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TRUNCATED VARIATION, UPWARD TRUNCATED VARIATION AND DOWNWARD TRUNCATED VARIATION OF BROWNIAN MOTION WITH DRIFT—THEIR MEAN VALUE AND THEIR APPLICATIONS

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Abstract. In [6] for $c > 0$ we defined the truncated variation, TV_μ^c, of a Brownian motion with drift, $W_t = B_t + \mu t, t \geq 0$, where (B_t) is a standard Brownian motion. In this article we define two related quantities: the upward truncated variation

$$UTV_\mu^c[a, b] = \sup_n \sup_{a \leq t_1 < s_1 < \ldots < t_n < s_n \leq b} \sum_{i=1}^n \max\{W_{s_i} - W_{t_i} - c, 0\}$$

and, analogously, the downward truncated variation

$$DTV_\mu^c[a, b] = \sup_n \sup_{a \leq t_1 < s_1 < \ldots < t_n < s_n \leq b} \sum_{i=1}^n \max\{W_{t_i} - W_{s_i} - c, 0\}.$$

We prove that the exponential moments of the above quantities are finite (in contrast to the regular variation, corresponding to $c = 0$, which is infinite almost surely). We present estimates of the expected value of UTV_μ^c up to universal constants.

As an application we give some estimates of the maximal possible gain from trading a financial asset in the presence of flat commission (proportional to the value of the transaction) when the dynamics of the prices of the asset follows a geometric Brownian motion process. In the presented estimates the upward truncated variation appears naturally.

1. Introduction. Let $(B_t, t \geq 0)$ be a standard Brownian motion, and $W_t = B_t + \mu t$ be a Brownian motion with drift μ.

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In [6] the truncated variation at the level $c > 0$ of the Brownian motion with drift μ on the interval $[a, b]$ was defined as

$$TV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 \le \cdots \le t_n \le b} \sum_{i=1}^{n-1} \max\{|W_{t_{i+1}} - W_{t_i}| - c, 0\}.$$

(Technical remark: for $a > b$ we set $TV_\mu^c[a,b] = 0$.)

There were also proved estimates of $\mathbf{E}TV_\mu^c[0,T]$ up to universal constants. Using similar techniques as in [6] we will prove existence of finite exponential moments of $TV_\mu^c[0,T]$, $\mathbf{E}\exp(\alpha TV_\mu^c[0,T])$, for any α, $T > 0$.

Further we will consider two related quantities:

- the upward truncated variation, defined as

$$UTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < s_1 < \cdots < t_n < s_n \le b} \sum_{i=1}^{n} \max\{W_{s_i} - W_{t_i} - c, 0\},$$

- analogously, the downward truncated variation, defined as

$$DTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < s_1 < \cdots < t_n < s_n \le b} \sum_{i=1}^{n} \max\{W_{t_i} - W_{s_i} - c, 0\}.$$

It is easy to see that all three above defined quantities have the following properties, which we state only for the truncated variation:

- shift invariance property in distributions:

$$\mathcal{L}(TV_\mu^c[a,b]) = \mathcal{L}(TV_\mu^c[a+\Delta, b+\Delta]),$$

- superadditivity: for any numbers $a \le a_1 < a_2 < \cdots < a_n \le b$

$$TV_\mu^c[a,b] \ge \sum_{i=1}^{n-1} TV_\mu^c[a_i, a_{i+1}].$$

It is also easy to see that the following relations hold

$$TV_\mu^c[0,T] \ge UTV_\mu^c[0,T], \tag{1}$$
$$TV_\mu^c[0,T] \ge DTV_\mu^c[0,T], \tag{2}$$
$$TV_\mu^c[0,T] \le UTV_\mu^c[0,T] + DTV_\mu^c[0,T],$$
$$\mathcal{L}(UTV_\mu^c[0,T]) = \mathcal{L}(DTV_{-\mu}^c[0,T]). \tag{3}$$

By (3) all estimates proved for the upward truncated variation have analogs for the downward truncated variation.

Analogously as in [6] we will prove some estimates of $\mathbf{E}UTV_\mu^c[0,T]$ (and thus for $\mathbf{E}DTV_\mu^c[0,T]$) up to universal constants. Unfortunately, the estimates involve the expected values of some other related variables.

Remark. In order to shorten the proofs we did not put much stress on obtaining the best possible constants.

Remark. K. Oleszkiewicz pointed out that it would be also interesting to have estimates for higher moments of the defined quantities. However, the author presumes that other methods than those used in this paper are needed to obtain such estimates.

REMARK. A. N. Chuprunov pointed out to the author that it would be also interesting to have estimates of the quadratic truncated variation, which one may define as

$$QTV_\mu^c[a,b] := \sup_n \sup_{a\le t_1\le\cdots\le t_n\le b} \sum_{i=1}^{n-1} \max\{|W_{t_{i+1}} - W_{t_i}|^2 - c^2, 0\}.$$

REMARK. A similar concept of truncation (or *shrinking*) of random variables on Hilbert spaces was investigated by Z. J. Jurek in a series of papers beginning with [2], [3], which now evolved into the theory of s-selfdecomposable distributions (see e.g. [4]).

2. Existence of exponential moments of truncated variation. Let us start with the existence of finite exponential moments of $TV_\mu^c[0,T]$. To prove this let us define

$$T_c = \inf\left\{t\ge 0 : \sup_{0\le s\le t} W_s \ge W_t + c\right\},$$

further let $T_c^{\sup}$ be the last instant when the maximum of W_t on $[0,T_c]$ is attained, and let $T_c^{\inf} \le T_c^{\sup}$ be such that $W_{T_c^{\inf}} = \inf_{0\le s\le T_c^{\sup}} W_s$.

Let us fix $\alpha > 0$ and let $\delta > 0$ be a small number such that

$$1 - \mathbf{E}\exp\left(\alpha \sup_{0\le t\le T} W_t + \alpha c\right) P(T_c < \delta) > 0.$$

By definition of T_c and $T_c^{\inf}$ we have $W_{T_c^{\inf}} > -c$ and $W_{T_c^{\sup}} - W_{T_c^{\inf}} - c \le W_{T_c^{\sup}}$. Now, by Lemma 1, Lemma 2 in [6] and independence of $W_t - W_{T_c}, t \ge T_c$, and T_c (strong Markov property of Brownian motion) for any $M > 0$ we have

$$\begin{aligned}
\mathbf{E}\exp(\alpha TV_\mu^c[0,T]\wedge M) &\le \mathbf{E}\exp(\alpha W_{T_c^{\sup}} + \alpha c + \alpha TV_\mu^c[T_c,T]\wedge M)\\
&\le \mathbf{E}\exp(\alpha W_{T_c^{\sup}} + \alpha c)\mathbf{E}\exp[\alpha TV_\mu^c[T_c,T]\wedge M; T_c<\delta]\\
&\quad + \mathbf{E}\exp(\alpha W_{T_c^{\sup}} + \alpha c)\mathbf{E}\exp[\alpha TV_\mu^c[T_c,T]\wedge M; T_c\ge\delta]\\
&\le \mathbf{E}\exp(\alpha W_{T_c^{\sup}} + \alpha c)\mathbf{E}\exp[\alpha TV_\mu^c[T_c,T+T_c]\wedge M; T_c<\delta]\\
&\quad + \mathbf{E}\exp(\alpha W_{T_c^{\sup}} + \alpha c)\mathbf{E}\exp[\alpha TV_\mu^c[T_c,T+T_c-\delta]\wedge M; T_c\ge\delta]\\
&\le \mathbf{E}\exp\left(\alpha \sup_{0\le t\le T} W_t + \alpha c\right)\mathbf{E}\exp(\alpha TV_\mu^c[0,T]\wedge M)P(T_c<\delta)\\
&\quad + \mathbf{E}\exp\left(\alpha \sup_{0\le t\le T} W_t + \alpha c\right)\mathbf{E}\exp(\alpha TV_\mu^c[0,T-\delta]\wedge M)P(T_c\ge\delta).
\end{aligned}$$

From the above we have

$$\begin{aligned}
&\mathbf{E}\exp(\alpha TV_\mu^c[0,T]\wedge M)\\
&\quad\le \frac{\mathbf{E}\exp(\alpha \sup_{0\le t\le T} W_t + \alpha c)P(T_c\ge\delta)}{1-\mathbf{E}\exp(\alpha \sup_{0\le t\le T} W_t + \alpha c)P(T_c<\delta)}\mathbf{E}\exp(\alpha TV_\mu^c[0,T-\delta]\wedge M).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbf{E}\exp(\alpha TV_\mu^c[0,T-\delta]\wedge M)\\
&\quad\le \frac{\mathbf{E}\exp(\alpha \sup_{0\le t\le T} W_t + \alpha c)P(T_c\ge\delta)}{1-\mathbf{E}\exp(\alpha \sup_{0\le t\le T} W_t + \alpha c)P(T_c<\delta)}\mathbf{E}\exp(\alpha TV_\mu^c[0,T-2\delta]\wedge M).
\end{aligned}$$

Iterating and putting together the above inequalities we finally obtain

$$\mathbf{E}\exp(\alpha TV_\mu^c[0,T]\wedge M) \le \left(\frac{\mathbf{E}\exp(\alpha\sup_{0\le t\le T} W_t+\alpha c)P(T_c\ge\delta)}{1-\mathbf{E}\exp(\alpha\sup_{0\le t\le T} W_t+\alpha c)P(T_c<\delta)}\right)^{\lceil T/\delta\rceil}.$$

Letting $M\to\infty$ we get $\mathbf{E}\exp(\alpha TV_\mu^c[0,T])<+\infty$.

By (1) and (2) we obtain the finiteness of exponential moments of $UTV_\mu^c[0,T]$ and $DTV_\mu^c[0,T]$ as well.

3. Estimates of expected value of upward and downward truncated variation

3.1. Preparatory lemmas. In order to obtain estimates of $\mathbf{E}UTV_\mu^c[0,T]$ (and analogously $\mathbf{E}DTV_\mu^c[0,T]$) we will use similar techniques as in [6]. Due to typographical reasons let us introduce the notation $\max\{x,0\}=:(x)_+$.

We will need the following analog of Lemma 2 from [6]:

LEMMA 3.1. *We have the following identity*

$$UTV_\mu^c[0,T]=\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+ + UTV_\mu^c[T_c,T]. \tag{4}$$

Proof. Let $0\le t_1<s_1<t_2<s_2<\cdots<t_n<s_n\le T$. We will prove that

$$\sum_{i=1}^n (W_{s_i}-W_{t_i}-c)_+ \le \sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+ + UTV_\mu^c[T_c,T]. \tag{5}$$

Let n_0 be the greatest number such that $s_{n_0}<T_c$ and let us assume that $n_0<n$ and $t_{n_0+1}<T_c$.

Let us consider several cases.

- $W_{t_{n_0+1}}\ge W_{T_c}$. In this case
$$(W_{s_{n_0+1}}-W_{t_{n_0+1}}-c)_+ \le (W_{s_{n_0+1}}-W_{T_c}-c)_+$$
and
$$\sum_{i=1}^n (W_{s_i}-W_{t_i}-c)_+ \le \sum_{i=1}^{n_0}(W_{s_i}-W_{t_i}-c)_+ + (W_{s_{n_0+1}}-W_{T_c}-c)_+ + \sum_{i=n_0+2}^{n}(W_{s_i}-W_{t_i}-c)_+. \tag{6}$$
- $W_{t_{n_0+1}}<W_{T_c}$ and $W_{s_{n_0+1}}\le W_{T_c^{\sup}}$. In this case $t_{n_0+1}<T_c^{\sup}$ (since for $T_c^{\sup}<t<T_c$, $W_t>W_{T_c}$) so
$$(W_{s_{n_0+1}}-W_{t_{n_0+1}}-c)_+ \le (W_{T_c^{\sup}}-W_{t_{n_0+1}}-c)_+$$
and
$$\sum_{i=1}^n (W_{s_i}-W_{t_i}-c)_+ \le \sum_{i=1}^{n_0}(W_{s_i}-W_{t_i}-c)_+ + (W_{T_c^{\sup}}-W_{t_{n_0+1}}-c)_+ + \sum_{i=n_0+2}^{n}(W_{s_i}-W_{t_i}-c)_+. \tag{7}$$

- $W_{t_{n_0+1}} < W_{T_c}$ and $W_{s_{n_0+1}} > W_{T_c^{\sup}} = W_{T_c} + c$. In this case

$$\begin{aligned}(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c)_+ &= W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\\ &= W_{T_c^{\sup}} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_c^{\sup}}\\ &= W_{T_c^{\sup}} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_c} - c\\ &= (W_{T_c^{\sup}} - W_{t_{n_0+1}} - c)_+ + (W_{s_{n_0+1}} - W_{T_c} - c)_+\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}(W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0}(W_{s_i} - W_{t_i} - c)_+ + (W_{T_c^{\sup}} - W_{t_{n_0+1}} - c)_+\\ + (W_{s_{n_0+1}} - W_{T_c} - c)_+ + \sum_{i=n_0+2}^{n}(W_{s_i} - W_{t_i} - c)_+. \quad (8)\end{aligned}$$

Thus for $t_{n_0+1} < T_c$ inequality (6), (7) or (8) holds and we may assume, adding in the case $t_{n_0+1} < T_c$ new terms in the partition and renaming the old ones, that

$$\begin{gathered}0 \le t_1 < s_1 < \cdots < t_{n_0} < s_{n_0} \le T_c,\\ T_c \le t_{n_0+1} < s_{n_0+1} < \cdots < t_n < s_n \le T.\end{gathered}$$

In order to prove (5) without loss of generality we may assume that for any $1 \le i \le n_0$, $(W_{s_i} - W_{t_i} - c)_+ > 0$ (otherwise we may omit the summand $(W_{s_i} - W_{t_i} - c)_+$). From the definition of T_c we have that for any $1 \le i \le n_0 - 1$, $W_{s_i} - W_{t_{i+1}} < c$, so

$$\begin{aligned}&(W_{s_i} - W_{t_i} - c)_+ + (W_{s_{i+1}} - W_{t_{i+1}} - c)_+\\ &= W_{s_i} - W_{t_i} - c + W_{s_{i+1}} - W_{t_{i+1}} - c\\ &= W_{s_{i+1}} - W_{t_i} - c + (W_{s_i} - W_{t_{i+1}} - c) < W_{s_{i+1}} - W_{t_i} - c.\end{aligned}$$

Iterating the above inequality, we obtain

$$\sum_{i=1}^{n_0}(W_{s_i} - W_{t_i} - c)_+ \le W_{s_{n_0}} - W_{t_1} - c \le \sup_{0 \le t < s \le T_c \wedge T}(W_s - W_t - c)_+.$$

This, together with the obvious inequality

$$\sum_{i=n_0+1}^{n}(W_{s_i} - W_{t_i} - c)_+ \le UTV_\mu^c[T_c, T]$$

proves (5). Taking the supremum over all partitions $0 \le t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n \le T$ we finally get

$$UTV_\mu^c[0,T] \le \sup_{0 \le t < s \le T_c \wedge T}(W_s - W_t - c)_+ + UTV_\mu^c[T_c, T].$$

Since the opposite inequality is obvious, we finally get (4). ■

Let us now define some auxiliary variables. Let $T_c^{(0)} \equiv 0$ and let $T_c^{(i)}, i = 1, 2, \ldots$ be defined recursively as

$$T_c^{(i)} = \inf\left\{t > T_c^{(i-1)} : \sup_{T_c^{(i-1)} \le s \le t} W_s \ge W_t + c\right\}.$$

(notice that $T_c^{(1)} = T_c$). We define a new variable

$$UTV_\mu^c(T) := \sum_{i=1}^{\infty} e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge (T_c^{(i-1)}+T)} (W_s - W_t - c)_+.$$

We have the following

Lemma 3.2. *The variables $UTV_\mu^c[0,T]$ and $UTV_\mu^c(T)$ are related by the following relations*

$$UTV_\mu^c[0,T] \le eUTV_\mu^c(T), \tag{9}$$

$$UTV_\mu^c[0,T] \succeq \frac{1-e^{-1}}{2} UTV_\mu^c(T), \tag{10}$$

where the first relation holds almost surely and the second holds in the sense of stochastic domination i.e. for every $y \ge 0$, $P(UTV_\mu^c[0,T] \ge y) \ge P(\frac{1-e^{-1}}{2} UTV_\mu^c(T) \ge y)$.

Proof. By the previous lemma, we have

$$\begin{aligned} UTV_\mu^c[0,T] &= \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c)_+ + UTV_\mu^c[T_c^{(1)}, T] \\ &= \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c)_+ + \sup_{T_c^{(1)} \le t < s \le T_c^{(2)} \wedge T} (W_s - W_t - c)_+ \\ &\quad + UTV_\mu^c[T_c^{(2)}, T] \\ &= \dots = \sum_{i \ge 1 : T_c^{(i-1)} \le T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge T} (W_s - W_t - c)_+. \end{aligned} \tag{11}$$

From (11) we almost immediately get (9)

$$\begin{aligned} UTV_\mu^c[0,T] &= \sum_{i \ge 1 : T_c^{(i-1)} \le T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge T} (W_s - W_t - c)_+ \\ &\le \sum_{i=1}^{\infty} e^{1-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge (T_c^{(i-1)}+T)} (W_s - W_t - c)_+ \\ &= eUTV_\mu^c(T). \end{aligned}$$

In order to prove the second relation let $i_0 \ge 1$ be the greatest index such that $T_c^{(i_0-1)} < T$ and let us consider the term

$$A = \sup_{T_c^{(i_0-1)} \le t < s \le T_c^{(i_0)} \wedge (T_c^{(i_0-1)}+T)} (W_s - W_t - c)_+.$$

If $i_0 = 1$ then $A = \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c, 0)_+$, otherwise A is independent of $B = \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c, 0)_+$ but has the same distribution as B.

By (11) we have

$$UTV_\mu^c[0,T] = \sum_{i\ge 1: T_c^{(i-1)}\le T} \sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge T} (W_s - W_t - c)_+ \tag{12}$$
$$= \sum_{i=1}^{i_0-1} \sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}} (W_s - W_t - c)_+ + \sup_{T_c^{(i_0-1)}\le t<s\le T} (W_s - W_t - c)_+ .$$

In both cases ($i_0 = 1$ and $i_0 > 1$) $2UTV_\mu^c[0,T]$ stochastically dominates the sum

$$S_1 = \sum_{i=1}^{i_0} e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge (T_c^{(i-1)}+T)} (W_s - W_t - c)_+ .$$

($\sum_{i=1}^{i_0-1} \sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}} (W_s - W_t - c)_+$ dominates the first $i_0 - 1$ terms in the above sum and B, which appears in the sum (12) dominates A.) Similarly, define i_k recursively as the greatest integer such that $T_c^{(i_k-1)} < T_c^{(i_{k-1})} + T$ and

$$S_k = \sum_{i=i_{k-1}+1}^{i_k} \exp\left(-\frac{T_c^{(i-1)} - T_c^{(i_{k-1})}}{T}\right) \sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge (T_c^{(i-1)}+T)} (W_s - W_t - c)_+ .$$

S_k is independent of $S_1, \ldots, S_{k-1}$, moreover it has the same distribution as S_1 and

$$UTV_\mu^c(T) = \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} S_k .$$

By the definition of $i_k, T_c^{(i_k)} \ge T_c^{(i_{k-1})} + T$, thus we have $T_c^{(i_k)} \ge (k-1)T$. Now, since $2UTV_\mu^c[0,T] \succeq S_k$, $k = 1, 2, \ldots$, we have that

$$\frac{2}{1-e^{-1}} UTV_\mu^c[0,T] = \sum_{k=1}^{\infty} e^{-(k-1)} 2UTV_\mu^c[0,T]$$
$$\succeq \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} 2UTV_\mu^c[0,T]$$
$$\succeq \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} S_k = UTV_\mu^c(T).$$

which proves (10). ■

Next, let us state a refinement of Lemma 3 from [6]:

LEMMA 3.3. *For any μ and $c > 0$*

$$P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \le \frac{7}{9}.$$

Proof. The proof follows exactly as in [6], since one can show that for any real μ

$$\frac{(\mathbf{E}T_c)^2}{\mathbf{E}T_c^2} = \frac{1}{2}\,\frac{(e^{2\mu c} - 1 - 2\mu c)^2}{e^{4\mu c} - 6e^{2\mu c}\mu c + e^{2\mu c} + 2\mu^2 c^2 - 2} \ge \frac{1}{2}$$

and, by the Paley-Zygmund inequality we obtain

$$P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right) \geq \left(1-\frac{1}{3}\right)^2 \frac{(\mathbf{E}T_c)^2}{\mathbf{E}T_c^2} \geq \frac{4}{9}\frac{1}{2} = \frac{2}{9}$$

and

$$P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) = 1 - P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right) \leq \frac{7}{9}. \blacksquare$$

3.2. Estimates for long and short time intervals. Now we are ready to prove estimates of the expected value of $UTV_\mu^c[0,T]$ for long and short time intervals ($T \geq \frac{1}{3}\mathbf{E}T_c$ and $T < \frac{1}{3}\mathbf{E}T_c$ respectively). We have

THEOREM 3.4. *For any $T \geq \frac{1}{3}\mathbf{E}T_c$ we have*

$$0.3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ \leq \mathbf{E}UTV_\mu^c[0,T]$$
$$\leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+.$$

Proof. By Lemma 3.1 and independence of $W_t - W_{T_c}, t \geq T_c$, and T_c (strong Markov property of Brownian motion) we calculate

$$\begin{aligned}
\mathbf{E}UTV_\mu^c[0,T] &= \mathbf{E}\sup_{0\leq t\leq s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[T_c\wedge T, T] \\
&\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c,T]; T_c < \frac{1}{3}\mathbf{E}T_c\right] \\
&\quad + \mathbf{E}\left[UTV_\mu^c[T_c,T]; \frac{1}{3}\mathbf{E}T_c \leq T_c \leq T\right] \\
&\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c,T+T_c]; T_c < \frac{1}{3}\mathbf{E}T_c\right] \\
&\quad + \mathbf{E}\left[UTV_\mu^c[T_c,T+T_c-\frac{1}{3}\mathbf{E}T_c]; \frac{1}{3}\mathbf{E}T_c \leq T_c \leq T\right] \\
&\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[0,T]P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \\
&\quad + \mathbf{E}UTV_\mu^c\left[0, T - \frac{1}{3}\mathbf{E}T_c\right]P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right).
\end{aligned}$$

Now, by the above inequality and Lemma 3.3

$$\begin{aligned}
\mathbf{E}UTV_\mu^c[0,T] &\leq \frac{\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+}{P(T_c \geq \frac{1}{3}\mathbf{E}T_c)} + \mathbf{E}UTV_\mu^c\left[0, T - \frac{1}{3}\mathbf{E}T_c\right] \\
&\leq \frac{9}{2}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c\left[0, T - \frac{1}{3}\mathbf{E}T_c\right].
\end{aligned}$$

Similarly

$$\mathbf{E}UTV_\mu^c\left[0, T - \frac{1}{3}\mathbf{E}T_c\right] \leq \frac{9}{2}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c\left[0, T - \frac{2}{3}\mathbf{E}T_c\right].$$

Iterating and putting together the above inequalities we obtain the estimate from above

$$\begin{aligned}\mathbf{E}UTV_\mu^c[0,T] &\le \left\lceil \frac{T}{\frac{1}{3}\mathbf{E}T_c}\right\rceil \frac{9}{2}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+\\ &\le \left(\frac{3T}{\mathbf{E}T_c}+1\right)\frac{9}{2}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+\\ &\le \frac{6T}{\mathbf{E}T_c}\frac{9}{2}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+\\ &\le 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+.\end{aligned}$$

The estimate from below is obtained from Lemma 3.2 (see also the comment after the calculation):

$$\begin{aligned}\mathbf{E}UTV_\mu^c[0,T] &\ge \frac{1-e^{-1}}{2}\mathbf{E}UTV_\mu^c(T)\ge 0.3\mathbf{E}UTV_\mu^c(T)\\ &= 0.3\sum_{i=1}^{\infty}\mathbf{E}e^{-T_c^{(i-1)}/T}\sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge(T_c^{(i-1)}+T)}(W_s-W_t-c)_+\\ &= 0.3\sum_{i=1}^{\infty}\mathbf{E}e^{-T_c^{(i-1)}/T}\mathbf{E}\sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge(T_c^{(i-1)}+T)}(W_s-W_t-c)_+\\ &= 0.3\left(\sum_{i=1}^{\infty}(\mathbf{E}e^{-T_c^{(1)}/T})^{i-1}\right)\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+\\ &= 0.3\frac{1}{1-\mathbf{E}e^{-T_c^{(1)}/T}}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+\\ &\ge 0.3\frac{1}{1-\mathbf{E}(1-T_c^{(1)}/T)}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+\\ &= 0.3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+.\end{aligned}$$

In the above calculations we used consecutively: the independence of $T_c^{(i-1)}$ and $W_s-W_{T_c^{(i-1)}}$, $s\ge T_c^{(i-1)}$, the equality of distributions of every term

$$\sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge(T_c^{(i-1)}+T)}(W_s-W_t-c)_+$$

for $i=1,2,\dots$, the definition of $T_c^{(i-1)}$, which implies the equality

$$\mathbf{E}e^{-T_c^{(i-1)}/T}=(\mathbf{E}e^{-T_c^{(1)}/T})^{i-1},$$

and finally we used the inequality $e^x\ge 1+x$. ∎

The estimates in Theorem 3.4 involve the expected value of the variable

$$\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+$$

the distribution of which, as far as the author knows, is not known, but it may be simulated numerically. We also have

COROLLARY 3.5. *For any $T \geq \frac{1}{3}\mathbf{E}T_c$ we have*

$$3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t\leq s\leq \frac{1}{3}\mathbf{E}T_c}(W_s - W_t - c)_+ \leq \mathbf{E}UTV_\mu^c[0,T] \leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t\leq s\leq T_c}(W_s - W_t - c)_+. \quad (13)$$

Proof. The estimate from above is a straightforward consequence of Theorem 3.4 and the estimate from below is obtained immediately by superadditivity:

$$\begin{aligned}\mathbf{E}UTV_\mu^c[0,T] &\geq \sum_{i=1}^{\lfloor 3T/\mathbf{E}T_c\rfloor}\mathbf{E}UTV_\mu^c\left[\frac{i-1}{3}\mathbf{E}T_c, \frac{i}{3}\mathbf{E}T_c\right] \\ &\geq \lfloor 3T/\mathbf{E}T_c\rfloor \mathbf{E}UTV_\mu^c\left[0, \frac{1}{3}\mathbf{E}T_c\right] \\ &\geq 3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t\leq s\leq \frac{1}{3}\mathbf{E}T_c}(W_s - W_t - c)_+. \ \blacksquare\end{aligned}$$

REMARK. Using the results of Hadjiliadis and Vecer [1] we are able to calculate exactly the estimate from above appearing in (13). Using the notation from [1], for $z > 0$ we have

$$\begin{aligned}P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t - c)_+ \geq z\right) &= P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t) \geq z + c\right) \\ &= P(T(c, z+c) = T_2(z+c))\end{aligned}$$

and by Theorem 2.1 from [1], for $y > c$ we have

$$P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t) \geq y\right) = \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\exp\left(-\frac{2\mu}{e^{2\mu c} - 1}(y - c)\right).$$

Hence

$$\begin{aligned}\mathbf{E}\sup_{0\leq t\leq s\leq T_c}(W_s - W_t - c)_+ &= \int_c^\infty P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t) \geq y\right)dy \\ &= \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\int_c^\infty \exp\left(-\frac{2\mu}{e^{2\mu c} - 1}(y - c)\right)dy \\ &= \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\frac{e^{2\mu c} - 1}{2\mu}.\end{aligned}$$

Estimates of $\mathbf{E}UTV_\mu^c[0,T]$ for short time intervals ($T < \frac{1}{2}\mathbf{E}T_c$) are the subject of the next theorem.

THEOREM 3.6. *For any $T < \frac{1}{3}\mathbf{E}T_c$ we have*

$$\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ \leq \mathbf{E}UTV_\mu^c[0,T] \leq 5\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+.$$

Proof. Applying Lemma 3.1 and the independence of $W_t - W_{T_c}, t \geq T_c$, and T_c we again calculate

$$\begin{aligned}\mathbf{E}UTV_\mu^c[0,T] &\le \mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+ + \mathbf{E}UTV_\mu^c[T_c\wedge T,T]\\ &\le \mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+ + \mathbf{E}[UTV_\mu^c[T_c,T];T_c<T]\\ &\le \mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+ + \mathbf{E}UTV_\mu^c[0,T]P\left(T_c<\frac{1}{3}\mathbf{E}T_c\right)\\ &\le \mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+ + \mathbf{E}UTV_\mu^c[0,T]\frac{7}{9}.\end{aligned}$$

Thus we get

$$\mathbf{E}UTV_\mu^c[0,T] \le \frac{9}{2}\mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+.$$

The estimate from above is self-evident

$$\mathbf{E}UTV_\mu^c[0,T] \ge \mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+. \blacksquare$$

REMARK. In order to calculate the quantity $\mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+$ for $T\le \frac{1}{3}\mathbf{E}T_c$, which appears in Corollary 3.5 and in Theorem 3.6, one may use the results of [5]. Let

$$G_{\bar{D}}(y) = 2e^{\mu y}\left\{L+\sum_{n=1}^{\infty}\frac{\theta_n\sin\theta_n}{\theta_n^2+\mu^2y^2+\mu y}\left(1-\exp\left(-\frac{\theta_n^2T}{2y^2}-\frac{\mu^2T}{2}\right)\right)\right\},$$

where θ_n are positive solutions of the eigenvalue condition $\tan\theta_n = -\frac{\theta_n}{\mu y}$,

$$L=\begin{cases} 0,\ 0<y<-\frac{1}{\mu};\\ \frac{3}{2}(1-e^{-\mu^2T/2}),\ y=-\frac{1}{\mu};\\ \frac{2\eta\sinh\eta}{\eta^2-\mu^2y^2-\mu y}\left(1-\exp\left(\frac{\eta^2T}{2y^2}-\frac{\mu^2T}{2}\right)\right),\ y>-\frac{1}{\mu};\end{cases}$$

and η is the unique positive solution of $\tanh\eta = -\frac{\eta}{\mu y}$. In the notation used in [5] for $z>0$ we have

$$\begin{aligned}P\left(\sup_{0\le t\le s\le T}(W_s-W_t-c)_+\ge z\right) &= P\left(\sup_{0\le t\le s\le T}(W_s-W_t)\ge z+c\right)\\ &= P(\bar{D}(T;-\mu,1)\ge z+c) = G_{\bar{D}}(z+c)\end{aligned}$$

and thus

$$\mathbf{E}\sup_{0\le t\le s\le T}(W_s-W_t-c)_+ = \int_0^\infty G_{\bar{D}}(z+c)dz = \int_c^\infty G_{\bar{D}}(z)dz.$$

However, the above formula is numerically very unstable and it does not seem to be a straightforward task to apply it to obtain good numerical or analytical estimates of the expected value of the variable $\sup_{0\le t\le s\le T}(W_s-W_t-c)_+$.

4. Example of application. As mentioned earlier, upward truncated variation appears naturally in the expression for **the least upper bound** for the rate of return from any trading of a financial asset, the dynamics of which follows geometric Brownian motion, in the presence of flat commission. A similar result was proved in [6] for truncated variation, however, truncated variation is not the least upper bound.

Indeed, similarly as in [6], let us assume that the dynamics of the prices P_t of some financial asset (e.g. stock) is the following $P_t = \exp(\mu t + \sigma B_t)$. We are interested in the maximal possible profit coming from trading this single instrument during the time interval $[0, T]$. We buy the instrument at times $0 \le t_1 < \cdots < t_n < T$ and sell it at $s_1 < \cdots < s_n \le T$, where $t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n$, in order to obtain the maximal possible profit. Furthermore we assume that for every transaction we have to pay a flat commission and γ is the ratio of the transaction value paid for the commission.

The maximal possible rate of return from our strategy is (cf. [6])

$$\sup_n \sup_{0\le t_1<s_1<\cdots<t_n<s_n\le T} \frac{P_{s_1}}{P_{t_1}}\frac{1-\gamma}{1+\gamma}\cdots\frac{P_{s_n}}{P_{t_n}}\frac{1-\gamma}{1+\gamma} - 1.$$

Let M_n be the set of all partitions

$$\pi = \{0 \le t_1 < s_1 < \cdots < t_n < s_n \le T\}.$$

To see that $\exp(\sigma UTV^{c/\sigma}_{\mu/\sigma}[0,T]) - 1$ with $c = \ln\frac{1+\gamma}{1-\gamma}$ is the least upper bound for maximal possible rate of return let us calculate

$$\begin{aligned}
\sup_n \sup_{M_n} \prod_{i=1}^n \left\{\frac{P_{s_i}}{P_{t_i}}\frac{1-\gamma}{1+\gamma}\right\} &= \sup_n \sup_{M_n} \prod_{i=1}^n \left\{\frac{\exp(\mu s_i + \sigma B_{s_i})}{\exp(\mu t_i + \sigma B_{t_i})} e^{-c}\right\} \\
&= \sup_n \sup_{M_n} \exp\left(\sigma \sum_{i=1}^n \left\{\left(\frac{\mu}{\sigma}s_i + B_{s_i}\right) - \left(\frac{\mu}{\sigma}t_i + B_{t_i}\right) - \frac{c}{\sigma}\right\}\right) \\
&= \exp\left(\sigma \sup_n \sup_{M_n} \sum_{i=1}^n \left\{\left(\frac{\mu}{\sigma}s_i + B_{s_i}\right) - \left(\frac{\mu}{\sigma}t_i + B_{t_i}\right) - \frac{c}{\sigma}\right\}\right) \\
&= \exp(\sigma UTV^{c/\sigma}_{\mu/\sigma}[0,T]).
\end{aligned}$$

This gives the claimed bound.

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