \times (0, \infty) \times \mathbb{R}_+$, and all $(p, A) \in \bar{\mathcal{P}}^{2,-} w_i(x, y)$ (resp. $\bar{\mathcal{P}}^{2,+} w_i(x, y)$):

$$F_i(x, y, w_i(x, y), p, A) + G_i(x, y, w) \geq 0 \quad (\text{resp. } \leq 0).$$

We then prove the following comparison theorem.

THEOREM 4.2. *Let $V = (V_i)_{i \in \mathbb{I}_d}$ (resp. $W = (W_i)_{i \in \mathbb{I}_d}$) be a viscosity subsolution (resp. supersolution) of (4.1), satisfying the growth condition (2.8), and the boundary conditions*

$$V_i(0, 0) \leq 0 \tag{4.6}$$

$$V_i(0, y) \leq \mathbb{E}_i \left[\hat{V}_{I_{\tau_1}} \left(y \frac{S_{\tau_1}}{S_0} \right) \right], \quad \forall y > 0, \tag{4.7}$$

(resp. \geq for W). Then $V \leq W$.

Proof. Step 1: Take $p' > p$ such that $k(p') < \rho$, and define $\psi_i(x, y) = (x + y)^{p'}$, $i \in \mathbb{I}_d$. Let us check that $W^n = W + \frac{1}{n}\psi$ is still a supersolution to (4.1). Notice that $\mathcal{P}^{2,-}W_i^n = \mathcal{P}^{2,-}W_i + \frac{1}{n}(D\psi_i, D^2\psi_i)$, and we have for all $(p, A) \in \mathcal{P}^{2,-}W_i(x, y)$:

$$\begin{aligned} & F_i \left(x, y, W_i^n(x, y), p + \frac{1}{n} D\psi_i, A + \frac{1}{n} D^2\psi_i \right) + G_i(x, y, W^n) \\ &= F_i(x, y, W_i(x, y), p, A) + G_i(x, y, W) \\ &+ \frac{1}{n}(x + y)^{p'} \left(\rho - p'b_i \frac{y}{x + y} + p'(1 - p') \frac{\sigma_i^2}{2} \left(\frac{y}{x + y} \right)^2 \right) \\ &- \sum_{j \neq i} q_{ij} \left(\left(1 - \frac{y}{x + y} \gamma_{ij} \right)^{p'} - 1 \right) + \tilde{U}(p_1) - \tilde{U} \left(p_1 + \frac{1}{n} p' x^{p'-1} \right) \geq 0. \end{aligned} \tag{4.8}$$

Indeed, the three lines in the right hand side of (4.8) are nonnegative: the first one since W is a supersolution, the second one by $k(p') < \rho$, and the last one since \tilde{U} is nonincreasing.

Moreover, by the growth condition (2.8) on V and W , we have:

$$\lim_{r \rightarrow \infty} \max_{i \in \mathbb{I}_d} (\hat{V}_i - \hat{W}_i^n)(r) = -\infty. \tag{4.9}$$

In the next step, our aim is to show that for all $n \geq 1$, $V \leq W^n$, which would imply that $V \leq W$. We shall argue by contradiction.

Step 2: Assume that there exists some $n \geq 1$ such that

$$M := \sup_{i \in \mathbb{I}_d, (x, y) \in \mathbb{R}_+^2} (V_i - W_i^n)(x, y) > 0.$$

By (4.9), there exist $i \in \mathbb{I}_d$, some compact subset \mathcal{C} of \mathbb{R}_+^2 , and $\bar{X} = (\bar{x}, \bar{y}) \in \mathcal{C}$ such that

$$M = \max_{\mathcal{C}} (V_i - W_i^n) = (V_i - W_i^n)(\bar{x}, \bar{y}). \tag{4.10}$$

Note that by (4.6), $(\bar{x}, \bar{y}) \neq (0, 0)$. We then have two possible cases:

Case 1. $\bar{x} = 0$. Notice that the boundary condition (4.7) implies the viscosity subsolution property for V_i also at $\bar{X} = (0, \bar{y})$:

$$F_i(\bar{X}, V_i(\bar{X}), p, A) + G_i(\bar{X}, V) \leq 0, \quad \forall (p, A) \in \overline{\mathcal{P}}^{2,+}V_i(\bar{X}).$$

However the viscosity supersolution property for W^n does not hold at $(0, \bar{y})$. Let $(X_k)_k = (x_k, y_k)_k$ be a sequence converging to \bar{X} , with $x_k > 0$, and $\varepsilon_k := |X_k - \bar{X}|$. We then consider the function

$$\begin{aligned} \Phi_k(X, X') &= V_i(X) - W_i^n(X') - \psi_k(X, X'), \\ \psi_k(x, y, x', y') &= x^4 + (y - \bar{y})^4 + \frac{|X - X'|^2}{2\varepsilon_k} + \left(\frac{x'}{x_k} - 1 \right)_-^3. \end{aligned}$$

Since Φ_k is continuous, there exists $(\widehat{X}_k, \widehat{X}'_k) \in \mathcal{C}^2$ such that

$$M_k := \sup_{\mathcal{C}^2} \Phi_k = \Phi_k(\widehat{X}_k, \widehat{X}'_k),$$

and a subsequence, still denoted by $(\widehat{X}_k, \widehat{X}'_k)$, converging to some $(\widehat{X}, \widehat{X}')$ as k goes to ∞ . By writing that $\Phi_k(\overline{X}, X_k) \leq \Phi_k(\widehat{X}_k, \widehat{X}'_k)$, we have

$$V_i(\overline{X}) - W_i^n(X_k) - \frac{|\overline{X} - X_k|}{2} \quad (4.11)$$

$$\leq V_i(\widehat{X}_k) - W_i^n(\widehat{X}'_k) - (\hat{x}_k^4 + (\hat{y}_k - \bar{y})^4) - R_k \quad (4.12)$$

$$\leq V_i(\widehat{X}_k) - W_i^n(\widehat{X}'_k) - (\hat{x}_k^4 + (\hat{y}_k - \bar{y})^4), \quad (4.13)$$

where we set

$$R_k = \frac{|\widehat{X}_k - \widehat{X}'_k|^2}{2\varepsilon_k} + \left(\frac{\hat{x}'_k}{x_k} - 1 \right)_-^3.$$

Since V_i and W_i^n are bounded on \mathcal{C} , we deduce by inequality (4.12) the boundedness of the sequence $(R_k)_{k \geq 0}$, which implies $\widehat{X} = \widehat{X}'$. Then by sending k to infinity in (4.11) and (4.13), with the continuity of V_i and W_i^n , we obtain $M = V_i(\overline{X}) - W_i^n(\overline{X}) \leq V_i(\widehat{X}) - W_i^n(\widehat{X}) - (\hat{x}_k^4 + (\hat{y}_k - \bar{y})^4)$, and by definition of M this shows

$$\widehat{X} = \widehat{X}' = \overline{X}. \quad (4.14)$$

Sending again k to infinity in (4.11)–(4.12)–(4.13), we obtain $M \leq M - \limsup_k R_k \leq M$, and so

$$\frac{|\widehat{X}_k - \widehat{X}'_k|^2}{2\varepsilon_k} + \left(\frac{\hat{x}'_k}{x_k} - 1 \right)_-^3 \rightarrow 0 \quad (4.15)$$

as k goes to infinity. In particular for k large enough $\hat{x}'_k \geq \frac{\varepsilon_k}{2} > 0$. We can then apply Ishii's lemma (see Theorem 3.2 in [2]) to obtain $A, A' \in \mathcal{S}^2$ such that

$$(p, A) \in \overline{\mathcal{P}}^{2,+} V_i(\widehat{X}_k), \quad (p', A') \in \overline{\mathcal{P}}^{2,-} W_i^n(\widehat{X}'_k) \quad (4.16)$$

$$\begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \leq D + \varepsilon_k D^2, \quad (4.17)$$

where

$$p = D_X \psi_k(\widehat{X}_k, \widehat{X}'_k), \quad p' = D_{X'} \psi_k(\widehat{X}_k, \widehat{X}'_k), \quad D = D_{X, X'}^2 \psi_k(\widehat{X}_k, \widehat{X}'_k).$$

Now, we write

$$\begin{aligned} \rho M &\leq \rho M_k \leq \rho(V_i(\widehat{X}_k) - W_i^n(\widehat{X}'_k)) \\ &= F_i(\widehat{X}_k, V_i(\widehat{X}_k), p, A) - F_i(\widehat{X}_k, W_i^n(\widehat{X}'_k), p, A) \\ &= F_i(\widehat{X}_k, V_i(\widehat{X}_k), p, A) + G_i(\widehat{X}_k, V) \\ &\quad - F_i(\widehat{X}'_k, W_i^n(\widehat{X}'_k), p', A') - G_i(\widehat{X}'_k, W^n) \\ &\quad + G_i(\widehat{X}'_k, W^n) - G_i(\widehat{X}_k, V) \\ &\quad + F_i(\widehat{X}'_k, W_i^n(\widehat{X}'_k), p', A') - F_i(\widehat{X}_k, W_i^n(\widehat{X}'_k), p, A). \end{aligned} \quad (4.18)$$

From the viscosity subsolution property for V at \widehat{X}_k , and the viscosity supersolution property for W^n at \widehat{X}'_k , the first two lines in the right hand side of (4.18) are nonpositive.

For the third line, by sending k to infinity, we have:

$$\begin{aligned} G_i(\widehat{X}'_k, W^n) - G_i(\widehat{X}_k, V) &\rightarrow G_i(\overline{X}, W^n) - G_i(\overline{X}, V) \\ &= \sum_{j \neq i} q_{ij} \left[(V_j - W_j^n)(\overline{x}, \overline{y}(1 - \gamma_{ij})) - (V_i - W_i^n)(\overline{x}, \overline{y}) \right] \\ &\quad + \lambda_i \left[(\widehat{V}_i - \widehat{W}_i^n)(\overline{x} + \overline{y}) - (V_i - W_i^n)(\overline{x}, \overline{y}) \right] \leq 0 \end{aligned}$$

by (4.10). For the fourth line of (4.18), we have

$$\begin{aligned} F_i(\widehat{X}'_k, W_i^n(\widehat{X}'_k), p', A') - F_i(\widehat{X}_k, W_i^n(\widehat{X}_k), p, A) \\ = b_i(\hat{y}_k p_2 - \hat{y}'_k p'_2) + \tilde{U}(p_1) - \tilde{U}(p'_1) + \frac{\sigma_i^2}{2}(\hat{y}_k^2 a_{22} - (\hat{y}'_k)^2 a'_{22}). \end{aligned}$$

Now

$$\begin{aligned} \hat{y}_k p_2 - \hat{y}'_k p'_2 &= \hat{y}_k \left(4(\hat{y}_k - \overline{y})^3 + \frac{\hat{y}_k - \hat{y}'_k}{\varepsilon_k} \right) - \hat{y}'_k \left(\frac{\hat{y}_k - \hat{y}'_k}{\varepsilon_k} \right) \\ &\leq 4\hat{y}_k(\hat{y}_k - \overline{y})^3 + \frac{|\widehat{X}_k - \hat{x}'_k|^2}{\varepsilon_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

by (4.14) and (4.15). Moreover,

$$\tilde{U}(p_1) - \tilde{U}(p'_1) = \tilde{U} \left(\frac{\hat{x}_k - \hat{x}'_k}{\varepsilon_k} + 4\hat{x}_k^3 \right) - \tilde{U} \left(\frac{\hat{x}_k - \hat{x}'_k}{\varepsilon_k} - \frac{3}{x_k} \left(\frac{\hat{x}'_k}{x_k} - 1 \right)^2 \right) \leq 0,$$

since \tilde{U} is nonincreasing. Finally,

$$\begin{aligned} \hat{y}_k^2 a_{22} - (\hat{y}'_k)^2 a'_{22} &= \begin{pmatrix} 0 & \hat{y}_k & 0 & \hat{y}'_k \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \begin{pmatrix} 0 \\ \hat{y}_k \\ 0 \\ \hat{y}'_k \end{pmatrix} \\ &\leq \begin{pmatrix} 0 & \hat{y}_k & 0 & \hat{y}'_k \end{pmatrix} (D + \varepsilon_k D^2) \begin{pmatrix} 0 \\ \hat{y}_k \\ 0 \\ \hat{y}'_k \end{pmatrix} \end{aligned}$$

by (4.17). Since

$$D^2 \psi_k(x, y, x', y') = \begin{pmatrix} 12x^2 & 0 & -\frac{1}{\varepsilon_k} & 0 \\ 0 & 12(y - \overline{y})^2 + \frac{1}{\varepsilon_k} & 0 & -\frac{1}{\varepsilon_k} \\ -\frac{1}{\varepsilon_k} & 0 & \frac{1}{\varepsilon_k} + \frac{6}{x_k^2} \left(\frac{x'}{x_k} - 1 \right) & 0 \\ 0 & -\frac{1}{\varepsilon_k} & 0 & \frac{1}{\varepsilon_k} \end{pmatrix},$$

a direct calculation gives

$$\begin{aligned} &\begin{pmatrix} 0 & \hat{y}_k & 0 & \hat{y}'_k \end{pmatrix} (D + \varepsilon_k D^2) \begin{pmatrix} 0 \\ \hat{y}_k \\ 0 \\ \hat{y}'_k \end{pmatrix} \\ &= \frac{3}{\varepsilon_k} (\hat{y}_k - \hat{y}'_k)^2 - 12(\hat{y}_k - \overline{y})^2 \hat{y}_k \hat{y}'_k + (36(\hat{y}_k - \overline{y})^2 + \varepsilon_k (12(\hat{y}_k - \overline{y})^2)^2) \hat{y}_k^2 \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

where we used again (4.14) and (4.15), and the boundedness of $(\widehat{y}_k, \widehat{y}'_k)$.

Finally by letting k go to infinity in (4.18) we obtain $\rho M \leq 0$, which is the required contradiction.

Case 2. $\bar{x} > 0$. This is the easier case, and we can obtain a contradiction similarly as in the first case, by considering for instance the function

$$\Phi_k(X, X') = V_i(X) - W_i^n(X') - (x - \bar{x})^4 - (y - \bar{y})^4 - k \frac{|X - X'|^2}{2}. \blacksquare$$

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