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INDEX PAIRINGS FOR PULLBACKS OF C*-ALGEBRAS

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Dedicated to S. L. Woronowicz on the occasion of his 70th birthday

Abstract. In this overview, we study how to reduce the index pairing for a fibre-product C*-algebra to the index pairing for the C*-algebra over which the fibre product is taken. As an example we analyze the case of suspensions and apply it to noncommutative instanton bundles of arbitrary charges over the suspension of quantum deformations of the 3-sphere.

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1. Introduction. The aim of this paper is to review and put in a general context a number of results concerning the index computations for different types of noncommutative instanton vector bundles. By the Serre-Swan theorem, the category of finite-dimensional vector bundles over compact Hausdorff spaces is equivalent to the category of finitely generated projective modules over algebras of all continuous functions on the corresponding compact Hausdorff spaces. The latter category makes perfect sense for arbitrary algebras, so that finitely generated projective modules over noncommutative algebras are considered as noncommutative vector bundles.

In this paper, we prove some general theorems concerning finitely generated projective modules over one-surjective pullbacks of C*-algebras. Then we specialize to non-reduced suspensions of C*-algebras viewed as the pullbacks of cones of C*-algebras. Choosing them to be noncommutative deformations of the algebra of all continuous functions on the 3-sphere, we can construct idempotents of noncommutative instanton vector bundles for an arbitrary "winding number" (cf. [1]) and determine their index pairing with an appropriate K-homology class ("instanton charges").

Our key idea is to take advantage of the compatibility of the index pairing with the connecting homomorphisms of the Mayer-Vietoris six-term exact sequences for K-theory and K-homology, which is a manifestation of the associativity of the Kasparov product. This allows us to reduce the computation of an index pairing from non-reduced suspensions of C*-algebras to the C*-algebras themselves. Geometrically speaking, we determine instanton charges by shifting calculations from a quantum 4-sphere to its equator quantum 3-sphere, which considerably simplifies the task.

Herein we focus on idempotents of noncommutative instanton vector bundles coming from a particular complex-parameter deformation of the algebra $C(S^3)$ of all continuous functions on the unital 3-dimensional sphere (cf. [6, 22]). The real and unitary cases of this deformation correspond respectively to Woronowicz's C*-algebra $C(SU_q(2))$ [35] of the quantum group $SU_q(2)$ and Matsumoto's C*-algebra $C(S^3_\theta)$ [25] of the quantum sphere S^3_θ built from noncommutative tori [32].

2. The Mayer-Vietoris six-term exact sequence. Let $B_1 \xrightarrow{\pi_1} A \xleftarrow{\pi_2} B_2$ be homomorphisms of C*-algebras. We denote by B the C*-algebra fiber product (pullback C*-algebra) [28]

$$B_1 \underset{A}{\times} B_2 := \{(b_1, b_2) \in B_1 \times B_2 \mid \pi_1(b_1) = \pi_2(b_2)\}. \tag{2.1}$$

With the obvious projection maps $\mathrm{pr}_1 : B \to B_1$ and $\mathrm{pr}_2 : B \to B_2$, there is the commutative diagram

$$\begin{tikzcd} & B \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ B_1 \arrow[dr, "\pi_1"'] & & B_2 \arrow[dl, "\pi_2"] , \\ & A & \end{tikzcd} \tag{2.2}$$

and B enjoys a universal property with respect to all such commutative diagrams.

To use Mayer-Vietoris arguments in K-theory, we need to assume that one of the defining homomorphisms is surjective. In this paper we choose π_2 to be surjective. To go beyond K-theory and use results of [4, Section 1.2.3], we also assume that π_2 is semi-split. This is always the case for surjections onto nuclear C*-algebras, which covers all our examples. First, there exist the following Mayer-Vietoris six-term exact sequences both in K-theory and K-homology:

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{(\mathrm{pr}_{1*},\mathrm{pr}_{2*})} & K_0(B_1)\oplus K_0(B_2) & \xrightarrow{\pi_{2*}-\pi_{1*}} & K_0(A) \\ {\scriptstyle\partial_{10}}\uparrow & & & & \downarrow{\scriptstyle\partial_{01}} \\ K_1(A) & \xleftarrow{\pi_{2*}-\pi_{1*}} & K_1(B_1)\oplus K_1(B_2) & \xleftarrow{(\mathrm{pr}_{1*},\mathrm{pr}_{2*})} & K_1(B), \end{array} \tag{2.3}$$

$$\begin{array}{ccccc} K^0(B) & \xleftarrow{\mathrm{pr}_1^*+\mathrm{pr}_2^*} & K^0(B_1)\oplus K^0(B_2) & \xleftarrow{(-\pi_1^*,\pi_2^*)} & K^0(A) \\ {\scriptstyle\delta_{01}}\downarrow & & & & \uparrow{\scriptstyle\delta_{10}} \\ K^1(A) & \xrightarrow{(-\pi_1^*,\pi_2^*)} & K^1(B_1)\oplus K^1(B_2) & \xrightarrow{\mathrm{pr}_1^*+\mathrm{pr}_2^*} & K^1(B). \end{array} \tag{2.4}$$

Moreover, our paper applies the fundamental result that the Mayer-Vietoris connecting homomorphisms are compatible with the index pairing [4, Section 1.2.3]. These connecting homomorphisms are determined uniquely up to a sign, so that their compatibility with the index pairing is also determined up to a sign. Herein we use an explicit construction of ∂_{10} and choose δ_{01} to be such that for any $\mathbf{x}\in K_1(A)$ and $\mathbf{w}\in K^0(B)$

$$\boxed{<\partial_{10}(\mathbf{x}),\mathbf{w}> = <\mathbf{x},\delta_{01}(\mathbf{w})>}\,. \tag{2.5}$$

2.1. Odd-to-even connecting homomorphism. Given a ring homomorphism $R\overset{f}{\to}S$ and a left R-module E, define f_*E to be the left S-module $S\otimes_R E$. Then there is a canonical R-linear map $f_*:E\to f_*E$ given by $f_*(e):=1\otimes_R e$. It is clear that, if E is free over R with basis $\{\hat{x}_\alpha\}_\alpha$, then f_*E is free over S with basis $\{x_\alpha:=f_*(\hat{x}_\alpha)=1\otimes_R\hat{x}_\alpha\}_\alpha$. Next, following [27], we take modules E_1 and E_2 over the rings B_1 and B_2 respectively, and observe that the Abelian group $E_1\times E_2$ is a left module over the pullback ring (2.1) via $b\cdot(\xi_1,\xi_2):=(\mathrm{pr}_1(b)\xi_1,\mathrm{pr}_2(b)\xi_2)$. Given an isomorphism $h:\pi_{1*}E_1\cong\pi_{2*}E_2$, we can now define

$$E:=M(E_1,E_2,h):=\{(\xi_1,\xi_2)\in E_1\times E_2 \mid (h\circ\pi_{1*})(\xi_1)=\pi_{2*}(\xi_2)\}. \tag{2.6}$$

Milnor's theorem [27, Theorem 2.1] asserts that, if E_1 and E_2 are finitely generated projective over B_1 and B_2 respectively, then E is finitely generated projective over B. In particular, choosing E_1 and E_2 to be the free modules B_1^N and B_2^N respectively, $N\in\mathbb{N}\setminus\{0\}$, this beautiful construction allows Milnor to define a connecting homomorphism [27, p. 28] for the Mayer-Vietoris long exact sequence in algebraic K-theory. Herein we prove the aforementioned special case of [27, Theorem 2.1] supplementing it with an explicit formula for an idempotent matrix representing the finitely generated projective module E. Our explicit formula agrees with the formula provided by Ranicki in his unpublished notes [30].[1]

[1]We are grateful to Ulrich Krähmer for making us aware of this reference.

THEOREM 2.1. *Let B be the pullback ring* (2.1) *and E be the left B-module* (2.6). *Assume that $E_i \cong B_i^N$, $i \in \{1,2\}$, $N \in \mathbb{N} \setminus \{0\}$, as left modules, and take $a := (a_{\alpha\beta}) \in GL_N(A)$ to be the matrix implementing an isomorphism $h\colon \pi_{1*}(E_1) \to \pi_{2*}(E_2)$. Then, if π_2 is surjective, $E \cong B^{2N}p$ as a left module, where*

$$p := \begin{bmatrix} (1, c(2-dc)d) & (0, c(2-dc)(1-dc)) \\ (0, (1-dc)d) & (0, (1-dc)^2) \end{bmatrix} \tag{2.7}$$

is an idempotent matrix in $M_{2N}(B)$ such that all entries of the sub-matrices $c, d \in M_N(B_2)$ satisfy $\pi_2(c_{\alpha\beta}) = a_{\alpha\beta}$ and $\pi_2(d_{\alpha\beta}) = (a^{-1})_{\alpha\beta}$ for all α and β.

Proof. Let $\{\hat{x}_\alpha\}_{1\leq\alpha\leq N}$ and $\{\hat{y}_\beta\}_{1\leq\beta\leq N}$ be bases of $E_1 \cong B_1^N$ and $E_2 \cong B_2^N$ respectively. Similarly, let $\{x_\alpha := \pi_{1*}(\hat{x}_\alpha)\}_\alpha$ and $\{y_\beta := \pi_{2*}(\hat{y}_\beta)\}_\beta$ be the corresponding bases for $\pi_{1*}E_1$ and $\pi_{2*}E_2$ respectively. Both of these modules are isomorphic to A^N. Therefore, the isomorphism $h : \pi_{1*}E_1 \to \pi_{2*}E_2$ is given by $h(x_\alpha) =: \sum_\beta a_{\alpha\beta} y_\beta$ for some unique $(a_{\alpha\beta}) \in GL_N(A)$. Now, the matrix c might be invertible or non-invertible. If it is, we quickly conclude that $E \cong B^N$. Without this assumption we need a doubling construction to prove (2.7).

Special case. Assume that c is invertible. For each α, define $\hat{y}'_\alpha := \sum_\beta c_{\alpha\beta}\hat{y}_\beta$. Since c is invertible, $\{\hat{y}'_\alpha\}_\alpha$ is a basis of B_2^N. Also, as $(h \circ \pi_{1*})(\hat{x}_\alpha) = h(x_\alpha) = \sum_\beta a_{\alpha\beta} y_\beta = \pi_{2*}(\hat{y}'_\alpha)$, it follows that $\hat{z}_\alpha := (\hat{x}_\alpha, \hat{y}'_\alpha) \in E := M(E_1, E_2, h)$ for all α. Finally, it is straightforward to check that $\{\hat{z}_\alpha\}_{1\leq\alpha\leq N}$ span E and are linearly independent, so that $E \cong B^N$ via the basis $\{\hat{z}_\alpha\}_\alpha$.

General case. Let us use the matrix a^{-1} to define the isomorphism

$$g\colon \pi_{1*}E_1 \to \pi_{2*}E_2, \quad g(x_\alpha) := \sum_\beta (a^{-1})_{\alpha\beta} y_\beta, \quad \alpha \in \{1, \cdots, N\}. \tag{2.8}$$

Then $h \oplus g$ defines an isomorphism $\pi_{1*}(E_1 \oplus E_1) \to \pi_{2*}(E_2 \oplus E_2)$. Next, as in Whitehead's lemma, we write

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2N}(A), \tag{2.9}$$

where 1 means the $N \times N$-identity matrix I_N. This lifts to

$$\begin{aligned} \mathcal{C} &:= \begin{bmatrix} \mathcal{C}^{11} & \mathcal{C}^{12} \\ \mathcal{C}^{21} & \mathcal{C}^{22} \end{bmatrix} \\ &:= \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (2-cd)c & cd-1 \\ 1-dc & d \end{bmatrix} \in M_{2N}(B_2), \end{aligned} \tag{2.10}$$

whose inverse is

$$\mathcal{D} := \begin{bmatrix} \mathcal{D}^{11} & \mathcal{D}^{12} \\ \mathcal{D}^{21} & \mathcal{D}^{22} \end{bmatrix} := \begin{bmatrix} d & 1-dc \\ cd-1 & (2-cd)c \end{bmatrix} \in M_{2N}(B_2). \tag{2.11}$$

This allows us to proceed as in the special case. First, however, we need to introduce more notation. We put

$$\hat{x}^L_\alpha := (\hat{x}_\alpha, 0), \quad \hat{x}^R_\alpha := (0, \hat{x}_\alpha), \quad \alpha \in \{1, \cdots, N\}, \tag{2.12}$$

to obtain $\{\hat{x}^L_\alpha, \hat{x}^R_\beta\}_{1\le\alpha,\beta\le N}$ as a basis of $E_1 \oplus E_1$, and

$$\hat{y}^L_\alpha := (\hat{y}_\alpha, 0), \quad \hat{y}^R_\alpha := (0, \hat{y}_\alpha), \quad \alpha \in \{1, \cdots, N\}, \tag{2.13}$$

to obtain $\{\hat{y}^L_\alpha, \hat{y}^R_\beta\}_{1\le\alpha,\beta\le N}$ as a basis of $E_2 \oplus E_2$. Furthermore, we put $(\mathcal{C}^{ij}_{\alpha\beta}) := \mathcal{C}^{ij} \in M_N(B_2)$ for all $i,j \in \{1,2\}$. Finally, we define

$$\hat{u}_\alpha := \sum_\mu \Big(\mathcal{C}^{11}_{\alpha\mu}\, \hat{y}^L_\mu + \mathcal{C}^{12}_{\alpha\mu}\, \hat{y}^R_\mu\Big)\,, \quad \hat{v}_\beta := \sum_\mu \Big(\mathcal{C}^{21}_{\beta\mu}\, \hat{y}^L_\mu + \mathcal{C}^{22}_{\beta\mu}\, \hat{y}^R_\mu\Big)\,. \tag{2.14}$$

Now we can argue as in the special case to conclude that $M(E_1 \oplus E_1, E_2 \oplus E_2, h \oplus g) \cong B^{2N}$ through the basis $\{(\hat{x}^L_\alpha, \hat{u}_\alpha), (\hat{x}^R_\beta, \hat{v}_\beta)\}_{1\le\alpha,\beta\le N}$. On the other hand, we have the isomorphism

$$\begin{aligned} M(E_1, E_2, h) \oplus M(E_1, E_2, g) &\longrightarrow M(E_1 \oplus E_1, E_2 \oplus E_2, h \oplus g),\\ ((e_1, e_2), (f_1, f_2)) &\longmapsto ((e_1, f_1), (e_2, f_2)). \end{aligned} \tag{2.15}$$

Consequently, $E := M(E_1, E_2, h)$ is a direct summand of a finitely generated free module, so that it is a finitely generated projective module, as claimed.

We proceed now to computing the idempotent matrix $p \in M_{2N}(B)$. Composing the projection and inclusion

$$M(E_1 \oplus E_1, E_2 \oplus E_2, h \oplus g) \xrightarrow{\pi} M(E_1, E_2, h) \overset{i}{\hookrightarrow} M(E_1 \oplus E_1, E_2 \oplus E_2, h \oplus g) \tag{2.16}$$

yields an idempotent: $(i \circ \pi)^2 = i \circ \pi$. To determine the matrix p of $i \circ \pi$ with respect to the basis $\{(\hat{x}^L_\alpha, \hat{u}_\alpha), (\hat{x}^R_\beta, \hat{v}_\beta)\}_{1\le\alpha,\beta\le N}$, we solve (2.14) for $\hat{y}^L_\mu$ with the help of (2.11)

$$\hat{y}^L_\mu = \sum_\nu \Big(\mathcal{D}^{11}_{\mu\nu}\hat{u}_\nu + \mathcal{D}^{12}_{\mu\nu}\hat{v}_\nu\Big), \tag{2.17}$$

and compute

$$\begin{aligned} (i\circ\pi)(\hat{x}^L_\alpha, \hat{u}_\alpha) &= (i\circ\pi)\Big(\hat{x}^L_\alpha\,,\ \sum_\mu \big(\mathcal{C}^{11}_{\alpha\mu}\, \hat{y}^L_\mu + \mathcal{C}^{12}_{\alpha\mu}\, \hat{y}^R_\mu\big)\Big) \\ &= \Big(\hat{x}^L_\alpha\,,\ \sum_\mu \mathcal{C}^{11}_{\alpha\mu}\, \hat{y}^L_\mu\Big) \\ &= \Big(\hat{x}^L_\alpha\,,\ \sum_{\mu,\nu} \mathcal{C}^{11}_{\alpha\mu}\big(\mathcal{D}^{11}_{\mu\nu}\hat{u}_\nu + \mathcal{D}^{12}_{\mu\nu}\hat{v}_\nu\big)\Big) \\ &= \sum_\nu \Big((I_N)_{\alpha\nu}\,,\ (\mathcal{C}^{11}\mathcal{D}^{11})_{\alpha\nu}\Big)(\hat{x}^L_\nu, \hat{u}_\nu) + \sum_\nu \Big(0\,,\ (\mathcal{C}^{11}\mathcal{D}^{12})_{\alpha\nu}\Big)(\hat{x}^R_\nu, \hat{v}_\nu), \end{aligned} \tag{2.18}$$

$$\begin{aligned} (i\circ\pi)(\hat{x}^R_\beta, \hat{v}_\beta) &= (i\circ\pi)\Big(\hat{x}^R_\beta\,,\ \sum_\mu \big(\mathcal{C}^{21}_{\beta\mu}\, \hat{y}^L_\mu + \mathcal{C}^{22}_{\beta\mu}\, \hat{y}^R_\mu\big)\Big) \\ &= \Big(0\,,\ \sum_\mu \mathcal{C}^{21}_{\beta\mu}\, \hat{y}^L_\mu\Big) \\ &= \Big(0\,,\ \sum_{\mu,\nu} \mathcal{C}^{21}_{\beta\mu}\big(\mathcal{D}^{11}_{\mu\nu}\hat{u}_\nu + \mathcal{D}^{12}_{\mu\nu}\hat{v}_\nu\big)\Big) \\ &= \sum_\nu \Big(0\,,\ (\mathcal{C}^{21}\mathcal{D}^{11})_{\beta\nu}\Big)(\hat{x}^L_\nu, \hat{u}_\nu) + \sum_\nu \Big(0\,,\ (\mathcal{C}^{21}\mathcal{D}^{12})_{\beta\nu}\Big)(\hat{x}^R_\nu, \hat{v}_\nu). \end{aligned} \tag{2.19}$$

Combining this with (2.10) and (2.11) yields

$$p=\begin{bmatrix} (1,\mathcal{C}^{11}\mathcal{D}^{11}) & (0,\mathcal{C}^{11}\mathcal{D}^{12}) \\ (0,\mathcal{C}^{21}\mathcal{D}^{11}) & (0,\mathcal{C}^{21}\mathcal{D}^{12}) \end{bmatrix} = \begin{bmatrix} (1,c(2-dc)d) & (0,c(2-dc)(1-dc)) \\ (0,(1-dc)d) & (0,(1-dc)^2) \end{bmatrix}, \tag{2.20}$$

as desired. ■

Furthermore, it is useful to observe that the idempotent p given by (2.7) can be written as

$$\boxed{p=XY,\quad X:=\begin{bmatrix} (1,c(2-dc)) \\ (0,1-dc) \end{bmatrix},\quad Y:=[(1,d),(0,1-dc)]}. \tag{2.21}$$

Since $YX=(1,1)\in B$, we immediately see that $p^2=p$. Note that the components of X and Y are in $B_1\times B_2$ but not necessarily in B, which allows the non-freeness of the projective module given by p.

Finally, let us use the above theorem to obtain an explicit formula for a connecting homomorphism. Herein we adapt Milnor's construction to unital C*-algebras as proposed by Nigel Higson in [18, Section 0.4].

THEOREM 2.2. *Let B be the pullback C*-algebra (2.1) of unital C*-homomorphisms $B_1 \xrightarrow{\pi_1} A \xleftarrow{\pi_2} B_2$. Assume that π_2 is surjective. Also, let $p\in M_{2N}(B)$ be an idempotent assigned to $a\in GL_N(A)$ by (2.7), and let $I_N\in M_N(B)$ denote the $N\times N$-identity matrix. Then the formula*

$$\partial_{10}\colon K_1(A)\ni[a]\longmapsto[p]-[I_N]\in K_0(B) \tag{2.22}$$

defines a connecting homomorphism of the Mayer-Vietoris six-term exact sequence (2.3).

2.2. Even-to-odd connecting homomorphism. Using the Bott periodicity to factorize ∂_{01} into

$$K_0(A)\xrightarrow{\cong}K_1(C_0(\mathbb{R})\otimes A)\longrightarrow K_1(B), \tag{2.23}$$

one can directly and explicitly compute the even-to-odd connecting homomorphism of the Mayer-Vietoris six-term exact sequence [3]. Alternatively, one can simply combine [4, Theorem 1.18] with [5, Section 9.3.2] to obtain:

THEOREM 2.3. *Let $p\in M_N(A)$ be a projection, and Q be a self-adjoint lifting of p to B_2, i.e. $\pi_2(Q)=p$ and $Q^*=Q$. Also, let $I_N\in M_N(B)$ denote the $N\times N$-identity matrix. Then the formula*

$$\boxed{\partial_{01}\colon K_0(A)\ni([p])\longmapsto[(I_N,\mathrm{e}^{2\pi iQ})]\in K_1(B)} \tag{2.24}$$

defines a connecting homomorphism of the Mayer-Vietoris six-term exact sequence (2.3).

2.3. Suspension case. In this section, we assume that B is the fibre product of two cones of A over A:

$$B_1:=\{f\in C([0,\tfrac12],A)\mid f(0)\in\mathbb{C}\},\qquad B_2:=\{f\in C([\tfrac12,1],A)\mid f(1)\in\mathbb{C}\}, \tag{2.25}$$

with $\pi_1:B_1\to A$ and $\pi_2:B_2\to A$ both given by $f\mapsto f(\frac12)$. Clearly, B can be identified with the non-reduced suspension of A (as in algebraic topology) via the isomorphism:

$$\Sigma A:=\{f\in C([0,1],A)\mid f(0),f(1)\in\mathbb{C}\}\ni f\longmapsto(f|_{[0,\frac12]},f|_{[\frac12,1]})\in B. \tag{2.26}$$

In this particular setting, the idempotent (2.7) can be given more explicitly. Given $a \in GL_N(A)$, we lift it to $c := \psi a \in M_N(B_2)$, where $\psi \in C([\frac{1}{2}, 1])$ satisfies $\psi(\frac{1}{2}) = 1$ and $\psi(1) = 0$. The inverse a^{-1} we lift to $d := \psi a^{-1} \in M_N(B_2)$. (In what follows we shall further restrict the choice of ψ to fit some known examples.) Plugging in c and d to (2.21) yields

$$\begin{aligned} X &= \begin{bmatrix} (1, \psi(2-\psi^2)a) \\ (0, (1-\psi^2)) \end{bmatrix} \in M_{2N\times N}(B_1 \times B_2), \\ Y &= \left[(1, \psi a^{-1}), (0, (1-\psi^2))\right] \in M_{N\times 2N}(B_1 \times B_2). \end{aligned} \tag{2.27}$$

Next, since cones are contractible,

$$K_j(B_i) = \begin{cases} \mathbb{Z} & \text{for } j = 0, \\ 0 & \text{for } j = 1, \end{cases} \qquad i = 1, 2\,. \tag{2.28}$$

Consequently, the six-term exact sequence (2.3) reduces to the exact sequence

$$0 \to K_1(A) \xrightarrow{\partial_{10}} K_0(B) \xrightarrow{(\mathrm{pr}_{1*}, \mathrm{pr}_{2*})} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_{2*} - \pi_{1*}} K_0(A) \xrightarrow{\partial_{01}} K_1(B) \to 0\,. \tag{2.29}$$

Assume furthermore that $K_0(A) \cong \mathbb{Z}$ via $[1] \mapsto 1$, and that $K_1(A) \cong \mathbb{Z}$ via $[\mathsf{u}] \mapsto 1$, for some $\mathsf{u} \in U_N(A)$. It follows that the map $\pi_{2*} - \pi_{1*} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is surjective, so that $K_1(B) = 0$. On the other hand, as the kernel of $\pi_{2*} - \pi_{1*} \cong \mathbb{Z}$, by standard homological algebra we conclude that $K_0(B) \cong \mathbb{Z}^2$ with one generator given by $1 \in B$ and the other given by $\partial_{10}([\mathsf{u}])$ defined in (2.22).

Now, by Rosenberg and Schochet's Universal Coefficient Theorem [33], if the K-groups K_j are free, then $K^j \cong K_j$ (as Abelian groups). Thus we obtain

$$K^j(A) = \begin{cases} \mathbb{Z} & \text{for } j = 0, \\ \mathbb{Z} & \text{for } j = 1, \end{cases} \quad K^j(B_i) = \begin{cases} \mathbb{Z} & \text{for } j = 0, \\ 0 & \text{for } j = 1, \end{cases} \quad K^j(B) = \begin{cases} \mathbb{Z}^2 & \text{for } j = 0, \\ 0 & \text{for } j = 1. \end{cases} \tag{2.30}$$

Combining it with the K-homology six-term exact sequence (2.4), we infer that the connecting homomorphism $\delta_{01} : K^0(B) \to K^1(A)$ is surjective. Hence, for any $\mathbf{z} \in K^1(A)$ we can choose $\mathbf{w} \in K^0(B)$ such that $\delta_{01}(\mathbf{w}) = \mathbf{z}$, and use the compatibility (2.5) of the index pairing with the Mayer-Vietoris six-term exact sequences to compute

$$< \partial_{10}([\mathsf{u}^n]), \mathbf{w} > = < [\mathsf{u}^n], \mathbf{z} > = n < [\mathsf{u}], \mathbf{z} >, \quad n \in \mathbb{Z}\,. \tag{2.31}$$

Remembering that the index pairing with a free module is always zero, we may now conclude that computing a non-trivial index pairing for an idempotent p_n corresponding to the unitary u^n, $n \in \mathbb{Z}$, via (2.7) boils down to finding an odd Fredholm module whose pairing with u equals one:

$$< [p_n], \mathbf{w} > = n < [\mathsf{u}], \mathbf{z} > . \tag{2.32}$$

Finally, let us remark that for the usual suspension of A, given by

$$SA := \{\, f \in C([0,1], A) \mid f(0) = f(1) = 0\}, \tag{2.33}$$

there is also a method to construct an idempotent $\widetilde{p} \in M_{2N}(SA)$ in terms of a unitary $\mathsf{u} \in U_N(A)$ [34, p. 139]. This method applied to our non-reduced suspension ΣA of A

yields $\widetilde{p} := TT^*$, where

$$T := \begin{bmatrix} \cos^2(\frac{\pi t}{2}) + \sin^2(\frac{\pi t}{2})\,\mathsf{u} \\ \cos(\frac{\pi t}{2})\sin(\frac{\pi t}{2})(\mathsf{u}^{-1} - 1) \end{bmatrix}, \quad \text{and } T^* \text{ is its Hermitian conjugate.} \tag{2.34}$$

3. Noncommutative instanton vector bundles of an arbitrary charge. In this section, we focus on the case when A is the C*-algebra of a quantum or classical S^3, so that the foregoing assumptions about the K-groups of A are satisfied. To begin with, we take $A := C(S^3)$. Then $B \cong \Sigma C(S^3) \cong C(S^4)$. We know that $K_1(C(S^3))$ is generated by the fundamental representation of $SU(2)$ identified with S^3. Explicitly, $u \in U_2(C(S^3))$ can be described as the continuous map

$$S^3 \ni x = (x_0, x_1, x_2, x_3) \stackrel{u}{\longmapsto} \begin{bmatrix} a, & -\bar{b} \\ b, & \bar{a} \end{bmatrix} \in SU_2(\mathbb{C}), \qquad \text{where} \quad a := x_0 + ix_3, \quad b := x_1 + ix_2 . \tag{3.1}$$

Likewise, we can introduce the coordinate functions

$$S^3 \ni x \stackrel{\alpha}{\longmapsto} a \in \mathbb{C}, \qquad S^3 \ni x \stackrel{\beta}{\longmapsto} b \in \mathbb{C}, \qquad u := \begin{bmatrix} \alpha, & -\beta^* \\ \beta, & \alpha^* \end{bmatrix}. \tag{3.2}$$

In what follows, we will also view u as a function on the space of unit quaternions.

As before, we can take p_n to be an idempotent corresponding via (2.7) to the unitary u^n, $n \in \mathbb{Z}$. Then, by the Serre-Swan Theorem, the finitely generated projective module $(C(S^4))^4\, p_n$ can be identified with the module of all continuous sections of the uniquely determined (up to an isomorphism) rank 2 complex vector bundle $E_n \to S^4$. These vector bundles are commonly referred to as *instanton* vector bundles. One can prove that the "winding number" $n \in \mathbb{Z}$ defining the vector bundles E_n coincides with a K_0-invariant, which we call the charge of an instanton.

Using homotopy and coordinate changes, we obtain many different but K_0-equivalent explicit descriptions of the idempotents p_n, $n \in \mathbb{Z}$. First, recall that identifying $\mathbb{R}^4$ with the space of quaternions $\mathbb{H}$ we can define the unit 4-sphere $S^4 \subseteq \mathbb{R}^5$ as

$$S^4 := \{(\tau, h) \in \mathbb{R} \times \mathbb{H} \mid \tau^2 + \|h\|^2 = 1\} . \tag{3.3}$$

Another way is to present S^4 via the non-reduced suspension of S^3. Since the latter can be identified with the space of quaternions of norm 1, we obtain

$$S^4 \cong \big\{(t, g) \in [0,1] \times \mathbb{H} \;\big|\; \|g\| = 1\big\} / \!\sim, \quad (0,g) \sim (0,g'), \quad (1,g) \sim (1,g'),\ \forall\, g, g' \in \mathbb{H} . \tag{3.4}$$

Finally, by removing the north or the south pole from S^4, we can cover it by two quaternionic charts via stereographic projections. Denoting one of these stereographic coordinates by $z \in \mathbb{H}$, we obtain the following relations between the above defined coordinate systems:

$$g = \frac{h}{\sqrt{1-\tau^2}}, \quad t = \frac{1+\tau}{2}, \qquad z = \sqrt{\frac{t}{1-t}}\, g, \qquad z = \frac{h}{1-\tau} . \tag{3.5}$$

It turns out that the instanton vector bundles defined above with the help of idempotents p_n, $n \in \mathbb{Z}$, can also be constructed by viewing S^4 as the quaternionic projective space $\mathbb{H}P^1$ and considering the tautological quaternionic line bundle. The bundle can be described by a projection which, in the foregoing three coordinate systems, at a generic point of S^4, can be respectively written as the following matrix in $M_2(\mathbb{H})$:

$$\frac{1}{2}\begin{bmatrix} 1-\tau & \bar{h} \\ h & 1+\tau \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1-\tau} \\ \frac{h}{\sqrt{1-\tau}} \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1-\tau} \\ \frac{h}{\sqrt{1-\tau}} \end{bmatrix}\right)^*, \tag{3.6}$$

$$\begin{bmatrix} 1-t & \sqrt{t(1-t)}\bar{g} \\ \sqrt{t(1-t)}g & t \end{bmatrix} = \begin{bmatrix} \sqrt{(1-t)} \\ \sqrt{t}g \end{bmatrix}\left(\begin{bmatrix} \sqrt{(1-t)} \\ \sqrt{t}g \end{bmatrix}\right)^*, \tag{3.7}$$

$$\frac{1}{1+z\bar{z}}\begin{bmatrix} 1 & \bar{z} \\ z & z\bar{z} \end{bmatrix} = \frac{1}{\sqrt{1+z\bar{z}}}\begin{bmatrix} 1 \\ z \end{bmatrix}\left(\frac{1}{\sqrt{1+z\bar{z}}}\begin{bmatrix} 1 \\ z \end{bmatrix}\right)^*, \tag{3.8}$$

where $^-$ denotes the quaternionic conjugation and * stands for the quaternionic-Hermitian adjoint. In what follows, we shall use the middle matrix (3.7). The aforementioned projection is given by varying this matrix over S^4 viewed as the non-reduced suspension of S^3 (see (3.4)):

$$S^4 \ni [(t,g)] \xmapsto{P_n} \begin{bmatrix} 1-t & \sqrt{t(1-t)}\bar{g}^n \\ \sqrt{t(1-t)}g^n & t \end{bmatrix} \in M_4(C(S^4)), \quad \forall\, n \in \mathbb{Z}. \tag{3.9}$$

THEOREM 3.1. *Let P_n, $n \in \mathbb{Z}$, be the projection* (3.9), *and let p_n, $n \in \mathbb{Z}$, be an idempotent corresponding by the formula* (2.7) *to v^n for some $v \in M_N(C(S^3))$. Then,*

$$K_1(C(S^3)) \ni [u] = [v^{-1}] \quad \Longrightarrow \quad \forall\, n \in \mathbb{Z}:\ [P_n] = [p_n] \in K_0(C(S^4)).$$

Proof. We prove the theorem by defining a homotopy between P_n and p_n. The latter is given by (2.27) with a replaced by v^n. Since $K_1(C(S^3)) \ni [u] = [v^{-1}]$ by assumption, the formula (2.27) with a substituted by u^{-n} yields an idempotent belonging to the class $[p_n] \in K_0(C(S^4))$. Next, we specify $\psi \in C([\frac{1}{2},1])$ appearing in (2.27) to be a $[0,1]$-valued function such that $\psi(\frac{1}{2}) = 1$ and $\psi(1) = 0$. Then we construct the following families of functions

$$\psi_s(t) := \psi\big(\tfrac{t+1-s}{2-s}\big), \quad t \in [\tfrac{s}{2},1], \quad s \in [0,1], \tag{3.10}$$

$$X_0^s(t) := \begin{cases} 1 & \text{for } 0 \le t < \frac{s}{2}, \\ 0 & \text{for } \frac{s}{2} \le t \le 1, \end{cases} \qquad X_1^s(t) := \begin{cases} 0 & \text{for } 0 \le t < \frac{s}{2}, \\ \psi_s(t)\big(2-\psi_s(t)^2\big) & \text{for } \frac{s}{2} \le t \le 1, \end{cases}$$

$$Y_1^s(t) := \begin{cases} 0 & \text{for } 0 \le t < \frac{s}{2}, \\ \psi_s(t) & \text{for } \frac{s}{2} \le t \le 1, \end{cases} \quad X_2^s(t) := \begin{cases} 0 & \text{for } 0 \le t < \frac{s}{2}, \\ 1-\psi_s(t)^2 & \text{for } \frac{s}{2} \le t \le 1, \end{cases} \tag{3.11}$$

and define the family of matrices resembling the matrices from (2.27)

$$X^s := \begin{bmatrix} X_0^s\, I_2 + X_1^s\, u^{-n} \\ X_2^s\, I_2 \end{bmatrix}, \qquad Y^s := [X_0^s\, I_2 + Y_1^s\, u^n\,,\, X_2^s\, I_2]\,, \tag{3.12}$$

where I_2 denotes the 2×2 identity matrix. The point of this construction is that $X^s\,Y^s$ is an idempotent matrix in $M_4(\Sigma C(S^3))$. Indeed, one easily checks that $Y^s\,X^s = I_2$, whence $(X^s\,Y^s)^2 = X^s\,Y^s$. Furthermore, by a straightforward computation one verifies that the entries of the matrix $X^s\,Y^s$ are continuous functions from $[0,1]$ to $C(S^3)$ taking numerical values at the endpoints. Now, applying the isomorphism (2.26) and comparing with (2.27), we see that $X^1\,Y^1 \cong p_n$. Hence p_n is isomorphic to an idempotent homotopic to $X^0\,Y^0$.

On the other hand, let us observe that $P_n = GG^*$, where

$$G := \begin{bmatrix} G_1\,u^{-n} \\ G_2\,I_2 \end{bmatrix}, \quad G_1(t) := \sqrt{1-t}, \quad G_2(t) := \sqrt{t}, \quad t \in [0,1]. \tag{3.13}$$

Next, we define another family of matrices

$$V^r := \sqrt{r}\,X^0 + \sqrt{1-r}\,G, \quad W^r := \sqrt{r}\,Y^0 + \sqrt{1-r}\,G^*, \quad r \in [0,1]. \tag{3.14}$$

Multiplying these matrices yields $W^rV^r = f^r I_2$, where

$$f^r(t) := 1 + \sqrt{r-r^2}\Big(G_1(t)\big(X_1^0(t) + Y_1^0(t)\big) + 2G_2(t)X_2^0(t)\Big), \quad t \in [0,1]. \tag{3.15}$$

Since $\psi([\frac{1}{2},1]) \subseteq [0,1]$, one can easily check that $f^r(t) \geq 1$, for all $t \in [0,1]$. Hence we can define $Q^r := \frac{1}{f^r}V^rW^r$. As before, by a straightforward computation we verify that that the entries of each matrix Q^r belong to $\Sigma C(S^3)$. Furthermore, all these matrices are evidently idempotent:

$$(Q^r)^2 = \tfrac{1}{(f^r)^2}V^rW^rV^rW^r = Q^r, \quad r \in [0,1]. \tag{3.16}$$

Combining it with the fact that $Q^0 = GG^* = P_n$ and $Q^1 = X^0\,Y^0$, we conclude that P_n is homotopic to $X^0\,Y^0$. Since the latter is homotopic to $X^1\,Y^1 \cong p_n$ by the first part of the proof, by the homotopy invariance of the K-groups, we infer that $[P_n] = [p_n]$ in $K_0(C(S^4))$, as claimed. ∎

To end with, let us note that there is yet another way to describe the idempotents P_n, $n \in \mathbb{Z}$. We view S^4 as the non-reduced suspension of S^3 via (3.4), and take advantage of the covering of S^4 by two open balls $U_1 := S^4 \setminus \{[(1,g)]\}$ and $U_2 := S^4 \setminus \{[(0,g)]\}$. Now we can take two continuous functions on S^4 given by $f_1([(t,g)]) := 1-t$ and $f_2([(t,g)]) := t$, and view them as a kind of a partition of unity subordinated to this covering, i.e. $f_1 + f_2 = 1$ and $f_i|_{S^4\setminus U_i} = 0$, $i \in \{1,2\}$. Next, we take the following family of continuous functions

$$\begin{aligned} &\phi^n_{ij} \colon U_i \cap U_j \longrightarrow GL_2(\mathbb{C}), \quad i,j \in \{1,2\}, \\ &\phi^n_{11}([(t,g)]) := 1 =: \phi^n_{22}([(t,g)]), \quad \phi^n_{21}([(t,g)]) := u(g)^n, \quad \phi^n_{12}([(t,g)]) := u(g)^{-n}, \end{aligned} \tag{3.17}$$

as transition functions corresponding to the same covering. Their supports are not in $U_i \cap U_j$, $i,j \in \{1,2\}$, so that their extensions by zero to functions on S^4 is no longer continuous. However, inserting them into Karoubi's formula [20, p. 35] together with f_1 and f_2 in place of a partition of unity subordinate to the covering $\{U_i\}_{i=1}^2$, we obtain continuous functions on S^4 that are precisely the entries of the idempotent matrix P_n:

$$P_n^{ij} := \sqrt{f_i}\,\phi^n_{ij}\,\sqrt{f_j}, \quad i,j = 1,2. \tag{3.18}$$

3.1. Instanton idempotents from Woronowicz's $SU_q(2)$. We take $A := C(S^3_q)$ to be the universal unital C*-algebra generated by α and β subject to the relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\beta^* = \bar{q}\beta^*\alpha, \quad \beta\beta^* = \beta^*\beta, \quad \alpha^*\alpha + \beta\beta^* = 1, \quad \alpha\alpha^* + |q|^2\beta\beta^* = 1, \tag{3.19}$$

where $q \in \mathbb{C}$ and $|q| \leq 1$. Since $C(S^3_q) \cong C(S^3_{q'})$ for any q and q' such that $|q|, |q'| < 1$, without the loss of generality we can restrict to the real $0 < q < 1$ and unitary $q \in U(1)$

cases[2] (cf. [12]). The case $q = 1$ is the classical case discussed earlier, and the case $q \in U(1) \setminus \{1\}$ is left for the subsequent section. Herein we focus on the case $0 < q < 1$, so that $C(S^3_q)$ is the C*-algebra of Woronowicz's quantum group $SU_q(2)$ [35]. There is a faithful representation π of $C(S^3_q)$ on the Hilbert space $\ell^2(\mathbb{N} \times \mathbb{Z})$ with an orthonormal basis $\{e_{m,n}\}_{m\in\mathbb{N},\, n\in\mathbb{Z}}$ given by

$$\begin{aligned} \pi(\alpha)e_{m,n} &:= \lambda_m e_{m-1,n}\,, & \pi(\alpha^*)e_{m,n} &:= \lambda_{m+1} e_{m+1,n}\,, \\ \pi(\beta)e_{m,n} &:= q^m e_{m,n+1}\,, & \pi(\beta^*)e_{m,n} &:= q^m e_{m,n-1}\,, \end{aligned} \tag{3.20}$$

where $\lambda_m := (1 - q^{2m})^{1/2}$ (see [24, Corollary 2.3]).

By [24, Theorem 2.4], the K-groups of $C(S^3_q)$ satisfy the assumptions of Section 2.3. Therefore, all we need to construct instanton idempotents is to find a generator of $K_1(C(S^3_q))$. Guided by the classical case, we presume that the K_1-class of the fundamental representation of $SU_q(2)$

$$u_q := \begin{bmatrix} \alpha & -\bar{q}\beta^* \\ \beta & \alpha^* \end{bmatrix} \in U_2(C(S^3_q)) \tag{3.21}$$

is a desired generator. It is stated in [10] that the class $[u_q]$ is non-trivial in $K_1(C(S^3_q))$, and its pairing with the K-homology class of the spectral triple of [8] is left as an exercise. In [17], the index pairing of u_q with the unbounded K-cycle (3-summable spectral triple) constructed in [16] was computed to be 1 (cf. [22]). From the integrality of the index pairing, we conclude that $[u_q]$ generates $K_1(C(S^3_q)) \cong \mathbb{Z}$. Thus we have shown that we can use the fundamental representation of $SU_q(2)$ to construct instanton idempotents of arbitrary charges. This goes along the lines of the pioneering Pflaum's construction [29] that was reformulated and continued in [15] and [14].

Since finding a K_1-generator is pivotal in our construction of instanton idempotents, we devote the remainder of this section to two computations showing that a given unitary represents a K_1-generator. As explained above, to prove that the fundamental representation u_q is such a representative, it suffices to show that its index pairing with a K-homology class equals to 1. In a simple and explicit way, we calculate the pairing of u_q with the bounded K-cycle (odd Fredholm module) given by [24, p. 176]

$$\mathbf{z} := (\ell^2(\mathbb{N} \times \mathbb{Z}), \pi, F)\,, \quad \text{where} \quad Fe_{m,n} := \begin{cases} e_{m,n} & \text{for} \quad n > 0, \\ -e_{m,n} & \text{for} \quad n \le 0. \end{cases} \tag{3.22}$$

Note first that it follows from the definition (3.20) of π that $[F, \pi(\alpha)] = 0 = [F, \pi(\alpha^*)]$ and

$$\pi(\beta^*)[F, \pi(\beta)]e_{m,n} = \begin{cases} 2\,q^{2m}\,e_{m,0} & \text{for} \quad n = 0, \\ 0 & \text{for} \quad n \neq 0, \end{cases} \tag{3.23}$$

$$\pi(\beta^*)[F, \pi(\beta)]e_{m,n} = \begin{cases} -2\,q^{2m}\,e_{m,1} & \text{for} \quad n = 1, \\ 0 & \text{for} \quad n \neq 1. \end{cases} \tag{3.24}$$

Furthermore, observe that all commutators of F with the generators of $\pi(C(S^3_q))$ are of trace class. Therefore, $\mathbf{z}$ is 1-summable over the *-algebra generated by α and β. Next, let Tr denote the operator trace on $\ell^2(\mathbb{N} \times \mathbb{Z})$ and Tr_2 the usual trace on 2×2-matrices.

[2] We are grateful to Piotr M. Sołtan for explaining this to us.

Since $F^2 = \mathrm{id}$, for a trace-class operator T we immediately infer that $\mathrm{Tr}(F(FT+TF)) = 2\mathrm{Tr}(T)$. Combining this equality with the 1-summability of $\mathbf{z}$, by [9, Definition IV.1.3, Propositions III.3.3 and IV.1.4], the index pairing takes the following form:

$$
\begin{aligned}
<[\mathbf{z}],[u_q]> &= \tfrac{1}{2}\mathrm{Tr}\left(\mathrm{Tr}_2\left(\begin{bmatrix} \pi(\alpha^*)-1 & \pi(\beta^*) \\ -q\pi(\beta) & \pi(\alpha)-1 \end{bmatrix}\begin{bmatrix} [F,\pi(\alpha)] & -q[F,\pi(\beta^*)] \\ [F,\pi(\beta)] & [F,\pi(\alpha^*)] \end{bmatrix}\right)\right) \\
&= \tfrac{1}{2}\mathrm{Tr}\left(\pi(\beta^*)[F,\pi(\beta)] - q^2\pi(\beta^*)[F,\pi(\beta)]\right) \\
&= (1-q^2)\sum_{m=0}^{\infty} q^{2m} \\
&= 1. \qquad (3.25)
\end{aligned}
$$

On the other hand, without taking advantage of the index pairing, it was already proven in [24, Theorem 2.4] that a generator of $K_1(C(S^3_q))$ can be represented by

$$w := \beta e(0) + (1-e(0)) \in C(S^3_q), \qquad (3.26)$$

where $e(0)$ is the spectral projection corresponding to the isolated point 1 of the spectrum of $\beta^*\beta$. The argument used in [24] relied on the six-term exact sequence in K-theory applied to the short exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}\otimes C(S^1) \longrightarrow C(S^3_q) \longrightarrow C(S^1) \longrightarrow 0, \qquad (3.27)$$

where $\mathcal{K}$ denotes the ideal of compact operators and $C(S^1)$ is the C*-algebra of all continuous functions on the unit circle. Herein we apply our pullback point of view also to $C(S^3_q)$. We present it as a pullback C*-algebra and compute an even-to-odd connecting homomorphism of the Mayer-Vietoris six term exact sequence using the formula (2.24). Thus we obtain an alternative proof that w represents a generating class of $K_1(C(S^3_q))$.

To begin with, recall that it was shown in [19] that $C(S^3_q)$ can be given by the following pullback diagram:

$$
\begin{tikzcd}[column sep=small]
& C(S^3_q) \cong C(S^1) \underset{C(S^1)\otimes C(S^1)}{\times} \mathcal{T}\otimes C(S^1) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
C(S^1) \arrow[dr, "\pi_1:=\Delta"'] & & \mathcal{T}\otimes C(S^1) \arrow[dl, "\pi_2:=\sigma\otimes\mathrm{id}"] \\
& C(T^2) = C(S^1)\otimes C(S^1)\,. &
\end{tikzcd}
\qquad (3.28)
$$

Here Δ is the pullback of the multiplication map restricted to unitary complex numbers and $\mathcal{T}$ denotes the Toeplitz algebra viewed as the C*-algebra generated by the unilateral shift S acting on the Hilbert space $\ell^2(\mathbb{N})$. The symbol map $\sigma : \mathcal{T} \to C(S^1)$ is given by $\sigma(S) = U$, where U denotes the unitary generator of $C(S^1)$, and pr_i, $i \in \{1,2\}$, are the restrictions of canonical projections.

For an explicit description of the isomorphism given in [19], we first realize $C(S^1)$ as an operator algebra on the Hilbert space $\ell^2(\mathbb{Z})$ via the faithful representation ρ identifying the unitary generator U with the negative bilateral shift given on an orthonormal basis of $\ell^2(\mathbb{Z})$ by

$$\rho(U) := V, \quad Ve_k := e_{k-1}\,, \quad k \in \mathbb{Z}. \tag{3.29}$$

On the other hand, since $\pi_1 := \Delta$ is injective, pr_2 corestricted to its image is an isomorphism. Combining the above identifications, we obtain

$$C(S^3_q) \cong P := \overline{\mathrm{span}}\{T \otimes V^N \in \mathcal{T} \otimes \rho(C(S^1)) \mid \sigma(T) = 0 \text{ or } \sigma(T) = U^N,\ N \in \mathbb{Z}\}, \tag{3.30}$$

where $\overline{\mathrm{span}}$ denotes the closed linear span. Thus we view $C(S^3_q)$ as an algebra of bounded operators on the Hilbert space $\ell^2(\mathbb{N} \otimes \mathbb{Z})$. To define an isomorphism $\psi\colon C(S^3_q) \to P$ (see [19]) on the generators from (3.19), consider the bounded linear operators y and z on the Hilbert space $\ell^2(\mathbb{N})$ given by

$$ye_n := q^n e_n, \quad ze_n := \lambda_{n+1} e_{n+1}\,, \quad n \in \mathbb{N}. \tag{3.31}$$

Here $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell^2(\mathbb{N})$ and the λ_n's are as in (3.20). Furthermore, recall that the symbol map yields the following well-known short exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \hookrightarrow \mathcal{T} \xrightarrow{\ \sigma\ } C(S^1) \longrightarrow 0\,. \tag{3.32}$$

Clearly, $y \in \mathcal{K}(\ell^2(\mathbb{N})) \subseteq \mathcal{T}$. Also, since $z - S \in \mathcal{K}(\ell^2(\mathbb{N}))$, we infer that $z \in \mathcal{T}$ and $\sigma(z) = \sigma(S) = U$. Putting it all together, we conclude that the isomorphism ψ is given by

$$\begin{aligned} \psi(\alpha) &:= z^* \otimes V^*, & \psi(\alpha^*) &:= z \otimes V, \\ \psi(\beta) &:= y \otimes V^*, & \psi(\beta^*) &:= y \otimes V. \end{aligned} \tag{3.33}$$

With the isomorphism ψ at hand, we can describe the unitary w from (3.26) in terms of generators of $C(S^3_q)$. Indeed, using $z|z|^{-1} = S$ and the fact the spectral projection of y^2 corresponding to the eigenvalue 1 equals $1 - SS^*$, we get

$$\begin{aligned} \psi(1 + (\beta - 1)(1 - \alpha^*(\alpha\alpha^*)^{-1}\alpha)) &= 1 \otimes 1 + (y \otimes V^* - 1 \otimes 1)(1 \otimes 1 - SS^* \otimes 1) \\ &= SS^* \otimes 1 + y(1 - SS^*) \otimes V^* \\ &= \psi(1 - e(0)) + \psi(\beta e(0)) \\ &= \psi(w), \end{aligned} \tag{3.34}$$

so that

$$1 + (\beta - 1)(1 - \alpha^*(\alpha\alpha^*)^{-1}\alpha) = w \in C(S^3_q). \tag{3.35}$$

We want to prove that $[w]$ is a generator of $K_1(C(S^3_q))$ by showing that it coincides with the even-to-odd connecting homomorphism (2.24) of the Mayer-Vietoris six-term exact sequence of the fibre product (3.28) applied to a generator of the non-trivial part of $K_0(C(T^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$. (The trivial part is generated by $1 \in C(T^2)$.) To this end, we follow [23] to explicitly represent such a generator by the projection

$$e := \begin{bmatrix} f & g + Uh \\ g + U^*h & 1 - f \end{bmatrix} \in M_2(C(T^2))\,. \tag{3.36}$$

Here we view functions on T^2 as periodic functions on $\mathbb{R}^2$. Thus the unitary generator of

$C(S^1)$ is now a function given by $U(t) := \mathrm{e}^{2\pi i t}$, and

$$f(s) := \begin{cases} 1-2s & \text{for} \quad s \in [0, \frac{1}{2}] \bmod 1, \\ 1+2s & \text{for} \quad s \in [-\frac{1}{2}, 0] \bmod 1, \end{cases} \tag{3.37}$$

$$h(s) := \chi_{[0,\frac{1}{2}]}(s)\sqrt{f(s)(1-f(s))}, \tag{3.38}$$

$$g(s) := \big(1-\chi_{[0,\frac{1}{2}]}(s)\big)\sqrt{f(s)(1-f(s))} \tag{3.39}$$

with

$$\chi_{[0,\frac{1}{2}]}(s) := \begin{cases} 1 & \text{for} \quad s \in [0, \frac{1}{2}] \bmod 1, \\ 0 & \text{for} \quad s \in (-\frac{1}{2}, 0) \bmod 1. \end{cases} \tag{3.40}$$

Note that $f, g, h \in C(S^1)$ and $e^* = e$. Furthermore, from

$$gh = 0, \qquad f^2 + g^2 + h^2 = f, \tag{3.41}$$

it follows that $e^2 = e$. As in [23], by checking that $\mathrm{ch}_1(e) = 1$, one can prove that $[e]$ generates the non-trivial part of $K_0(C(T^2))$.

THEOREM 3.2. *Let α and β be the generators of $C(S^3_q)$ given in* (3.19), *and let e be the projection generating the non-trivial part of the K_0-group of the 2-torus provided in* (3.36). *Then the even-to-odd connecting homomorphism* (2.24) *applied to the K_0-class of e yields*

$$\partial_{01}([e]) = [1 + (\beta - 1)(1 - \alpha^*(\alpha\alpha^*)^{-1}\alpha)] = 1 \in K_1(C(S^3_q) = \mathbb{Z}\,. \tag{3.42}$$

Proof. We begin by showing that $\partial_{01}([e])$ generates $K_1(C(S^3_q)$. To this end, we consider the Mayer-Vietoris six-term exact sequence of the fibre product (3.28):

$$\begin{array}{ccccc} K_0(C(S^3_q)) & \xrightarrow{(\mathrm{pr}_{1*},\mathrm{pr}_{2*})} & K_0(C(S^1)) \oplus K_0(\mathcal{T} \otimes C(S^1)) & \xrightarrow{\pi_{2*}-\pi_{1*}} & K_0(C(T^2)) \\ \partial_{10}\Big\uparrow & & & & \Big\downarrow\partial_{01} \\ K_1(C(T^2)) & \xleftarrow{\pi_{2*}-\pi_{1*}} & K_1(C(S^1)) \oplus K_1(\mathcal{T} \otimes C(S^1)) & \xleftarrow{(\mathrm{pr}_{1*},\mathrm{pr}_{2*})} & K_1(C(S^3_q))\,. \end{array} \tag{3.43}$$

The K-groups of $C(S^3_q))$ were computed in [24, Theorem 2.4] to be $\mathbb{Z}$, and other K-groups are well-known to be $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ [34, p. 123]. Since both $K_0(C(S^1))$ and $K_0(\mathcal{T} \otimes C(S^1))$ are generated by the 1's of the C*-algebras, the induced map to $K_0(C(T^2))$ is given by $(m,n) \mapsto (m-n, 0)$. On the other hand, since $K_1(C(S^1))$ is generated by $[U]$, $K_1(\mathcal{T} \otimes C(S^1))$ is generated by $[1 \otimes U]$, and $K_1(C(T^2)) = K_1(C(S^1) \otimes C(S^1))$ is generated by $[U \otimes 1]$ and $[1 \otimes U]$, the induced map to $K_1(C(T^2))$ is given by the formula $(m,n) \mapsto (-m, n-m)$. Consequently, the exactness of the sequence forces the preceding map to be zero. Summarizing, we have arrived at the following exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(n,m) \mapsto (n-m,0)} & \mathbb{Z} \oplus \mathbb{Z} \\ \Big\uparrow & & & & \Big\downarrow\partial_{01} \\ \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{(-m,n-m) \leftarrow\!\shortmid (m,n)} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{0} & \mathbb{Z}\,. \end{array} \tag{3.44}$$

The exactness of the above sequence immediately implies that $\partial_{01}([e])$ generates the group $K_1(C(S^3_q))$, as claimed.

It remains to compute $\partial_{01}([e])$ in terms of the generators α and β. We do it with the help of (2.24). First, we lift $U \in C(S^1)$ to $S \in \mathcal{T}$, and then $e \in M_2(C(T^2))$ to

$$Q := \begin{bmatrix} 1 \otimes f & 1 \otimes g + S \otimes h \\ 1 \otimes g + S^* \otimes h & 1 \otimes (1-f) \end{bmatrix} \in M_2(\mathcal{T} \otimes C(S^1)). \tag{3.45}$$

Next, remembering $S^*S = 1$ and (3.41), we obtain

$$\begin{aligned} Q^2 &= \begin{bmatrix} 1 \otimes (f^2+g^2) + SS^* \otimes h^2 & 1 \otimes g + S \otimes h \\ 1 \otimes g + S^* \otimes h & 1 \otimes (1-f) \end{bmatrix} \\ &= Q - \begin{bmatrix} (1-SS^*) \otimes h^2 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.46}$$

Furthermore, observe that for any $\varphi \in C(S^1)$ with $\mathrm{supp}(\varphi) \subseteq [0, \frac{1}{2}]$ mod 1, we have

$$Q \begin{bmatrix} (1-SS^*) \otimes \varphi & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (1-SS^*) \otimes f\varphi & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (1-SS^*) \otimes \varphi & 0 \\ 0 & 0 \end{bmatrix} Q \tag{3.47}$$

because $(1-SS^*)S = S^*(1-SS^*) = 0$ and $g\varphi = 0$. For the sake of the induction argument, assume that

$$Q^n = Q - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-f^n) & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.48}$$

Then combining (3.46) with $h^2 = \chi_{[0,\frac{1}{2}]}(f-f^2)$ and (3.47), we compute:

$$\begin{aligned} Q^{n+1} &= Q^2 - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-f^n) & 0 \\ 0 & 0 \end{bmatrix} Q \\ &= Q - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-f^2) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}f(f-f^n) & 0 \\ 0 & 0 \end{bmatrix} \\ &= Q - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-f^{n+1}) & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.49}$$

By induction, this proves the formula (3.48) for Q^n for all $n \in \mathbb{N}$. Hence, noting the equality $\chi_{[0,\frac{1}{2}]}(\exp(2\pi i f) - 1) = \exp(2\pi i \chi_{[0,\frac{1}{2}]} f) - 1$, we can compute:

$$\begin{aligned} \exp(2\pi i Q) &= \begin{bmatrix} 1 \otimes 1 & 0 \\ 0 & 1 \otimes 1 \end{bmatrix} + \sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} \left(Q - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-f^n) & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 \otimes 1 & 0 \\ 0 & 1 \otimes 1 \end{bmatrix} + \sum_{n=0}^{\infty} \frac{(2\pi i)^n}{n!} \left(Q - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-f^n) & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &\quad - \left(Q - \begin{bmatrix} (1-SS^*) \otimes \chi_{[0,\frac{1}{2}]}(f-1) & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 \otimes 1 + \big(1-SS^*\big) \otimes \big(\exp(2\pi i \chi_{[0,\frac{1}{2}]} f) - 1\big) & 0 \\ 0 & 1 \otimes 1 \end{bmatrix}. \end{aligned} \tag{3.50}$$

Now, by inserting the definition (3.37) of f, remembering that the pullback C*-algebra and P defined in (3.30) are isomorphic via the corestriction of $(\mathrm{id} \otimes \rho) \circ \mathrm{pr}_2$ to its image, and removing the lower diagonal entry $1\otimes 1$ by the defining equivalence relation of the K_1-group, we can write

$$\begin{aligned}\partial_{01}([e]) &= [((\mathrm{id}\otimes\rho)\circ \mathrm{pr}_2)(I,\exp(2\pi i Q))]\\ &= [1\otimes 1+\big(1-SS^*\big)\otimes\rho\big(\exp(-4\pi i\chi_{[0,\frac{1}{2}]}\mathrm{id}_{\mathbb{R}})-1\big)]\,.\end{aligned}\tag{3.51}$$

Since $1-SS^*$ is a projection, one easily checks that $v:=1\otimes 1+\big(1-SS^*\big)\otimes\big(\nu-1\big)$ is unitary for any unitary element $\nu\in C(S^1)$. Furthermore, v belongs to $C(S^3_q)$ by (3.30) because $\sigma(1-SS^*)=0$. Therefore we can define a homotopy of unitaries in $C(S^3_q)$ by

$$v_t := 1\otimes 1+\big(1-SS^*\big)\otimes\rho\left(\exp\left(-\tfrac{4\pi i\chi_t\mathrm{id}_{\mathbb{R}}}{1+t}\right)-1\right),\quad t\in[0,1],\tag{3.52}$$

where

$$\chi_t(s):=\begin{cases}1 & \text{for}\quad s\in[0,\frac{1}{2}(1+t)] \bmod 1,\\ 0 & \text{for}\quad s\notin[0,\frac{1}{2}(1+t)] \bmod 1.\end{cases}\tag{3.53}$$

Hence, by the homotopy invariance of K-theory, it follows from (3.51) that $\partial_{01}([e])=[v_0]=[v_1]$ in $K_1(C(S^3_q))$. Now it suffices to note that $\rho(\exp(-2\pi i\,\mathrm{id}_{\mathbb{R}}))=\rho(U^*)=V^*$ to conclude that

$$\partial_{01}([e])=[v_1]=[1\otimes 1+(1-SS^*)\otimes(V^*-1)]=[SS^*\otimes 1+(1-SS^*)\otimes V^*].\tag{3.54}$$

Combining it with the fact that $1-SS^*=y(1-SS^*)$ and (3.34) ends the proof. ■

3.2. Instanton projections from Matsumoto's S^3_θ. In this section, we take our deformation parameter q to be unitary, and write it $q:=\mathrm{e}^{2\pi i\theta}$, $\theta\in[0,1]$. Thus our main building block is the C*-algebra $A:=C(S^3_\theta)$ of the Matsumoto quantum 3-sphere [25]. It is defined as the universal unital C*-algebra generated by α, β, satisfying the relations (3.19) for $q\in U(1)$. By [26, Proposition 2.1.(i)] it is a nuclear C*-algebra, so that (2.5) holds for its non-reduced suspension pullback diagram.

As in the previous case, the K-groups of $C(S^3_\theta)$ satisfy the assumptions of Section 2.3. We know that the unitary matrix

$$\mathrm{u}_\theta := \begin{bmatrix}\alpha & -\bar{\lambda}\beta^*\\ \beta & \alpha^*\end{bmatrix}\in U_2(C(S^3_\theta))\tag{3.55}$$

generates $K_1(C(S^3_\theta))$ because its pairing with a certain spectral triple (unbounded Fredholm module) over $C(S^3_\theta)$ is 1. This was shown in [21] using a Dirac operator isospectral to the standard classical Dirac operator on S^3. Therefore, by plugging in u_θ^n, $n\in\mathbb{Z}$, for a in (2.27) and applying (2.32), we conclude that XY is a θ-instanton idempotent of charge n with entries in $C(S^4_\theta):=\Sigma C(S^3_\theta)$. Finally, by a general argument, we can always choose a projection in the K_0-class of any of our idempotents.

This gives a significant simplification of a more direct argument combining [7, Proposition 4.14] with the computation of the index pairing of the basic ($n=1$) θ-instanton projection with an appropriate spectral triple over $C(S^4_\theta)$. The aforementioned index pairing was calculated in [11] by applying the local index theorem of Connes and Moscovici, which required direct computation containing a few hundreds terms. This determined the charge of the basic θ-instanton to be equal to 1. With this result at hand, θ-instanton projections of arbitrary charge were produced in [7].

3.3. Other instanton idempotents. The two cases of quantum deformations of S^3 presented in the previous sections yield different types of C*-algebras. However, they find a common denominator in the C*-algebra of the Heegaard quantum S^3 [2]. The even-to-odd connecting homomorphism (2.24) applied to the Powers-Rieffel idempotent [31] yields a unitary whose powers allow one to construct a new type of noncommutative instantons [13].

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