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A NONCOMMUTATIVE 2-SPHERE GENERATED BY THE QUANTUM COMPLEX PLANE

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Dedicated to S. L. Woronowicz on the occasion of his 70th birthday

Abstract. S. L. Woronowicz's theory of C*-algebras generated by unbounded elements is applied to q-normal operators satisfying the defining relation of the quantum complex plane. The unique non-degenerate C*-algebra of bounded operators generated by a q-normal operator is computed and an abstract description is given by using crossed product algebras. If the spectrum of the modulus of the q-normal operator is the positive half line, this C*-algebra will be considered as the algebra of continuous functions on the quantum complex plane vanishing at infinity, and its unitization will be viewed as the algebra of continuous functions on a quantum 2-sphere.

1. Introduction. In his seminal paper [12], S. L. Woronowicz introduced the concept of C*-algebras generated by unbounded elements. His main motivation was to provide a proper mathematical framework for a topological theory of non-compact quantum groups (cf. [8, 11]). The basic idea is to use an affiliation relation to give a precise meaning to the statement that a finite set of (unbounded) operators satisfying certain relations generates a given C*-algebra. Naturally, one expects that a C*-algebra generated by unbounded

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operators is non-unital. This algebra is then viewed as the algebra of continuous functions vanishing at infinity on the underlying quantum space.

Since compact spaces are better behaved than non-compact ones, we can pass to the compact case by adjoining a unit which corresponds to the one-point compactification of a locally compact space. The compactification may also turn a topologically trivial space into a non-trivial one, as it is the case when considering for instance a sphere as the one-point compactification of the Euclidean plane. It is a non-commutative counterpart of this example that we want to study within Woronowicz's framework in the present paper.

Our starting point is the quantum complex plane given as the complex *-algebra $\mathcal{O}(\mathbb{C}_q)$ generated by ζ and ζ^* satisfying the relation

$$\zeta\zeta^* = q^2\zeta^*\zeta, \tag{1}$$

where, throughout the paper, we assume that $q \in (0,1)$. A representation of $\mathcal{O}(\mathbb{C}_q)$ is given by a densely defined closed linear operator ζ on a separable Hilbert space satisfying (1). Such operators are known as q-normal operators (or q^2-normal operators). On the contrary to usual normal operators, non-zero q-normal operators are never bounded. In particular, they do not generate themselves a C*-algebra. This naturally motivates the use of Woronowicz's theory for the study of the quantum complex plane $\mathcal{O}(\mathbb{C}_q)$ in the C*-algebra setting.

The main result of this paper is an explicit description of the unique non-degenerate C*-algebra of bounded operators generated by a q-normal operator. However, since the algebra of polynomial functions on the quantum complex plane $\mathcal{O}(\mathbb{C}_q)$ is defined in a purely algebraic manner without referring to a Hilbert space, we prefer to give an (almost) Hilbert space free description and state our main theorem (Theorem 3.2) in terms of crossed product C*-algebras. If the spectrum of the modulus $|\zeta|$ of the q-normal operator ζ is the positive half line $[0,\infty)$, the generated C*-algebra has an obvious interpretation as the algebra of continuous functions on the quantum complex plane vanishing at infinity. Its one-point compactification, given by adjoining a unit, will be viewed as the C*-algebra of continuous functions on a quantum 2-sphere.

2. Representations of the quantum complex plane. Although its main motivation comes from abstract algebras given by generators and relations, Woronowicz's framework starts by considering a set of unbounded operators on a Hilbert space affiliated with a C*-algebra of bounded operators [10]. For this reason, we are interested in operators on a separable Hilbert space satisfying the relation (1) in an appropriate sense. A natural choice is to assume ζ to be q-normal. Such operators have been studied, e. g., in [1, 5, 6]. In this section, we will collect some facts about q-normal operators following closely the lines of [1, Section 2].

PROPOSITION 2.1 ([1, 5]). *Let ζ be a densely defined closed operator on a Hilbert space $\mathcal{H}$ and $\zeta = u|\zeta|$ its polar decomposition. Let E denote the projection-valued measure on the Borel σ-algebra $\Sigma([0,\infty))$ such that $|\zeta| = \int \lambda\, \mathrm{d}E(\lambda)$. Then the following statements are equivalent:*

i) ζ *is a q-normal operator, that is,* $\zeta\zeta^* = q^2\zeta^*\zeta$.
ii) $u|\zeta|u^* = q|\zeta|$.
iii) $uE(M)u^* = E(q^{-1}M)$ *for all* $M \in \Sigma([0,\infty))$.
iv) $uf(|\zeta|)u^* = f(q|\zeta|)$ *for every Borel function* f *on* $[0,\infty)$*, where* $f(|\zeta|) := \int f(\lambda)\mathrm{d}E(\lambda)$.

For a more explicit description of q-normal operators, note that

$$\ker(\zeta) = \ker(\zeta^*) = \ker(|\zeta|) = E(\{0\})\mathcal{H}.$$

Set

$$\mathcal{H}_n := E((q^{n+1}, q^n])\mathcal{H}, \quad n \in \mathbb{Z}. \tag{2}$$

By Proposition 2.1 iii), $u : \mathcal{H}_n \to \mathcal{H}_{n-1}$ is an isomorphism. Therefore we can write

$$\mathcal{H}_n = \{h_n := u^{*n}h \,:\, h \in \mathcal{H}_0\}. \tag{3}$$

Define $A : \mathcal{H}_0 \to \mathcal{H}_0$ by $A := \int_{(q,1]} \lambda\, \mathrm{d}E(\lambda)$. Then, by Proposition 2.1 iv),

$$\zeta h_n = u|\zeta|u^{*n}h = q^n u^{*(n-1)}Ah = q^n(Ah)_{n-1} \tag{4}$$

on $\ker(\zeta)^\perp = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n$. From the preceding, we get the following description of q-normal operators.

Corollary 2.2. *Let* ζ *be a non-zero q-normal operator on a Hilbert space* $\mathcal{H}$*. Up to unitary equivalence, the action of* ζ *is determined by the following formulas: There exists a Hilbert space* $\mathcal{H}_0$ *such that* $\mathcal{H}$ *decomposes into the direct sum* $\mathcal{H} = \ker(\zeta) \oplus \oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ *with* $\mathcal{H}_n = \mathcal{H}_0$*. For* $h \in \mathcal{H}_0$*, let* h_n *denote the vector in* $\mathcal{H}$ *which has* h *in the n-th component of the direct sum* $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ *and* 0 *elsewhere. Then there exist a self-adjoint operator* A *on* $\mathcal{H}_0$*, satisfying that* $\mathrm{spec}(A) \subset [q,1]$ *with* q *not being an eigenvalue, such that*

$$\zeta h_n = q^n A h_{n-1} \quad \text{for all} \ \ h_n \in \mathcal{H}_n.$$

It is well known [7, Theorem VII.3] that each self-adjoint operator T on a separable Hilbert space is unitarily equivalent to a direct sum of multiplication operators $\hat{x}$ on $\mathcal{L}_2(\mathrm{spec}(T), \mu)$, where $(\hat{x}f)(x) := xf(x)$. We will apply this fact to the operators $|\zeta|$ and A from the last corollary. Setting $\mu([0,\infty) \setminus \mathrm{spec}(|\zeta|)) = 0$, we may assume that $\mathcal{H} = \mathcal{L}_2([0,\infty), \mu)$. The spectral projections $E(M)$, $M \in \Sigma([0,\infty))$, from Proposition 2.1 are then given by multiplication with the indicator function χ_M, that is, $(E(M)f)(x) = \chi_M(x)f(x)$. Applying Proposition 2.1 iv) to the Borel functions χ_M shows that μ is q-invariant: $\mu(qM) = \mu(M)$ for all $M \in \Sigma([0,\infty))$. As a consequence, $u : \mathcal{H} \to \mathcal{H}$, $(uf)(x) := f(qx)$ is unitary. One easily checks that $\zeta := u\hat{x}$ defines a q-normal operator. It has been shown in [1] that any q-normal operator is unitarily equivalent to a direct sum of such operators.

Theorem 2.3 ([1]). *Any q-normal operator is unitarily equivalent to a direct sum of operators of the following form:* $\mathcal{H} = \mathcal{L}_2([0,\infty), \mu)$*, where* μ *is a q-invariant Borel measure on* $[0,\infty)$*,* $\mathrm{dom}(\zeta) = \{f \in \mathcal{H} \,:\, \int_{[0,\infty)} x^2\,|f(x)|^2\,\mathrm{d}\mu(x) < \infty\}$*, and*

$$(\zeta f)(x) = qxf(qx), \quad (\zeta^* f)(x) = xf(q^{-1}x) \quad \text{for all} \ \ f \in \mathrm{dom}(\zeta). \tag{5}$$

Moreover, for each q-invariant Borel measure μ *on* $[0,\infty)$*, Equation* (5) *defines a q-normal operator.*

Note that a q-invariant measure μ is uniquely determined by the value $\mu(\{0\})$ and its restriction $\mu_0 := \mu\lceil_{\Sigma([0,\infty))\cap(q,1]}$ via

$$\mu(M) = \sum_{k\in\mathbb{Z}} \mu_0(q^{-k}(M \cap (q^{k+1}, q^k])) + \mu(M \cap \{0\}). \tag{6}$$

On the other hand, this formula defines for any measure μ_0 on the Borel σ-algebra $\Sigma((q,1])$ a q-invariant measure μ on $\Sigma([0,\infty))$. Once the value $\mu(\{0\})$ is fixed, this measure is unique.

Suppose we are given a closed non-empty q-invariant set $X \subset [0,\infty)$. Here, the q-invariance means $qX = X$. Since $K := [q,1] \cap X$ is compact (and thus has a countable dense subset), there exists a finite Borel measure ν on $[q,1]$ such that $\operatorname{supp}(\nu) = K$, see [3]. Let δ_x denote the Dirac measure at $x \in [0,\infty)$ and define μ_0 to be the restriction of $\nu + \nu(\{q\})\delta_1 + \nu(\{1\})\delta_q$ to $\Sigma([0,\infty)) \cap (q,1]$. If $X = \{0\}$, set $\mu(\{0\}) := 1$. Then the measure μ determined by the formula in (6) is a q-invariant σ-finite Borel measure on $[0,\infty)$ such that $\operatorname{supp}(\mu) = X$.

Recall that the multiplication operator $\hat{x}$ on $\mathcal{L}_2([0,\infty),\mu)$ satisfies $\operatorname{spec}(\hat{x}) = \operatorname{supp}(\mu)$. Combining the discussion of the last paragraph with Theorem 2.3 gives the following corollary.

Corollary 2.4. *For each non-empty q-invariant closed subset $X \subset [0,\infty)$, there exists a q-normal operator ζ such that* $\operatorname{spec}(|\zeta|) = X$.

3. C*-algebra generated by q-normal operators. In this section, we will determine the C*-algebra generated by a q-normal operator. It turns out that this C*-algebra is closely related to crossed product C*-algebras. Since our transformation group will always be $\mathbb{Z}$, we use a slightly more direct (but equivalent) definition of crossed product C*-algebras than the usual one (cf. [9]).

Let $X \subset [0,\infty)$ be a closed non-empty q-invariant set. Then

$$\alpha_q : C_0(X) \longrightarrow C_0(X), \quad (\alpha_q(f))(x) := f(qx), \tag{7}$$

defines an automorphism of $C_0(X)$. Let U be an abstract unitary element. Consider the *-algebra

$$\text{*-alg}\{C_0(X), U\} := \Big\{ \sum_{\text{finite}} f_k U^k \;:\; f_k \in C_0(X),\;\; k \in \mathbb{Z} \Big\}, \tag{8}$$

with multiplication and involution determined by

$$fU^n gU^m = f\alpha_q^n(g)U^{n+m}, \quad (fU^n)^* = \alpha_q^{-n}(\bar{f})U^{-n}, \quad f,g \in C_0(X),\;\; n,m \in \mathbb{Z}, \tag{9}$$

where $\bar{f}$ denotes the complex conjugate of f. Note that $U \notin \text{*-alg}\{C_0(X),U\}$.

We remark that the use of the unitary U in the definition of $\text{*-alg}\{C_0(X),U\}$ in Equation (8) is superfluous since the way the functions f_k and g_j multiply in the product $(\sum_{\text{finite}} f_k U^k)(\sum_{\text{finite}} g_j U^j)$ is known as the convolution product. However, later on, we will replace U by a unitary operator u and f_k by the operator $f_k(|\zeta|)$, where $|\zeta|$ and u are defined by the polar decomposition $\zeta = u|\zeta|$ of an appropriate q-normal operator ζ, so our notation is more suggestive.

For a Hilbert space $\mathcal{H}$, we denote by $B(\mathcal{H})$ the C*-algebra of bounded operators. A covariant representation of *-alg$\{C_0(X), U\}$ is given by a unitary operator $V \in B(\mathcal{H})$ and a *-representation $\pi_0 : C_0(X) \to B(\mathcal{H})$ such that $V\pi_0(f) = \pi_0(\alpha_q(f))V$ for all $f \in C_0(X)$. Setting $\pi(U) := V$ and $\pi(fU^n) := \pi_0(f)\pi(U)^n$, we obtain a *-representation $\pi :$ *-alg$\{C_0(X), U\} \to B(\mathcal{H})$. Therefore we can view covariant representations as *-representations of *-alg$\{C_0(X), U\}$ satisfying $\pi(fU^n) := \pi_0(f)\pi(U)^n$ with $\pi(U)$ unitary. Now we define the crossed product C*-algebra $C_0(X) \rtimes \mathbb{Z}$ as the closure of *-alg$\{C_0(X), U\}$ under the norm

$$\|a\|_{\text{univ}} := \sup\{\|\pi(a)\| \,:\, \pi \text{ is a covariant representation}\}, \quad a \in \text{*-alg}\{C_0(X), U\}. \tag{10}$$

The existence of the norm defined in (10) follows from general considerations, see [9].

Note that $X\backslash\{0\}$ remains to be q-invariant for any q-invariant set $X \subset [0, \infty)$. The crossed product C*-algebra $C_0(X\backslash\{0\}) \rtimes \mathbb{Z}$ is defined similarly to the above with X replaced by $X\backslash\{0\}$. That is,

$$C_0(X \setminus \{0\}) \rtimes \mathbb{Z} := \|\cdot\|_{\text{univ}}\text{-cls}\Big\{ \sum_{\text{finite}} f_k U^k \,:\, f_k \in C_0(X \setminus \{0\}),\ k \in \mathbb{Z}\Big\},$$

where the multiplication and involution are determined by (9), and the norm is given as in (10).

Suppose that $\zeta = u|\zeta|$ is a q-normal operator with $\text{spec}(|\zeta|) = X \neq \{0\}$. It follows from Proposition 2.1 iv) that we obtain a covariant representation of *-alg$\{C_0(X), U\}$ and *-alg$\{C_0(X \setminus \{0\}), U\}$ on $\ker(|\zeta|)^\perp$ by setting $\pi_0(f) = f(|\zeta|)$ and $\pi(U) = u$. The next proposition shows that we can always think of the corresponding crossed product C*-algebras as the closure of the image of these covariant representations.

In the proof of the proposition, we will need the following notation: Given a C*-algebra $\mathcal{A}$, we denote by $M(\mathcal{A})$ its multiplier C*-algebra (see e.g. [9, Section 1.5]). Let $\mathcal{H}$ be a Hilbert space. We say that a *-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is non-degenerate, if the set $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$. It is known that each non-degenerate π admits a unique extension (denoted by the same symbol) $\pi : M(\mathcal{A}) \to B(\mathcal{H})$.

Proposition 3.1. *Let $X \neq \{0\}$ be a non-empty q-invariant closed subset of* $[0, \infty)$. *Then*

$$C_0(X) \rtimes \mathbb{Z} \cong \mathcal{B}_1 := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k(|\zeta|)\, u^k \,:\, f_k \in C_0(X),\ k \in \mathbb{Z}\Big\},$$

$$C_0(X \setminus \{0\}) \rtimes \mathbb{Z} \cong \mathcal{B}_0 := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k(|\zeta|)\, u^k : f_k \in C_0(X \setminus \{0\}),\ k \in \mathbb{Z}\Big\},$$

where u and $|\zeta|$ are defined by the polar decomposition $\zeta = u|\zeta|$ of a q-normal operator ζ such that $\text{spec}(|\zeta|) = X$ *and* $\ker(|\zeta|) = \{0\}$.

Proof. First we remark that a q-normal operator with $\text{spec}(|\zeta|) = X$ exists by Corollary 2.4. Taking its restriction to $\ker(|\zeta|)^\perp$, we may assume that $\ker(|\zeta|) = \{0\}$. For brevity, set $X_0 := X\backslash\{0\}$ and $X_1 := X$. The proposition will be proven by invoking the universal property of crossed product C*-algebras. That is, if we show that

(a) there is covariant representation $\rho_i :$ *-alg$\{C_0(X_i), U\} \to M(\mathcal{B}_i)$,

(b) given a covariant representation $\pi_i : {}^*\text{-alg}\{C_0(X_i), U\} \to B(\mathcal{H})$, there is a non-degenerate *-representation $\Pi_i : \mathcal{B}_i \to B(\mathcal{H})$ such that its unique extension to $M(\mathcal{B}_i)$ satisfies $\Pi_i \circ \rho_i = \pi_i$,

(c) $\mathcal{B}_i = \|\cdot\|\text{-cls}\{\rho_i(f)\rho_i(U)^k \,:\, f \in C_0(X_i),\ k \in \mathbb{Z}\}$,

then $\mathcal{B}_i \cong C_0(X_i) \rtimes \mathbb{Z}$, $i = 0, 1$, by Raeburn's Theorem [9, Theorem 2.61]. Setting $\rho_i(f) := f(|\zeta|)$, $f \in C_0(X_i)$, and $\rho_i(U) = u$, the item (a) follows from Proposition 2.1 iv) and (c) from the definition of $\mathcal{B}_i$.

To prove (b), we show that $\Pi_i(\sum_{\text{finite}} f_k(|\zeta|)\, u^k) := \sum_{\text{finite}} \pi_i(f_k)\pi_i(U)^k$ is well defined. For this, it suffices to verify that $\sum_{k=M}^{N} f_k(|\zeta|)\, u^k = 0$ implies $f_k = 0$ for all k, where $f_k \in C_0(X_i)$, $M, N \in \mathbb{Z}$ and $M \le N$. By unitary equivalence, we may assume that the action of $\zeta = u|\zeta|$ is given by the formulas of Corollary 2.2. Recall that $\mathcal{H}_n \perp \mathcal{H}_m$ for $n \ne m$, $f_k(|\zeta|) : \mathcal{H}_n \to \mathcal{H}_n$ and $u^k h_n = h_{n-k} \in \mathcal{H}_{n-k}$. Suppose now that $f_M \ne 0$. Then there exist $n \in \mathbb{Z}$ and $g_n, h_n \in \mathcal{H}_n$ such that $\langle g_n, f_M(|\zeta|)h_n\rangle \ne 0$. Hence

$$\Big\langle g_n, \sum_{k=M}^{N} f_k(|\zeta|)\, u^k\, h_{n+M} \Big\rangle = \Big\langle g_n, \sum_{k=M}^{N} f_k(|\zeta|)\, h_{n+M-k} \Big\rangle = \langle g_n, f_M(|\zeta|)h_n\rangle \ne 0.$$

Therefore $\sum_{k=M}^{N} f_k(|\zeta|)\, u^k = 0$ implies $f_M = 0$. By induction on $m = M, M+1, \ldots,$ we conclude that $f_m = 0$ for all $m = M, M+1, \ldots, N$, so Π_i is well defined. The non-degeneracy is easily shown by applying Π_i to an approximate unit of $C_0(X_i)$. ∎

Before stating our main theorem, we recall Woronowicz's definition [12, Definition 3.1] of a C*-algebra generated by unbounded elements. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{A} \subset B(\mathcal{H})$ be a C*-algebra. We say that a densely defined closed operator T acting on $\mathcal{H}$ is affiliated with $\mathcal{A}$ if its z-transform

$$z_T := T(1 + T^*T)^{-1/2} \tag{11}$$

belongs to the multiplier algebra $M(\mathcal{A})$ and if the set $(1 - z_T^* z_T)\mathcal{A}$ is dense in $\mathcal{A}$. Suppose that $\mathcal{A}$ is non-degenerate, i.e., the set $\mathcal{A}\mathcal{H}$ is dense in $\mathcal{H}$. Then, given a C*-algebra $\mathcal{A}_0$, the set of morphisms from $\mathcal{A}_0$ into $\mathcal{A}$ is defined as

$$\begin{aligned}\operatorname{Mor}(\mathcal{A}_0, \mathcal{A}) := \{\pi : \mathcal{A}_0 \longrightarrow M(\mathcal{A}) \subset B(\mathcal{H}) \,:\, &\pi \text{ is a *-homomorphism} \\ &\text{and } \pi(\mathcal{A}_0)\mathcal{A} \text{ is dense in } \mathcal{A}\}.\end{aligned}$$

Let $\mathcal{K}$ be a separable Hilbert space and $\pi : \mathcal{A} \to B(\mathcal{K})$ a non-degenerate representation. Using the fact that π admits a unique extension $\pi : M(\mathcal{A}) \to B(\mathcal{K})$, we can define the π-image of an operator T affiliated with $\mathcal{A}$ by

$$\pi(T) := \pi(z_T)\big(1 - \pi(z_T)^*\pi(z_T)\big)^{-1/2}. \tag{12}$$

Note that $z_{\pi(T)} = \pi(z_T)$ and that $\pi(T)$ is uniquely determined by $\pi(z_T)$. Now, given a C*-algebra $\mathcal{A}$ and a finite set of elements $T_1, \ldots, T_N$ affiliated with $\mathcal{A}$, it is said that $T_1, \ldots, T_N$ generate $\mathcal{A}$ if for any non-degenerate representation $\pi : \mathcal{A} \to B(\mathcal{K})$ and any non-degenerate C*-algebra $\mathcal{B} \subset B(\mathcal{K})$, one has

$$\pi(T_1), \ldots, \pi(T_N) \text{ are affiliated with } \mathcal{B} \qquad \Longrightarrow \qquad \pi \in \operatorname{Mor}(\mathcal{A}, \mathcal{B}). \tag{13}$$

It follows immediately from [12, Proposition 3.2] that a non-degenerate C*-algebra $\mathcal{A}$ generated by $T_1, \ldots, T_N$ is unique; see also the comments before [12, Theorem 4.2] for the more general case of a C*-algebra $\mathcal{A}$ generated by a quantum family of affiliated elements.

Our main goal, achieved in the next theorem, is to give an explicit description of the C*-algebra generated by a non-trivial q-normal operator. Since q-normal operators act on Hilbert spaces, it would be natural to search for a subalgebra of bounded operators. However, the use of crossed product algebras and the last proposition allow us to describe the generated C*-algebra without reference to a Hilbert space.

THEOREM 3.2. *Let ζ be a q-normal operator on a separable Hilbert space $\mathcal{H}$ such that $X := \operatorname{spec}(|\zeta|) \neq \{0\}$. Then the unique non-degenerate C*-algebra in $B(\mathcal{H})$ generated by ζ is isomorphic to*

$$C_0^*(\zeta,\zeta^*) := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k U^k \in C_0(X) \rtimes \mathbb{Z} \ :\ k \in \mathbb{Z},\ \ f_k(0) = 0 \ \text{ for all } \ k \neq 0 \Big\}, \quad (14)$$

where the norm closure is taken in the C-algebra $C_0(X) \rtimes \mathbb{Z}$.*

Proof. The first step of the proof consists in identifying the abstractly defined C*-algebra $C_0^*(\zeta,\zeta^*)$ with a non-degenerate C*-subalgebra of $\mathcal{A} \subset B(\mathcal{H})$. Let $\zeta = u|\zeta|$ be the polar decomposition of ζ. Up to unitary equivalence, we may assume that $\mathcal{H}$ and ζ are given by the formulas of Corollary 2.2. Then $\ker(\zeta) = \ker(|\zeta|)$ and $u\restriction_{\ker(|\zeta|)^\perp}$ is a unitary operator. Furthermore, by Proposition 3.1, we have $C_0^*(\zeta,\zeta^*) \cong \mathcal{B} \subset B(\ker(|\zeta|)^\perp)$, where

$$\mathcal{B} := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k(|\zeta|)\, u^k\restriction_{\ker(|\zeta|)^\perp} \ :\ f_k \in C_0(\operatorname{spec}(|\zeta|)),\ \ f_k(0) = 0 \ \text{ for all } \ k \neq 0 \Big\}.$$

On $\mathcal{H} = \ker(|\zeta|) \oplus \ker(|\zeta|)^\perp$, set

$$\mathcal{A} := \|\cdot\|\text{-cls}\Big\{ f_0(|\zeta|)\restriction_{\ker(|\zeta|)} \oplus \sum_{\text{finite}} f_k(|\zeta|)\, u^k\restriction_{\ker(|\zeta|)^\perp} \ :\ f_k \in C_0(\operatorname{spec}(|\zeta|)) \ \text{ and } \ f_k(0) = 0 \ \text{ for all } \ k \neq 0 \Big\}. \quad (15)$$

Recall from Corollary 2.2 that $\mathcal{H} = \ker(|\zeta|) \oplus \oplus_{n\in\mathbb{Z}} \mathcal{H}_n$, where $\mathcal{H}_n \cong \mathcal{H}_0$ is the image of the spectral projection of $|\zeta|$ corresponding to the Borel set $(q^{n+1}, q^n] \cap \operatorname{spec}(|\zeta|)$. Since

$$\begin{aligned} \|f_0(|\zeta|)\restriction_{\ker(|\zeta|)}\| = |f_0(0)| &\leq \sup\{|f_0(x)| \ :\ x \in \operatorname{spec}(|\zeta|)\backslash\{0\}\} \\ &= \sup\{\|f_0(|\zeta|)h_n\| \ :\ h_n \in \mathcal{H}_n,\ \ \|h_n\| = 1,\ \ n \in \mathbb{Z}\} \\ &\leq \Big\| \sum_{\text{finite}} f_k(|\zeta|)\, u^k\restriction_{\ker(|\zeta|)^\perp} \Big\|, \end{aligned}$$

one easily checks that $\Psi : \mathcal{A} \to \mathcal{B}$, given by

$$\Psi\Big(f_0(|\zeta|)\restriction_{\ker(|\zeta|)} \oplus \sum_{\text{finite}} f_k(|\zeta|)\, u^k\restriction_{\ker(|\zeta|)^\perp} \Big) = \sum_{\text{finite}} f_k(|\zeta|)\, u^k\restriction_{\ker(|\zeta|)^\perp}, \quad (16)$$

defines an isometric *-isomorphism. Thus $C_0^*(\zeta,\zeta^*) \cong \mathcal{A}$. The non-degeneracy of $\mathcal{A}$ follows from the fact that for each $m \in \mathbb{N}$, there exists a $\varphi_m \in C_0(\operatorname{spec}(|\zeta|))$ satisfying $\varphi_m(t) = 1$ for all $t \in [0, q^{-m})$ so that $\varphi_m(|\zeta|) \in \mathcal{A}$ acts as the identity on $\ker(|\zeta|) \oplus \oplus_{n=-m}^{\infty} \mathcal{H}_n$.

The theorem will now be proven by applying [12, Theorem 3.3] to $\mathcal{A} \subset B(\mathcal{H})$, so it suffices to show that

(a) ζ is affiliated with $\mathcal{A}$,
(b) ζ separates the representations of $\mathcal{A}$,
(c) $(1+\zeta^*\zeta)^{-1} \in \mathcal{A}$.

Observe that (c) holds trivially since the function f_0, defined by $f_0(t) := (1+t^2)^{-1}$, belongs to $C_0(\mathrm{spec}(|\zeta|))$ and $(1+\zeta^*\zeta)^{-1} = f_0(|\zeta|) = f_0(|\zeta|)\!\upharpoonright_{\ker(|\zeta|)} \oplus f_0(|\zeta|)\!\upharpoonright_{\ker(|\zeta|)^\perp}$.

Next we verify (a). By the definition of the affiliation relation, this means that

$$z_\zeta := \zeta(1+\zeta^*\zeta)^{-1/2} \in M(\mathcal{A}), \tag{17}$$

and that

$$\|\cdot\|\text{-cls}\big\{(1-z_\zeta^* z_\zeta)a \,:\, a \in \mathcal{A}\big\} = \mathcal{A}. \tag{18}$$

By the formulas in Corollary 2.2, $z_\zeta = 0 \oplus \big(u|\zeta|(1+|\zeta|^2)^{-1/2}\big)\!\upharpoonright_{\ker(|\zeta|)^\perp}$. On $\ker(|\zeta|)^\perp$, we get from Proposition 2.1 iv) $u|\zeta|(1+|\zeta|^2)^{-1/2} f_k(|\zeta|)u^k = q|\zeta|(1+q^2|\zeta|^2)^{-1/2} f_k(q|\zeta|)u^{k+1}$. Since the function $\tilde{f}_k$ given by $\tilde{f}_k(t) := \frac{qt}{\sqrt{1+q^2t^2}} f_k(qt)$ belongs to $C_0(\mathrm{spec}(|\zeta|))$ for any $f_k \in C_0(\mathrm{spec}(|\zeta|))$ and satisfies $\tilde{f}_k(0) = 0$, we see that multiplying from the left with z_ζ maps the defining set of $\mathcal{A}$ on the right-hand side of (15) into itself. Taking the closure yields $z_\zeta \in M(\mathcal{A})$.

To show (18), note that $1 - z_\zeta^* z_\zeta = (1+|\zeta|^2)^{-1}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an approximate unit for $C_0(\mathrm{spec}(|\zeta|))$ such that each φ_n has compact support. Set $\phi_n(t) := (1+t^2)\varphi_n(t)$. Then $\phi_n \in C_0(\mathrm{spec}(|\zeta|))$ and $\lim_{n\to\infty}(1 - z_\zeta^* z_\zeta)\phi_n(|\zeta|) f(|\zeta|) = \lim_{n\to\infty} \varphi_n(|\zeta|) f(|\zeta|) = f(|\zeta|)$ for all $f \in C_0(\mathrm{spec}(|\zeta|))$. From this, one easily concludes that the closure on the left-hand side of (18) contains the defining set of $\mathcal{A}$ on the right-hand side of (15) which proves (18).

We turn now to the proof of (b). Since $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, it suffices to consider representations of $\mathcal{B}$. Set $\tilde{\zeta} := \zeta\!\upharpoonright_{\ker(|\zeta|)^\perp}$. Note that $\mathrm{spec}(|\tilde{\zeta}|) = \mathrm{spec}(|\zeta|)$ by the q-invariance of $\mathrm{spec}(|\zeta|) \neq \{0\}$. Therefore $\mathcal{B}$ is generated by $\sum_{\text{finite}} f_k(|\tilde{\zeta}|)u^k$ with $f_k \in C_0(\mathrm{spec}(|\zeta|))$ and $f_k(0) = 0$ for all $k \neq 0$. The same arguments as above show that

$$z_{\tilde{\zeta}} := \tilde{\zeta}(1+\tilde{\zeta}^*\tilde{\zeta})^{-1/2} \in M(\mathcal{B}). \tag{19}$$

Let $\mathcal{K}$ be a Hilbert space and let $\pi : \mathcal{B} \to B(\mathcal{K})$ be a non-degenerate *-representation. As mentioned before, π admits a unique extension $\pi : M(\mathcal{B}) \to B(\mathcal{K})$. By (12) and the isomorphism (16), the π-image of ζ is $\pi(\zeta) = \pi(z_{\tilde{\zeta}})\big(1 - \pi(z_{\tilde{\zeta}})^*\pi(z_{\tilde{\zeta}})\big)^{-1/2}$. On the other hand, plugging $\pi(\zeta)$ into (11) yields $z_{\pi(\zeta)} = \pi(z_{\tilde{\zeta}})$, so $\pi(\zeta)$ is uniquely determined by $\pi(z_{\tilde{\zeta}}) = z_{\pi(\tilde{\zeta})}$ (see the comment below Equation (12)).

Suppose that we are given two representations $\pi_i : \mathcal{B} \to B(\mathcal{K})$, $i = 1, 2$. Then the statement "ζ separates the representations of $\mathcal{B}$" means $\pi_1(\zeta) \neq \pi_2(\zeta)$ whenever $\pi_1 \neq \pi_2$. By the previous discussion, this is equivalent to $\pi_1(z_{\tilde{\zeta}}) = \pi_2(z_{\tilde{\zeta}})$ implies $\pi_1 = \pi_2$.

Our first aim is to show that $|\pi_1(z_{\tilde{\zeta}})| = |\pi_2(z_{\tilde{\zeta}})|$ entails $\pi_1(f(|\tilde{\zeta}|)) = \pi_2(f(|\tilde{\zeta}|))$ for an appropriate class of continuous functions f on $\mathrm{spec}(|\zeta|)$. To begin with, observe that $\pi(z_{\tilde{\zeta}})^*\pi(z_{\tilde{\zeta}}) = \pi(z_{\tilde{\zeta}}^* z_{\tilde{\zeta}}) = \pi(|z_{\tilde{\zeta}}|^2) = \pi(|z_{\tilde{\zeta}}|)^2$ gives $|\pi(z_{\tilde{\zeta}})| = \pi(|z_{\tilde{\zeta}}|)$. Here the fact that $z_{\tilde{\zeta}} \in M(\mathcal{B})$ implies $z_{\tilde{\zeta}}^*$, $|z_{\tilde{\zeta}}| \in M(\mathcal{B})$ is used. Next, we invoke the Gelfand representation to obtain an isometric embedding

$$\iota : C_0(\mathrm{spec}(|\zeta|)) \hookrightarrow \mathcal{B}, \quad \iota(f) := f(|\tilde{\zeta}|). \tag{20}$$

Combining it with $\pi : M(\mathcal{B}) \to B(\mathcal{K})$ yields a representation $\pi : \iota(C_0(\mathrm{spec}(|\zeta|))) \to B(\mathcal{K})$. From (19), we see that $|z_{\tilde\zeta}| = |\tilde\zeta|(1+|\tilde\zeta|^2)^{-1/2}$. The function $z(t) := t(1+t^2)^{-1/2}$ separates the points of the one-point compactification $\mathrm{spec}(|\zeta|) \cup \{\infty\}$. By the Stone–Weiertrass Theorem, the functions 1 and z generate $C(\mathrm{spec}(|\zeta|) \cup \{\infty\})$. Viewing $C_0(\mathrm{spec}(|\zeta|))$ as a subalgebra of $C(\mathrm{spec}(|\zeta|) \cup \{\infty\})$ and extending $\pi \circ \iota$ to the multiplier algebra $M(C_0(\mathrm{spec}(|\zeta|)))$, it follows that $\pi \circ \iota : C_0(\mathrm{spec}(|\zeta|)) \to B(\mathcal{K})$ is uniquely determined by its value on $z \in M(C_0(\mathrm{spec}(|\zeta|)))$, and so is its extension to $M(C_0(\mathrm{spec}(|\zeta|)))$. Finally, it follows from (20), that $\pi(\iota(z)) = \pi(|\tilde\zeta|(1 + |\tilde\zeta|^2)^{-1/2}) = \pi(|z_{\tilde\zeta}|) = |\pi(z_{\tilde\zeta})|$. Therefore, given two representations $\pi_i : \mathcal{B} \to B(\mathcal{K})$, $i = 1, 2$, we have $|\pi_1(z_{\tilde\zeta})| = |\pi_2(z_{\tilde\zeta})|$ if and only if $\pi_1 \circ \iota = \pi_2 \circ \iota$, and the same is true for their extensions to $M(C_0(\mathrm{spec}(|\zeta|)))$.

It still remains to show that $\pi_1(z_{\tilde\zeta}) = \pi_2(z_{\tilde\zeta})$ implies $\pi_1(f(\tilde\zeta)u^k) = \pi_2(f(\tilde\zeta)u^k)$ for all $k \in \mathbb{Z}\backslash\{0\}$ and $f \in C_0([0,\infty))$ with $f(0) = 0$. So consider again a representation π lifted to the multiplier C*-algebra $\pi : M(\mathcal{B}) \to B(\mathcal{K})$, and write $\pi(z_{\tilde\zeta})$ in its polar decomposition $\pi(z_{\tilde\zeta}) = v|\pi(z_{\tilde\zeta})|$. If u belonged to $M(\mathcal{B})$, then it would suffice to show that $\pi(u) = v$, since then $\pi(f(\tilde\zeta)u^k) = \pi(f(\tilde\zeta))v^k$ would be uniquely determined by v and the arguments from the last paragraph. Unfortunately, $u \notin M(\mathcal{B})$ since $f(|\tilde\zeta|) \in \mathcal{B}$ for $f \in C_0([0,\infty))$ with $f(0) \neq 0$, but $f(|\tilde\zeta|)u \notin \mathcal{B}$. For this reason, our proof is more complex, involving the spectral theorem of self-adjoint operators.

Let F denote the unique projection-valued measure on the Borel σ-algebra $\Sigma([0,1])$ such that $|\pi(z_{\tilde\zeta})| = \int z\,\mathrm{d}F(z)$. Using $|\pi(z_{\tilde\zeta})| = \pi(|z_{\tilde\zeta}|)$ from the paragraph below Equation (20), we get $|\pi(z_{\tilde\zeta})| = \pi(|\tilde\zeta(1+\tilde\zeta^*\tilde\zeta)^{-1/2}|) = \pi(|\tilde\zeta|(1+|\tilde\zeta|^2)^{-1/2}) = \pi(z_{|\tilde\zeta|})$. By the definition of the π-image of operators affiliated to $\mathcal{B}$, we have $\pi(|\tilde\zeta|) = \pi(z_{|\tilde\zeta|})(1 - \pi(z_{|\tilde\zeta|})^2)^{-1/2}$. Therefore we can write $\pi(|\tilde\zeta|) = \int z(1 - z^2)^{-1/2}\mathrm{d}F(z)$. The function $z : [0,\infty] \to [0,1]$, $z(\tau) = \frac{\tau}{\sqrt{1+\tau^2}}$ is a homeomorphism with inverse $\tau : [0,1] \to [0,\infty]$, $\tau(z) = \frac{z}{\sqrt{1-z^2}}$. Let E denote projection-valued measure on $\Sigma([0,\infty])$ given by the pull-back of F under z, i.e., $E(M) := F(z(M))$ for all $M \in \Sigma([0,\infty])$. Then

$$\pi(|\tilde\zeta|) = \int \tau\,\mathrm{d}E(\tau) \quad\text{and}\quad \pi(z_{|\tilde\zeta|}) = \int z(\tau)\,\mathrm{d}E(\tau) = \int \frac{\tau}{\sqrt{1+\tau^2}}\,\mathrm{d}E(\tau). \tag{21}$$

Observe also that $E(\{\infty\}) = 0$ since $F(\{1\}) = 0$.

We claim that $\pi(f(|\tilde\zeta|)) = \int f(\tau)\,\mathrm{d}E(\tau)$ for all $f \in C([0,\infty])$. Note that $\pi(f(|\tilde\zeta|))$ is well defined for such an f since then $f(|\tilde\zeta|) \in M(\mathcal{B})$. As explained above, the function $z \in C([0,\infty])$ separates the points of $[0,\infty]$ so that for each $f \in C([0,\infty])$ there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p_n(z) = f$ in the supremum norm. Using uniform convergence, $z_{|\tilde\zeta|} = z(|\tilde\zeta|)$ and the second equation in (21), we get

$$\pi(f(|\tilde\zeta|)) = \lim_{n\to\infty} \pi(p_n(z(|\tilde\zeta|))) = \lim_{n\to\infty} p_n(\pi(z_{|\tilde\zeta|})) = \lim_{n\to\infty} \int p_n(z(\tau))\mathrm{d}E(\tau) = \int f(\tau)\mathrm{d}E(\tau),$$

which proves our claim. Combining it with (21) gives $\pi(f(|\tilde\zeta|)) = f(\pi(|\tilde\zeta|))$.

Now let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_0((0,\infty))$ be a sequence of functions of rapid decay, i.e.,

$$\sup\{\, t^k|\varphi_n(t)| \,:\, t \in (0,\infty)\,\} < \infty \quad \text{for all} \quad k \in \mathbb{Z}, \tag{22}$$

such that $0 \le \varphi_1 \le \varphi_2 \le \ldots \le 1$ and $\lim_{n\to\infty} \varphi_n(t) = 1$ for all $t \in (0,\infty)$. With the obvious extension to continuous function on $[0,\infty]$, it follows that

$$\lim_{n\to\infty}\pi(\varphi_n(|\tilde{\zeta}|)) = \lim_{n\to\infty}\varphi_n(\pi(|\tilde{\zeta}|)) = E((0,\infty))$$

in the operator weak topology (even in the operator strong topology).

Let us recall that $\tilde{\zeta} = u|\tilde{\zeta}|$ defines a q-normal operator on $\ker(|\zeta|)^\perp$ so that, by Proposition 2.1, $uf(|\tilde{\zeta}|)u^* = f(q|\tilde{\zeta}|)$ for every Borel function f on $[0,\infty]$. Furthermore,

$$z_{\tilde{\zeta}} = \tilde{\zeta}(1+\tilde{\zeta}^*\tilde{\zeta})^{-1/2} = u|\tilde{\zeta}|(1+|\tilde{\zeta}|^2)^{-1/2} = u\,z(|\tilde{\zeta}|) = z(q|\tilde{\zeta}|)\,u.$$

As a consequence, for all $f\in C_0([0,\infty))$ and $k\in\mathbb{N}$,

$$\begin{aligned}
\varphi_n(|\tilde{\zeta}|)f(|\tilde{\zeta}|)u^k &= \varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=1}^{k} z(q^j|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\big(u\,z(|\tilde{\zeta}|)\big)^k\\
&= \varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=1}^{k} z(q^j|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\,z_{\tilde{\zeta}}^k\,, \qquad (23)\\
\varphi_n(|\tilde{\zeta}|)f(|\tilde{\zeta}|)u^{-k} &= \varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=0}^{k-1} z(q^{-j}|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\big(z(|\tilde{\zeta}|)\,u^*\big)^k\\
&= \varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=0}^{k-1} z(q^{-j}|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\,z_{\tilde{\zeta}}^{*k}. \qquad (24)
\end{aligned}$$

From (22), it follows that the function $(0,t)\ni t\mapsto \varphi_n(t)(\prod_{j=1}^{k} z(q^jt)^{-1})$ belongs to $C_0((0,\infty))$ and can therefore also be considered as an element of $C_0([0,\infty))$. The same holds for the function $(0,t)\ni t\mapsto \varphi_n(t)(\prod_{j=0}^{k-1} z(q^{-j}t)^{-1})$.

To finish the proof of (b), assume that $\pi_i:\mathcal{B}\to B(\mathcal{K})$, $i=1,2$, are two representations satisfying $\pi_1(z_{\tilde{\zeta}}) = \pi_2(z_{\tilde{\zeta}})$. In particular, we have $|\pi_1(z_{\tilde{\zeta}})| = |\pi_2(z_{\tilde{\zeta}})|$. It has already be shown that then $\pi_1(f(|\tilde{\zeta}|)) = \pi_2(f(|\tilde{\zeta}|))$ for all $f\in C([0,\infty])$. This also implies $\pi_1(|\tilde{\zeta}|) = \pi_2(|\tilde{\zeta}|)$ by the definition of the π-image of $|\tilde{\zeta}|$. As in (21), write $\pi_i(|\tilde{\zeta}|) = \int\tau\,\mathrm{d}E(\tau)$. Using $\pi_i(f(|\tilde{\zeta}|)) = f(\pi_i(|\tilde{\zeta}|))$, it follows that $E(\{0\})\pi_i(f(|\tilde{\zeta}|)) = f(0)E(\{0\})$. As a consequence, $\pi_i(f(|\tilde{\zeta}|)) = E((0,\infty))\pi_i(f(|\tilde{\zeta}|))$ for all $f\in C_0([0,\infty))$ with $f(0)=0$. Given such an f, we compute for all $k\in\mathbb{N}$ by taking operator weak limits

$$\begin{aligned}
\pi_1(f(|\tilde{\zeta}|)u^k) &= E((0,\infty))\,\pi_1(f(|\tilde{\zeta}|)u^k) = \lim_{n\to\infty}\pi_1(\varphi_n(|\tilde{\zeta}|))\,\pi_1(f(|\tilde{\zeta}|)u^k)\\
&= \lim_{n\to\infty}\pi_1\Big(\varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=1}^{k} z(q^j|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\,z_{\tilde{\zeta}}^k\Big)\\
&= \lim_{n\to\infty}\pi_1\Big(\varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=1}^{k} z(q^j|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\Big)\pi_1(z_{\tilde{\zeta}})^k\\
&= \lim_{n\to\infty}\pi_2\Big(\varphi_n(|\tilde{\zeta}|)\Big(\prod_{j=1}^{k} z(q^j|\tilde{\zeta}|)^{-1}\Big)f(|\tilde{\zeta}|)\Big)\pi_2(z_{\tilde{\zeta}})^k\\
&= \pi_2(f(|\tilde{\zeta}|)u^k). \qquad (25)
\end{aligned}$$

Here, we applied (23) in the passage from the first to the second line, and used the property that π defines a representation of $M(\mathcal{B})$ in the next line. The crucial step from the third to the fourth line follows from $\pi_1(z_{\tilde\zeta}) = \pi_2(z_{\tilde\zeta})$ by invoking the assumption and from the previously proven fact that $\pi_1(g(|\tilde\zeta|)) = \pi_2(g(|\tilde\zeta|))$ for all $g \in C([0,\infty])$. The last equality is obtained by repeating all steps performed for π_1 backwards.

The same arguments with (23) replaced by (24) show that (25) holds also for $k \in \mathbb{Z}$, $k < 0$. To sum up, we have shown for all $k \in \mathbb{Z}$ and all $f_k \in C_0([0,\infty))$ satisfying $f_k(0) = 0$ if $k \neq 0$ that $\pi_1(z_{\tilde\zeta}) = \pi_2(z_{\tilde\zeta})$ implies $\pi_1(f_k(|\tilde\zeta|)u^k) = \pi_2(f_k(|\tilde\zeta|)u^k)$. Since these elements generate $\mathcal{B}$, we finally conclude that $\pi_1 = \pi_2$. ■

We remark that if we had considered the C*-algebra generated by $|\zeta|$ and u, where $\zeta = u|\zeta|$, then the generated C*-algebra would be $C_0(X) \rtimes \mathbb{Z}$ by the simple argument given in the second paragraph following Equation (20). A more general construction with $|\zeta|$ having discrete spectrum can be found in [8].

Assume now that Z is a q-normal operator such that $\mathrm{spec}(|Z|) = [0,\infty)$. Then

$$C_0^*(Z,Z^*) := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k U^k \in C_0([0,\infty)) \rtimes \mathbb{Z} \ :\ k \in \mathbb{Z},\ \ f_k(0) = 0 \ \text{ for all }\ k \neq 0 \Big\}$$

can be viewed as an universal object of (the category of) C*-algebras generated by q-normal operators since

$$C_0^*(Z,Z^*) \ni \sum_{\text{finite}} f_k U^k \ \longmapsto\ \sum_{\text{finite}} f_k\!\restriction_X U^k \in C_0^*(\zeta,\zeta^*)$$

yields a well-defined *-homorphism for all C*-algebras $C_0^*(\zeta,\zeta^*)$ from (14), see the proof of Proposition 3.1.

For a geometric interpretation of $C_0^*(Z,Z^*)$, observe that the definitions of the crossed product C*-algebra $C_0([0,\infty)) \rtimes \mathbb{Z}$ and of $C_0^*(Z,Z^*)$ still make sense if we set $q = 1$. In this case, $\alpha_q = \mathrm{id}$ and both algebras become commutative. If $f_k \in C_0([0,\infty))$ and $f_k(0) = 0$ for $k \neq 0$, then $f_k U^k$ can be viewed as a function in $C_0(\mathbb{C})$ by using Euler's formula $Z = |Z|\mathrm{e}^{\mathrm{i}\theta}$ and assigning $|Z|\mathrm{e}^{\mathrm{i}\theta} \mapsto f_k(|Z|)\mathrm{e}^{\mathrm{i}\theta k}$. Note that $f_k(0) = 0$ if $k \neq 0$ is crucial since $|Z| = 0$ corresponds to the unique point $Z = 0$ so that the function must be independent from θ. Furthermore, one easily checks that the algebra of functions $|Z|\mathrm{e}^{\mathrm{i}\theta} \mapsto \sum_{\text{finite}} f_k(|Z|)\mathrm{e}^{\mathrm{i}\theta k}$ separates the points of $\mathbb{C} \cup \{\infty\}$. Here, it is crucial to include functions $f_0 \in C_0([0,\infty))$ satisfying $f(0) \neq 0$ since otherwise the points 0 and ∞ could not be distinguished. By the Stone–Weierstrass theorem, the algebra of functions just defined generates $C_0(\mathbb{C})$. Finally, since $C_0(\mathbb{C})$ is commutative, the universal norm coincides with the supremum norm. Therefore, we obtain $C_0^*(Z,Z^*) = C_0(\mathbb{C})$ for $q = 1$. This motivates the following definition.

Definition 3.3. We say that

$$C_0(\mathbb{C}_q) := \|\cdot\|\text{-cls}\Big\{ \sum_{k=M}^{N} f_k U^k \in C_0([0,\infty)) \rtimes \mathbb{Z} \ :\ f_k(0) = 0 \ \text{ for all }\ k \neq 0 \Big\}$$

is the C*-algebra of continuous functions vanishing at infinity on the quantum complex plane $\mathcal{O}(\mathbb{C}_q)$. Its unitization

$$C(\mathrm{S}_q^2) := C_0(\mathbb{C}_q) \oplus \mathbb{C}$$

is called the quantum sphere generated by $\mathcal{O}(\mathbb{C}_q)$.

4. Final remarks. The calculations in [2] show that $C(\mathrm{S}_q^2)$ has the same K-groups as the classical 2-sphere, that is, $K_0(C(\mathrm{S}_q^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(C(\mathrm{S}_q^2)) \cong 0$. It is expected that the non-trivial part of $K_0(C(\mathrm{S}_q^2))$ can be described by analogues of the classical Bott projections

$$P_n := \begin{pmatrix} 1 \\ \zeta^{*n} \end{pmatrix} \tfrac{1}{1+q^{n(n+1)}|\zeta|^{2n}} \begin{pmatrix} 1 & \zeta^n \end{pmatrix} = \begin{pmatrix} \frac{1}{1+q^{n(n+1)}|\zeta|^{2n}} & \frac{1}{1+q^{n(n+1)}|\zeta|^{2n}}\,\zeta^n \\ \frac{1}{1+q^{-n(n-1)}|\zeta|^{2n}}\,\zeta^{*n} & \frac{q^{-n(n-1)}|\zeta|^{2n}}{1+q^{-n(n-1)}\,|\zeta|^{2n}} \end{pmatrix},$$

$$P_{-n} := \begin{pmatrix} 1 \\ \zeta^{n} \end{pmatrix} \tfrac{1}{1+q^{-n(n-1)}|\zeta|^{2n}} \begin{pmatrix} 1 & \zeta^{*n} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+q^{-n(n-1)}|\zeta|^{2n}} & \frac{1}{1+q^{-n(n-1)}|\zeta|^{2n}}\,\zeta^{*n} \\ \frac{1}{1+q^{n(n+1)}|\zeta|^{2n}}\,\zeta^{n} & \frac{q^{n(n+1)}|\zeta|^{2n}}{1+q^{n(n+1)}|\zeta|^{2n}} \end{pmatrix},$$

and that the pairing with the generators of the K-homology group $K^0(C(\mathrm{S}_q^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ computes the rank (equal to 1) and the "winding number" $\pm n$ of projective module given by $P_{\pm n}$, $n \in \mathbb{N}$.

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