STABILITY IN PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 90
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

ON THE INFINITE DIVISIBILITY OF SCALE MIXTURES OF SYMMETRIC α-STABLE DISTRIBUTIONS, $\alpha \in (0,1]$

G. MAZURKIEWICZ

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Szafrana 4a, 65-516 Zielona Góra, Poland
E-mail: G.Mazurkiewicz@wmie.uz.zgora.pl

Abstract. The paper contains a new and elementary proof of the fact that if $\alpha \in (0,1]$ then every scale mixture of a symmetric α-stable probability measure is infinitely divisible. This property is known to be a consequence of Kelker's result for the Cauchy distribution and some nontrivial properties of completely monotone functions. It is known that this property does not hold for $\alpha = 2$. The problem discussed in the paper is still open for $\alpha \in (1,2)$.

1. Introduction. Throughout the paper we denote by $\mathcal{L}(X)$ the distribution of the random variable X. If random variables X and Y have the same distribution we will write $X \stackrel{d}{=} Y$. By X' we denote a random variable which is independent of X and such that $X' \stackrel{d}{=} X$. The set $\mathcal{P}$ contains all probability measures on $\mathbb{R}$.

For every $a \in \mathbb{R}$ and every probability measure $\mu \in \mathcal{P}$ we define the rescaling operator $T_a \colon \mathcal{P} \to \mathcal{P}$ by the formula:

$$T_a\mu(A) = \begin{cases} \mu(A/a) & \text{for } a \neq 0, \\ \delta_0(A) & \text{for } a = 0, \end{cases}$$

for every Borel set $A \subseteq \mathbb{R}$. Equivalently $T_a\mu$ is the distribution of the random variable aX if μ is the distribution of X.

The scale mixture $\mu \circ \lambda$ of the measure $\mu \in \mathcal{P}$ with respect to the measure $\lambda \in \mathcal{P}$ is defined by the formula:

$$\mu \circ \lambda(A) \stackrel{def}{=} \int_{\mathbb{R}} T_s\mu\,(A)\,\lambda(ds).$$

2010 *Mathematics Subject Classification*: 60A10, 60B05, 60E05, 60E07, 60E10.
Key words and phrases: stable distribution, scale mixture, variance mixture.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc90-0-5

Notice that $\mu \circ \lambda$ is the distribution of the random variable ΘX if $\mu = \mathcal{L}(X)$, $\lambda = \mathcal{L}(\Theta)$, X and Θ are independent.

Let us recall (see e.g. [4, 6]) that a symmetric random variable X is α-stable, $\alpha \in (0, 2]$, if for every choice of $a, b \in (0, \infty)$ there exists a constant $c = (a^\alpha + b^\alpha)^{1/\alpha}$ such that

$$aX + bX' \stackrel{d}{=} cX.$$

For $\alpha \in (0, 2]$ we denote by γ_α the standard symmetric α-stable distribution with the characteristic function $\varphi_{S_\alpha}(t) = \exp\{-|t|^\alpha\}$. The corresponding random variable is denoted by S_α.

A random variable X is infinitely divisible (see e.g. [1]) if for every $n \in \mathbb{N}$ there exist a sequence of independent, identically distributed random variables $Y_{1,n}, Y_{2,n}, \ldots, Y_{n,n}$ such that

$$X \stackrel{d}{=} Y_{1,n} + Y_{2,n} + \ldots + Y_{n,n}.$$

Infinite divisibility is of interest in view of its applications especially in Lévy processes, in waiting-time theory and most notably in modeling problems.

2. Infinite divisibility of scale mixtures of symmetric α-stable distributions, $\alpha \in (0, 1]$. In this section, using elementary criterions and a very useful method, we show that if $\alpha \in (0, 1]$ and $\lambda \in \mathcal{P}$ then every scale mixture of the form $\gamma_\alpha \circ \lambda$ is infinitely divisible. This property is known and can be found in [8] as Corollary 10.6. This corollary follows from Theorem 10.5 and by combining (III.3.8) and Example III.4.9, or from Proposition A.3.7(vi) and Bernstein's theorem.

We give here a direct proof of this fact. But we do not know whether this surprising property stays true for $\alpha \in (1, 2)$ and this problem seems to be open since 1972 (see [2]). The cases $\alpha = 1$ and $\alpha = 2$ were previously considered in [3]. In this paper Kelker showed that every scale mixture of a symmetric Cauchy distribution is infinitely divisible. He showed also that a scale mixture of a symmetric Gaussian measure need not be infinitely divisible. Another proof of this fact was given by Rosiński in [5].

THEOREM 2.1. *Let Θ be a real random variable and S_α be independent of Θ. If $\alpha \in (0, 1]$ then ΘS_α is infinitely divisible.*

Proof. For $\alpha \in (0, 1]$ let

$$C(t) = \mathsf{E} \exp\{it\Theta S_\alpha\} = \mathsf{E} \exp\{it|\Theta| S_\alpha\} = \int_0^\infty \exp\{-|t|^\alpha u\} \lambda(du),$$

where $\lambda = \mathcal{L}(|\Theta|^\alpha)$. Using the criterion from [7] it is enough to show that the real characteristic function $C(t)$ is log-convex on $(0, \infty)$, that is, $\log C(t)$ is convex for $t \in (0, \infty)$. Notice first that $t \mapsto C(t)$ is at least twice differentiable on $(0, \infty)$. Hence for every $t > 0$ we obtain

$$\frac{d^2}{dt^2}\log C(t) = \alpha^2 t^{2\alpha-2}\Bigg[\int_0^\infty u^2\exp\{-t^\alpha u\}\lambda(du)/C(t) - \Bigg[\int_0^\infty u\exp\{-t^\alpha u\}\lambda(du)/C(t)\Bigg]^2$$
$$- \frac{\alpha-1}{\alpha}t^{-\alpha}\Bigg[\int_0^\infty u\exp\{-t^\alpha u\}\lambda(du)/C(t)\Bigg]\Bigg]$$
$$= \alpha^2 t^{2\alpha-2}\Bigg[\mathrm{Var}Y_t - \frac{\alpha-1}{\alpha}t^{-\alpha}\mathsf{E}Y_t\Bigg],$$

where the nonnegative random variable Y_t, $t > 0$, has the cumulative distribution function $H_t(y) = \mathrm{P}\{Y_t < y\}$ defined in the following way

$$H_t(y) = \begin{cases} \int_0^y \exp\{-t^\alpha u\}\lambda(du)/C(t) & \text{for } y > 0, \\ 0 & \text{for } y \le 0. \end{cases}$$

For every $\alpha \in (0,1]$ and $t > 0$ we have

$$\int_0^\infty u^2\exp\{-t^\alpha u\}\lambda(du) \le \int_0^\infty u^2\exp\{-t^\alpha u\}|_{u=2t^{-\alpha}}\lambda(du) = 4\exp\{-2\}t^{-2\alpha} < \infty,$$

which means $\mathsf{E}Y_t^2 < \infty$. Since $-\frac{\alpha-1}{\alpha} \ge 0$ we obtain $\frac{d^2}{dt^2}\log C(t) \ge 0$ for every $t > 0$. ■

REMARK. Log-convexity criterion is based on Pólya's criterion ([2]). Therefore it is also worth noticing that Theorem 2.1 directly follows from Pólya's sufficient condition. Then it is enough to show that the function $t \mapsto C(t)^{1/n}$ is nonincreasing and convex on $(0,\infty)$ for every $n \in \mathbb{N}$. Notice first that $\frac{d}{dt}C(t)^{1/n} \le 0$ for every $t > 0$. Using analogous substitutions and calculations as in the proof of Theorem 2.1 we show that for every $t > 0$

$$\frac{d^2}{dt^2}C(t)^{1/n}$$
$$= \frac{1}{n}\alpha^2 t^{2(\alpha-1)}C(t)^{1/n}\Bigg[\mathrm{Var}Y_t - \frac{\alpha-1}{\alpha}t^{-\alpha}\mathsf{E}Y_t + \frac{1}{n}(\mathsf{E}Y_t)^2\Bigg] \ge 0.$$

We do not know whether Theorem 2.1 stays true also for $\alpha \in (1,2)$. It is clear that if Θ^α is infinitely divisible and independent of S_α, $\alpha \in (0,2]$, then the scale mixture ΘS_α is infinitely divisible. Theorem 2.1 is not true if $\alpha = 2$ since for Θ nondegenerate, nonnegative, bounded almost everywhere and independent of S_2 the random variable ΘS_2 is not infinitely divisible (see [3, 5]).

References

[1] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 2, John Wiley, New York, 1966.

[2] J. Keilson and F. W. Steutel, *Families of infinitely divisible distributions closed under mixing and convolution*, Annals of Math. Statist. 43 (1972), 242–250.

[3] D. Kelker, *Infinite divisibility and variance mixtures of the normal distribution*, Ann. Math. Statist. 42 (1971), 802–808.

[4] E. Lukács, *Characteristic Functions*, Griffin, London, 1960.

[5] J. Rosiński, *On a class of infinitely divisible processes represented as mixtures of Gaussian processes*, in: Stable Processes and Related Topics, Progress in Probability 25, Birkhäuser, 1991, 27–41.

[6] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, 1994.

[7] F. W. Steutel, *Infinite divisibility in theory and practice*, Scand. J. Statist. 6 (1979), 57–64.

[8] F. W. Steutel and K. Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, CRC Press, 2004.