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MORITA EQUIVALENCE OF MEASURED QUANTUM GROUPOIDS. APPLICATION TO DEFORMATION OF MEASURED QUANTUM GROUPOIDS BY 2-COCYCLES

MICHEL ENOCK

Institut de Mathématiques de Jussieu
Unité Mixte Paris 6 / Paris 7 / CNRS Unité de Recherche 7586
Case 247, 4 place Jussieu, 75252 Paris Cedex, France
E-mail: enock@math.jussieu.fr

Abstract. In a recent article, Kenny De Commer investigated Morita equivalence between locally compact quantum groups, in which a measured quantum groupoid, of basis $\mathbb{C}^2$, was constructed as a linking object. Here, we generalize all these constructions and concepts to the level of measured quantum groupoids. As for locally compact quantum groups, we apply this construction to the deformation of a measured quantum groupoid by a 2-cocycle.

1. Introduction

1.1. In two articles ([Val1], [Val2]), J.-M. Vallin has introduced two notions (pseudo-multiplicative unitary, Hopf bimodule), in order to generalize, to the groupoid case, the classical notions of multiplicative unitary [BS] and of Hopf-von Neumann algebras [ES] which were introduced to describe and explain duality of groups, and led to appropriate notions of quantum groups ([ES], [W1], [W2], [BS], [MN], [W3], [KV1], [KV2], [MNW]).

In another article [EVal], J.-M. Vallin and the author have constructed, from a depth 2 inclusion of von Neumann algebras $M_0 \subset M_1$, with an operator-valued weight T_1 satisfying a regularity condition, a pseudo-multiplicative unitary, which led to two structures of Hopf bimodules, dual to each other. Moreover, we have then obtained an action of one of these structures on the algebra M_1 such that M_0 is the fixed point subalgebra, the algebra M_2 given by the basic construction being then isomorphic to the crossed product. There is on M_2 an action of another structure, which can be considered as the dual action.

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If the inclusion $M_0 \subset M_1$ is irreducible, we recovered quantum groups, as proved and studied in former papers ([EN], [E2]).

Therefore, this construction leads to a notion of "quantum groupoid", and a duality within "quantum groupoids".

1.2. In a finite-dimensional setting, this construction can be much simplified, and is studied in [NV2], [BSz1], [BSz2], [Sz], [Val3], [Val4], [Val5], and examples are described. In [NV3], the link between these "finite quantum groupoids" and depth 2 inclusions of II_1 factors is given, and in [D] it has been proved that any finite-dimensional connected $\mathbb{C}^*$-quantum groupoid can act outerly on the hyperfinite II_1 factor.

1.3. In [E3], the author studied, in whole generality, the notion of pseudo-multiplicative unitary introduced by J.-M. Vallin in [Val2]; following the strategy given by [BS], with the help of suitable fixed vectors, he introduced a notion of "measured quantum groupoid of compact type". Then F. Lesieur in [L], starting from a Hopf bimodule (as introduced in [Val1]), when there exist a left-invariant operator-valued weight and a right-invariant operator-valued weight, mimicking in this wider setting the technics of Kustermans and Vaes ([KV1], [KV2]), obtained a pseudo-multiplicative unitary, which, as in the quantum group case, "contains" all the information about the object (the von Neumann algebra, the coproduct, the antipode, the co-inverse). Lesieur gave the name of "measured quantum groupoids" to these objects. A new set of axioms for these have been given in an appendix of [E5]. Moreover, in [E4] it has been shown that, with suitable conditions, the objects constructed in [EVal] from depth 2 inclusions, are "measured quantum groupoids" in the sense of Lesieur.

1.4. In [E5] have been developed the notions of action (already introduced in [EVal]), crossed product, etc, following what has been done for locally compact quantum groups in ([E1], [ES1], [V2]); a biduality theorem for actions has been obtained in ([E5], 11.6). Moreover, we proved in ([E5] 13.9) that, for any action of a measured quantum groupoid, the inclusion of the initial algebra (on which the measured quantum groupoid is acting) into the crossed product is depth 2, which leads, thanks to [E4], to the construction of another measured quantum groupoid ([E5] 14.2). In [E6] was proved a generalization of Vaes' theorem ([V2], 4.4) on the standard implementation of an action of a locally compact quantum group; namely, we have obtained such a result when there exists a normal semi-finite faithful operator-valued weight from the von Neumann algebra on which the measured quantum groupoid is acting, onto the copy of the basis of this measured quantum groupoid which is put inside this algebra by the action.

1.5. In [E7] were studied outer actions of measured quantum groupoids. This notion was used to prove that any measured quantum groupoid can be constructed from a depth 2 inclusion.

1.6. In [DC1], Kenny De Commer introduced a notion of monoidal equivalence between two locally compact quantum groups, and constructed, in that situation, a measured quantum groupoid of basis $\mathbb{C}^2$ as a linking object between these two locally compact quantum groups. More precisely, from a locally compact quantum group $\mathbf{G_1}$ having a specific action $\mathfrak{a}_1$, called a Galois action, on a von Neumann algebra A, he was able to construct an important bunch of structures on A, and by a reflexion technique, inspired by the work of P. Shauenburg in an algebraic context ([Sc]), a second locally compact quantum group $\mathbf{G_2}$, and more precisely, a measured quantum groupoid linking $\mathbf{G_1}$ and $\mathbf{G_2}$. This leads to an equivalence relation between locally compact quantum groups.

1.7. In that article, we generalize De Commer's construction to measured quantum groupoids. We call this equivalence relation Morita equivalence; two measured quantum groupoids $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are Morita equivalent if there exists a von Neumann algebra on which $\mathfrak{G}_1$ acts on the right, $\mathfrak{G}_2$ acts on the left, and the two actions commute and being Galois, roughly speaking in a similar sense as De Commer's. This von Neumann algebra is then called an imprimitivity bi-comodule for these two measured quantum groupoids. This definition is similar to Renault's equivalence of locally compact groupoids, as defined in [R1], and developed in [R2], in which he proved that the $\mathbf{C}^*$-algebras of these two locally compact groupoids are then Morita equivalent. This is why we have chosen this terminology of "Morita equivalence". In [DC2], De Commer uses also this terminology, but two quantum groups are Morita equivalent in his sense if and only if their duals are Morita equivalent in ours.

1.8. In fact, De Commer's technics remain unchanged in the measured quantum groupoid context, if we start from a measured quantum groupoid $\mathfrak{G}$, and a Galois action $\mathfrak{a}$ of $\mathfrak{G}$ on a von Neumann algebra A, such that the invariant subalgebra $A^{\mathfrak{a}}$ is a finite sum of factors. This was remarked also in [DC4]. In the general context, some extra hypothesis is needed, and we had to introduce what we called a "Galois system", which is, roughly speaking, a Galois action, equipped with an invariant weight.

1.9. De Commer used his construction to solve the problem of deforming a locally compact quantum group by a 2-cocycle. Namely, if $\mathbf{G}$ is a locally compact quantum group, and Ω a 2-cocycle, it has been observed since years that it is possible to deform the coproduct by using Ω. Is the deformation still a locally compact quantum group? or, equivalently, is there, in that case, an existence theorem for a left (resp. right) Haar weight? This problem was solved in several particular cases and examples ([EV], [V], [FV]) and De Commer answered positively to this question in whole generality. Of course, the same problem holds for measured quantum groupoids, and the answer is still positive when the basis of the measured quantum groupoid is a finite sum of factors. In the general case, we were able to give different sufficient conditions on the 2-cocycle, and give some examples, based on the construction of matched pairs of groupoids ([Val6]).

1.10. This article is organized as follows:

In chapter 2, we recall as quickly as possible all the notations and results needed in that article; we emphazise that this article should be understood as the continuation of [E5] and [E6], and that reading this article needs having [E5] in hand.

In chapter 3, inspired by [V2] and [DC1], we prove specific results on integrable actions of a measured quantum groupoid $\mathfrak{G}$ and define Galois actions of $\mathfrak{G}$ and Galois systems for $\mathfrak{G}$.

In chapter 4, inspired by [DC1], we associate to a Galois action of $\mathfrak{G}$ several data which will be useful in the sequel. In particular we discuss how it is possible to construct a Galois system from a Galois action.

In chapter 5, we use the reflexion technique introduced in [DC1], in order to construct, "through the Galois system", another measured quantum groupoid $\mathfrak{G}_1$, and more precisely, a measured quantum groupoid linking $\mathfrak{G}$ and $\mathfrak{G}_1$.

Chapter 6 is devoted to several equivalent definitions of Morita equivalence of measured quantum groupoids. We finish that chapter by giving some examples and constructions of locally compact quantum groups Morita equivalent to measured quantum groupoids (6.12).

In chapter 7, following K. De Commer, we tried to use Morita equivalence to solve the problem of deforming a measured quantum groupoid by a 2-cocycle. This problem is here solved if the basis of the measured quantum groupoid is a finite sum of factors. In the general case, we obtain sufficient conditions, which will help, in chapter 8, to give a new example of construction of measured quantum groupoids, using J.-M. Vallin's construction of matched pairs of groupoids ([Val6]).

2. Preliminaries. This article is a continuation of [E5]; preliminaries are to be found in [E5], and we just recall herafter the following definitions and notations:

2.1. Spatial theory; relative tensor products of Hilbert spaces and fiber products of von Neumann algebras ([C1], [S], [T], [EVal]). Let N be a von Neumann algebra, ψ a normal semi-finite faithful weight on N; we shall denote by $H_\psi, \mathfrak{N}_\psi, \dots$ the canonical objects of the Tomita-Takesaki theory associated to the weight ψ; let α be a non-degenerate faithful representation of N on a Hilbert space $\mathcal{H}$; the set of ψ-bounded elements of the left-module ${}_\alpha\mathcal{H}$ is

$$D({}_\alpha\mathcal{H}, \psi) = \{\xi \in \mathcal{H}; \exists C < \infty, \|\alpha(y)\xi\| \leq C\|\Lambda_\psi(y)\|, \forall y \in \mathfrak{N}_\psi\}$$

Then, for any ξ in $D({}_\alpha\mathcal{H}, \psi)$, there exists a bounded operator $R^{\alpha,\psi}(\xi)$ from H_ψ to $\mathcal{H}$, defined, for all y in $\mathfrak{N}_\psi$ by

$$R^{\alpha,\psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi$$

which intertwines the actions of N.

If ξ, η are bounded vectors, we define the operator product

$$< \xi, \eta >_{\alpha,\psi} = R^{\alpha,\psi}(\eta)^* R^{\alpha,\psi}(\xi)$$

belongs to $\pi_\psi(N)'$, which, thanks to Tomita-Takesaki theory, will be identified to the opposite von Neumann algebra N^o.

If now β is a non-degenerate faithful antirepresentation of N on a Hilbert space $\mathcal{K}$, the relative tensor product $\mathcal{K} {}_\beta\!\underset{\psi}{\otimes}{}_\alpha \mathcal{H}$ is the completion of the algebraic tensor product $\mathcal{K} \odot D({}_\alpha\mathcal{H}, \psi)$ by the scalar product defined, if ξ_1, ξ_2 are in $\mathcal{K}$, η_1, η_2 are in $D({}_\alpha\mathcal{H}, \psi)$, by the following formula:

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2) = (\beta(< \eta_1, \eta_2 >_{\alpha,\psi})\xi_1 | \xi_2)$$

If $\xi \in \mathcal{K}$, $\eta \in D(_\alpha\mathcal{H}, \psi)$, we shall denote by $\xi \underset{\psi}{_\beta\otimes_\alpha} \eta$ the image of $\xi \odot \eta$ into $\mathcal{K} \underset{\psi}{_\beta\otimes_\alpha} \mathcal{H}$, and writing $\rho_\eta^{\beta,\alpha}(\xi) = \xi \underset{\psi}{_\beta\otimes_\alpha} \eta$, we get a bounded linear operator from $\mathcal{K}$ into $\mathcal{K} \underset{\nu}{_\beta\otimes_\alpha} \mathcal{H}$, which is equal to $1_\mathcal{K} \otimes_\psi R^{\alpha,\psi}(\eta)$.

Changing the weight ψ will give an isomorphic Hilbert space, but the isomorphism will not exchange elementary tensors!

We shall denote by σ_ψ the relative flip, which is a unitary sending $\mathcal{K} \underset{\psi}{_\beta\otimes_\alpha} \mathcal{H}$ onto $\mathcal{H} \underset{\psi^o}{_\alpha\otimes_\beta} \mathcal{K}$, defined, for any ξ in $D(\mathcal{K}_\beta, \psi^o)$, η in $D(_\alpha\mathcal{H}, \psi)$, by

$$\sigma_\psi(\xi \underset{\psi}{_\beta\otimes_\alpha} \eta) = \eta \underset{\psi^o}{_\alpha\otimes_\beta} \xi.$$

In $x \in \beta(N)'$, $y \in \alpha(N)'$, it is possible to define an operator $x \underset{\psi}{_\beta\otimes_\alpha} y$ on $\mathcal{K} \underset{\psi}{_\beta\otimes_\alpha} \mathcal{H}$, with natural values on the elementary tensors. As this operator does not depend upon the weight ψ, it will be denoted $x \underset{N}{_\beta\otimes_\alpha} y$.

We define a relative flip ς_N from $\mathcal{L}(\mathcal{K}) \underset{N}{_\beta*_\alpha} \mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H}) \underset{N^o}{_\alpha*_\beta} \mathcal{L}(\mathcal{K})$ by $\varsigma_N(X) = \sigma_\psi X(\sigma_\psi)^*$, for any $X \in \mathcal{L}(\mathcal{K}) \underset{N}{_\beta*_\alpha} \mathcal{L}(\mathcal{H})$ and any normal semi-finite faithful weight ψ on N.

If P is a von Neumann algebra on $\mathcal{H}$, with $\alpha(N) \subset P$, and Q a von Neumann algebra on $\mathcal{K}$, with $\beta(N) \subset Q$, then we define the fiber product $Q \underset{N}{_\beta*_\alpha} P$ as $\{x \underset{N}{_\beta\otimes_\alpha} y, x \in Q'$, $y \in P'\}'$.

Moreover, this von Neumann algebra can be defined independently of the Hilbert spaces on which P and Q are represented; if $(i = 1, 2)$, α_i is a faithful non-degenerate homomorphism from N into P_i, β_i is a faithful non-degenerate antihomomorphism from N into Q_i, and Φ (resp. Ψ) a homomorphism from P_1 to P_2 (resp. from Q_1 to Q_2) such that $\Phi \circ \alpha_1 = \alpha_2$ (resp. $\Psi \circ \beta_1 = \beta_2$), then, it is possible to define a homomorphism $\Psi \underset{N}{_{\beta_1}*_{\alpha_1}} \Phi$ from $Q_1 \underset{N}{_{\beta_1}*_{\alpha_1}} P_1$ into $Q_2 \underset{N}{_{\beta_2}*_{\alpha_2}} P_2$.

The operators $\theta^{\alpha,\psi}(\xi, \eta) = R^{\alpha,\psi}(\xi)R^{\alpha,\psi}(\eta)^*$, for all ξ, η in $D(_\alpha\mathcal{H}, \psi)$, generates a weakly dense ideal in $\alpha(N)'$. Moreover, there exists a family $(e_i)_{i\in I}$ of vectors in $D(_\alpha\mathcal{H}, \psi)$ such that the operators $\theta^{\alpha,\psi}(e_i, e_i)$ are pairwise orthogonal projections ($\theta^{\alpha,\psi}(e_i, e_i)$ being then the projection on the closure of $\alpha(N)e_i$). Such a family is called an orthogonal (α, ψ)-basis of $\mathcal{H}$.

2.2. Measured quantum groupoids ([L], [E5]). Following ([Val2], [EVal] 6.5), a quintuplet $(N, M, \alpha, \beta, \Gamma)$ will be called a Hopf bimodule,, if N, M are von Neumann algebras, α a faithful non-degenerate representation of N into M, β a faithful non-degenerate antirepresentation of N into M, with commuting ranges, and Γ an injective involutive homomorphism from M into $M \underset{N}{_\beta*_\alpha} M$ such that, for all X in N:

(i) $\Gamma(\beta(X)) = 1 \underset{N}{_\beta\otimes_\alpha} \beta(X)$.

(ii) $\Gamma(\alpha(X)) = \alpha(X) \underset{N}{_\beta\otimes_\alpha} 1$.

(iii) Γ satisfies the co-associativity relation:

$$(\Gamma \underset{N}{{}_\beta *_\alpha} id)\Gamma = (id \underset{N}{{}_\beta *_\alpha} \Gamma)\Gamma.$$

This last formula makes sense, thanks to the two preceding ones and 2.1. The von Neumann algebra N will be called the basis of $(N, M, \alpha, \beta, \Gamma)$.

If $(N, M, \alpha, \beta, \Gamma)$ is a Hopf bimodule, it is clear that $(N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma)$ is another Hopf bimodule, we shall call the symmetrized of the first one. (Recall that $\varsigma_N \circ \Gamma$ is a homomorphism from M to $M \underset{N^o}{{}_\alpha *_\beta} M$).

If N is abelian, $\alpha = \beta$, $\Gamma = \varsigma_N \circ \Gamma$, then the quadruplet $(N, M, \alpha, \alpha, \Gamma)$ is equal to its symmetrized Hopf bimodule, and we shall say that it is a symmetric Hopf bimodule.

A measured quantum groupoid is an octuplet $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ such that ([E5], 3.8):

(i) $(N, M, \alpha, \beta, \Gamma)$ is a Hopf bimodule.

(ii) T is a left-invariant normal, semi-finite, faithful operator valued weight T from M to $\alpha(N)$ (to be more precise, from M^+ to the extended positive elements of $\alpha(N)$ (cf. [T] IX.4.12)), which means that, for any $x \in \mathfrak{M}_T^+$, we have $(id \underset{\nu}{{}_\beta *_\alpha} T)\Gamma(x) = T(x) \underset{N}{{}_\beta\otimes_\alpha} 1$.

(iii) T' is a right-invariant normal, semi-finite, faithful operator-valued weight T' from M to $\beta(N)$, which means that, for any $x \in \mathfrak{M}_{T'}^+$, we have $(T' \underset{\nu}{{}_\beta *_\alpha} id)\Gamma(x) = 1 \underset{N}{{}_\beta\otimes_\alpha} T'(x)$.

(iv) ν is normal semi-finite faithful weight on N, which is relatively invariant with respect to T and T', which means that the modular automorphism groups of the weights $\Phi = \nu \circ \alpha^{-1} \circ T$ and $\Psi = \nu^o \circ \beta^{-1} \circ T'$ commute.

We shall write $H = H_\Phi$, $J = J_\Phi$, and for all $n \in N$, $\hat\beta(n) = J\alpha(n^*)J$, $\hat\alpha(n) = J\beta(n^*)J$. The weight Φ will be called the left-invariant weight on M.

Examples are described and explained in 2.3.

Then, $\mathfrak{G}$ can be equipped with a pseudo-multiplicative unitary W which is a unitary from $H \underset{\nu}{{}_\beta\otimes_\alpha} H$ onto $H \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} H$ ([E5], 3.6), which intertwines α, $\hat\beta$, β in the following way: for all $X \in N$, we have

$$W(\alpha(X) \underset{N}{{}_\beta\otimes_\alpha} 1) = (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \alpha(X))W,$$

$$W(1 \underset{N}{{}_\beta\otimes_\alpha} \beta(X)) = (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \beta(X))W,$$

$$W(\hat\beta(X) \underset{N}{{}_\beta\otimes_\alpha} 1) = (\hat\beta(X) \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)W,$$

$$W(1 \underset{N}{{}_\beta\otimes_\alpha} \hat\beta(X)) = (\beta(X) \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)W,$$

and the operator W satisfies:

$$(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} W)(W \underset{N}{{}_\beta\otimes_\alpha} 1_{\mathfrak{H}}) = (W \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)\sigma^{2,3}_{\alpha,\beta}(W \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o})(1 \underset{N}{{}_\beta\otimes_\alpha} W).$$

Here, $\sigma^{2,3}_{\alpha,\beta}$ goes from $(H \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} H) \underset{\nu}{{}_\beta\otimes_\alpha} H$ to $(H \underset{\nu}{{}_\beta\otimes_\alpha} H) \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} H$, and $1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o}$ goes from $H \underset{\nu}{{}_\beta\otimes_\alpha} (H \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} H)$ to $H \underset{\nu}{{}_\beta\otimes_\alpha} H \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} H$.

All the intertwining properties properties allow us to write such a formula, which will be called the "pentagonal relation". Moreover, W, M and Γ are related by the following results:

(i) M is the weakly closed linear space generated by all operators of the form $(id * \omega_{\xi,\eta})(W)$, where $\xi \in D({}_\alpha H, \nu)$, and $\eta \in D(H_{\hat\beta}, \nu^o)$ ([E5], 3.8(vii)).

(ii) For any $x \in M$, we have $\Gamma(x) = W^*(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} x)W$ ([E5], 3.6).

2.2.1. Lemma. *Let $\mathfrak{G}$ be a measured quantum groupoid, W its pseudo-multiplicative unitary, $\xi \in D({}_\alpha H, \nu)$ and $\eta \in D(H_{\hat\beta}, \nu^o)$. Then*

$$\Gamma((id * \omega_{\xi,\eta})(W)) = (id \underset{N}{{}_\beta *_\alpha} id * \omega_{\xi,\eta})(\sigma^{2,3}_{\alpha,\beta}(W \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o})(1 \underset{N}{{}_\beta\otimes_\alpha} W)).$$

Proof. This is clear, using the pentagonal relation, and the formula linking Γ and W. ■

Moreover, it is also possible to construct many other data, namely a co-inverse R, a scaling group τ_t, an antipode S, a modulus δ, a scaling operator λ, a managing operator P, and a canonical one-parameter group γ_t of automorphisms on the basis N ([E5], 3.8). Instead of $\mathfrak{G}$, we shall mostly use $(N, M, \alpha, \beta, \Gamma, T, RTR, \nu)$ which is another measured quantum groupoid, denoted $\underline{\mathfrak{G}}$, which is equipped with the same data $(W, R, \ldots)$ as $\mathfrak{G}$.

A dual measured quantum group $\widehat{\mathfrak{G}}$, denoted $(N, \widehat{M}, \alpha, \hat\beta, \widehat\Gamma, \widehat T, \widehat R \widehat T \widehat R, \nu)$, can be constructed, and we have $\widehat{\widehat{\mathfrak{G}}} = \underline{\mathfrak{G}}$.

In particular, from the fact that ν is relatively invariant with respect to T and $R \circ T \circ R$, is obtained the definition of the modulus and the scaling operator by the formula

$$(D\Phi \circ R : D\Phi)_t = \lambda^{it^2/2}\delta^{it}.$$

Then, thanks to [V1], we obtain that, if $a \in M$ is such that the operator $a\delta^{1/2}$ is bounded and its closure $\overline{a\delta^{1/2}}$ belongs to $\mathfrak{N}_\Phi$, then a belongs to $\mathfrak{N}_{\Phi\circ R}$, and that we can identify $H_{\Phi\circ R}$ with H by writing then $\Lambda_{\Phi\circ R}(a) = \Lambda_\Phi(\overline{a\delta^{1/2}})$.

Canonically associated to $\mathfrak{G}$, can be defined also the opposite measured quantum groupoid is $\mathfrak{G}^o = (N^o, M, \beta, \alpha, \varsigma_N\Gamma, RTR, T, \nu^o)$ and the commutant measured quantum groupoid $\mathfrak{G}^c = (N^o, M', \hat\beta, \hat\alpha, \Gamma^c, T^c, R^cT^cR^c, \nu^o)$; we have $(\mathfrak{G}^o)^o = (\mathfrak{G}^c)^c = \underline{\mathfrak{G}}$, $\widehat{\mathfrak{G}^o} = (\widehat{\mathfrak{G}})^c$, $\widehat{\mathfrak{G}^c} = (\widehat{\mathfrak{G}})^o$, and $\mathfrak{G}^{oc} = \mathfrak{G}^{co}$ is canonically isomorphic to $\underline{\mathfrak{G}}$ ([E5], 3.12).

The pseudo-multiplicative unitary of $\widehat{\mathfrak{G}}$ (resp. $\mathfrak{G}^o$, $\mathfrak{G}^c$) will be denoted $\widehat W$ (resp. W^o, W^c). The left-invariant weight on $\widehat{\mathfrak{G}}$ (resp. $\mathfrak{G}^o$, $\mathfrak{G}^c$) will be denoted $\widehat\Phi$ (resp. Φ^o, Φ^c). For simplification, we shall write $\hat J$ for $J_{\widehat\Phi}$.

We have $\widehat W = \sigma_{\nu^o} W^* \sigma_\nu$ which is a unitary from $H \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} H$ onto $H \underset{\nu^o}{{}_\alpha\otimes_\beta} H$. The algebra $\widehat M$ is generated by the operators $(\omega_{\xi,\eta} * id)(W)$, where ξ belongs to $D(H_{\hat\beta}, \nu^o)$ and η belongs to $D({}_\alpha H, \nu)$. In ([E5]4.8) was proved that such an element belongs to $\mathfrak{N}_{\widehat\Phi}$ if and only if ξ belongs to $\mathcal{D}(\pi'(\eta)^*)$, where $\pi'(\eta)^*$ is the adjoint of the (densely defined) operator $\pi'(\eta)$ defined on $\Lambda_\Phi(\mathfrak{N}_\Phi)$ by $\pi'(\eta)\Lambda_\Phi(x) = x\eta$, and we have then:

$$\widehat{\Phi}[(\omega_{\xi,\eta} * id)(W)^*(\omega_{\xi,\eta} * id)(W)] = \|\pi'(\eta)^*\xi\|^2$$

which allows us to identify $H_{\widehat{\Phi}}$ with H by writing $\Lambda_{\widehat{\Phi}}((\omega_{\xi,\eta} * id)(W)) = \pi'(\eta)^*\xi$, or, for any $x \in \mathfrak{N}_\Phi$:

$$(\Lambda_\Phi(x)|\Lambda_{\widehat{\Phi}}((\omega_{\xi,\eta} * id)(W))) = (x\eta|\xi).$$

The pseudo-multiplicative unitary W^o is equal to $(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})$ ([E5], 3.12(v)), which is a unitary from $H \underset{\nu^o}{{}_\alpha\otimes_\beta} H$ onto $H \underset{\nu}{{}_\beta\otimes_{\hat{\alpha}}} H$, where, for all $n \in N$, $\hat{\alpha}(n) = J\beta(n^*)J$. Therefore, applying this result about $\mathfrak{N}_{\widehat{\Phi}}$ to the duality between $\mathfrak{G}^o$ and $\widehat{\mathfrak{G}}^c$, we obtain that the operator $(\omega_{\xi,\eta} * id)(W^o) = (\omega_{\xi,\eta} * id)[(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})]$ belongs to $\mathfrak{N}_{\widehat{\Phi}^c}$ if and only if $\hat{J}\xi$ belongs to $\mathcal{D}(\pi'(\hat{J}\eta)^*)$, and then we get

$$\Lambda_{\widehat{\Phi}^c}(\omega_{\xi,\eta} * id)(W^o) = \Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})]] = \hat{J}\pi'(\hat{J}\eta)^*\hat{J}\xi$$

and if moreover η belongs to $\mathcal{D}(\delta^{-1/2})$, we get, for any $x \in \mathfrak{N}_\Phi$:

$$\begin{aligned}(\Lambda_\Phi(x)|\Lambda_{\widehat{\Phi}^c}(\omega_{\xi,\eta} * id)(W^o)) &= (\Lambda_\Phi(x)|\Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})]]) \\ &= (x\delta^{-1/2}\eta|\xi).\end{aligned}$$

Let ${}_a\mathfrak{H}_b$ be an $N-N$-bimodule, i.e. a Hilbert space $\mathfrak{H}$ equipped with a normal faithful non-degenerate representation a of N on $\mathfrak{H}$ and a normal faithful non-degenerate antirepresentation b on $\mathfrak{H}$, such that $b(N) \subset a(N)'$. A corepresentation of $\mathfrak{G}$ on ${}_a\mathfrak{H}_b$ is a unitary V from $\mathfrak{H} \underset{\nu^o}{{}_a\otimes_\beta} H$ onto $\mathfrak{H} \underset{\nu}{{}_b\otimes_\alpha} H$, satisfying, for all $n \in N$:

$$V(b(n) \underset{N^o}{{}_a\otimes_\beta} 1) = (1 \underset{N}{{}_b\otimes_\alpha} \beta(n))V,$$

$$V(1 \underset{N^o}{{}_a\otimes_\beta} \alpha(x)) = (a(n) \underset{N}{{}_b\otimes_\alpha} 1)V,$$

$$V(1 \underset{N^o}{{}_a\otimes_\beta} \hat{\beta}(n)) = (1 \underset{N}{{}_b\otimes_\alpha} \hat{\beta}(n))V,$$

such that, for any $\xi \in D({}_a\mathfrak{H}, \nu)$ and $\eta \in D(\mathfrak{H}_b, \nu^o)$, the operator $(\omega_{\xi,\eta} * id)(V)$ belongs to M (then, it is possible to define $(id * \theta)(V)$, for any θ in $M_*^{\alpha,\beta}$ which is the linear set generated by the ω_ξ, with $\xi \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$), and such that the map $\theta \to (id * \theta)(V)$ from $M_*^{\alpha,\beta}$ into $\mathcal{L}(\mathfrak{H})$ is multiplicative ([E5], 5.1, 5.5).

2.2.2. Lemma. *Let $\mathfrak{G}$ be a measured quantum groupoid; we have, for any $\xi \in D(H_\beta, \nu^o)$ and $\eta \in D({}_\alpha H, \nu)$, $t \in \mathbb{R}$:*

$$\sigma_t^{\widehat{\Phi}}[(\omega_{\xi,\eta} * id)(W)] = (\omega_{P^{it}\xi, \delta^{-it}P^{it}\eta} * id)(W).$$

Proof. Let $\zeta_1 \in D({}_\alpha H, \nu)$, and $\zeta_2 \in D(H_{\hat{\beta}}, \nu^o)$; we have, using successively [E5], 3.10(vii), 3.8(vii), 3.11(iii), 3.8(vi) and again 3.11(iii):

$$\begin{aligned}\sigma_t^{\widehat{\Phi}}((\omega_{\xi,\eta} * id)(W)\zeta_1|\zeta_2) &= (W(\xi \underset{\nu}{_\beta\otimes_\alpha} \Delta_{\widehat{\Phi}}^{-it}\zeta_1)|\eta \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} \Delta_{\widehat{\Phi}}^{-it}\zeta_2)\\
&= (W(\xi \underset{\nu}{_\beta\otimes_\alpha} P^{-it}J\delta^{it}J\zeta_1)|\eta \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} P^{-it}J\delta^{it}J\zeta_2)\\
&= (W(P^{it}\xi \underset{\nu}{_\beta\otimes_\alpha} J\delta^{it}J\zeta_1)|P^{it}\eta \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} J\delta^{it}J\zeta_2)\\
&= (\hat{J}P^{it}\eta \underset{\nu}{_\beta\otimes_\alpha} \delta^{it}J\zeta_2|W^*(\hat{J}P^{it}\xi \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} \delta^{it}J\zeta_1))\\
&= (\hat{J}P^{it}\eta \underset{\nu}{_\beta\otimes_\alpha} \delta^{it}J\zeta_2|(\delta^{it} \underset{N}{_\beta\otimes_\alpha} \delta^{it})W^*(\hat{J}P^{it}\xi \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} J\zeta_1))\\
&= (\hat{J}\delta^{-it}P^{it}\eta \underset{\nu}{_\beta\otimes_\alpha} J\zeta_2|W^*(\hat{J}P^{it}\xi \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} J\zeta_1))\\
&= (W(P^{it}\xi \underset{\nu}{_\beta\otimes_\alpha} \zeta_1)|\delta^{-it}P^{it}\eta \underset{\nu^o}{_\alpha\otimes_{\hat\beta}} \zeta_2)\end{aligned}$$

from which we get the result. ■

2.2.3. Lemma. *Let $\mathfrak{G}$ be a measured quantum groupoid, and $m \in \widehat{M}'$. Then*

$$\widehat{W}(1 \underset{N}{_{\hat\beta}\otimes_\alpha} m)\widehat{W}^* = W^{o*}(\hat{J}\hat{R}^c(m^*)\hat{J} \underset{N}{_\beta\otimes_{\hat\alpha}} 1)\widehat{W}^o.$$

Proof. By definition, we have

$$\widehat{W}(1 \underset{N}{_{\hat\beta}\otimes_\alpha} m)\widehat{W}^* = \sigma W^*\sigma((1 \underset{N}{_{\hat\beta}\otimes_\alpha} m)\sigma W\sigma = \sigma W^*(m \underset{N^o}{_\alpha\otimes_{\hat\beta}} 1)W\sigma$$

and using ([E5], 3.11(iii), 3.10 (iii), 3.12(v) and 3.11(iii) again), we get it is equal to

$$\begin{aligned}\sigma(\hat{J} \underset{N}{_\beta\otimes_\alpha} J)W(\hat{J}m\hat{J} \underset{N}{_\beta\otimes_\alpha} 1)W^*(\hat{J} \underset{N}{_\beta\otimes_{\hat\alpha}} J)\sigma &= \sigma(\hat{J} \underset{N}{_\beta\otimes_\alpha} J)\varsigma\widehat{\Gamma}(\hat{J}m\hat{J})(\hat{J} \underset{N}{_\beta\otimes_{\hat\alpha}} J)\sigma\\
&= (J \underset{N^o}{_\alpha\otimes_\beta} \hat{J})\widehat{\Gamma}(\hat{J}m\hat{J})(J \underset{N^o}{_{\hat\alpha}\otimes_\beta} \hat{J})\\
&= W^{o*}(\hat{J}\hat{R}^c(m^*)\hat{J} \underset{N}{_\beta\otimes_{\hat\alpha}} 1)\widehat{W}^o. \ ■\end{aligned}$$

2.3. Examples of measured quantum groupoids. Examples of measured quantum groupoids are the following:

(i) Locally compact quantum groups, as defined and studied by J. Kustermans and S. Vaes ([KV2], [KV2], [V2]); these are, trivially, the measured quantum groupoids with the basis $N = \mathbb{C}$.

(ii) Measured groupoids, equipped with a left Haar system and a quasi-invariant measure on the set of units, as studied mostly by T. Yamanouchi ([Y1], [Y2], [Y3], [Y4]); it was proved in [E8] that these measured quantum groupoids are exactly those whose underlying von Neumann algebra is abelian. This example has been presented in full details in ([E5], 3.4 and 3.13).

(iii) The finite dimensional case has been studied by D. Nikshych and L. Vainermann ([NV2]) and J.-M. Vallin ([Val3], [Val4]); in that case, non-trivial examples are given.

(iv) Continuous fields of ($\mathbf{C}^*$-version of) locally compact quantum groups, as studied by E. Blanchard in ([Bl1], [Bl2]); it was proved in [E8] that these measured quantum groupoids are exactly those whose basis is central in the underlying von Neumann algebras of both the measured quantum groupoid and its dual.

(v) In ([L], 17.1), be given a family $\mathfrak{G}_i = (N_i, M_i, \alpha_i, \beta_i, \Gamma_i, T_i, T'_i, \nu_i)$ a measured quantum groupoids, Lesieur showed that it is possible to construct another measured quantum groupoid

$$\oplus_{i\in I}\mathfrak{G}_i = (\oplus_{i\in I}N_i, \oplus_{i\in I}M_i, \oplus_{i\in I}\alpha_i, \oplus_{i\in I}\beta_i, \oplus_{i\in I}\Gamma_i, \oplus_{i\in I}T_i, \oplus_{i\in I}T'_i, \oplus_{i\in I}\nu_i).$$

(vi) In [DC1], K. De Commer proved that, in the case of a monoidal equivalence between two locally compact quantum groups (which means that each of these locally compact quantum group has an ergodic and integrable action on the other one), it is possible to construct a measured quantum groupoid of basis $\mathbb{C}^2$ which contains all the data. Moreover, he proved that such measured quantum groupoids are exactly those whose basis $\mathbb{C}^2$ is central in the underlying von Neumann algebra of the measured quantum groupoid, but not in the underlying von Neumann algebra of the dual measured quantum groupoid.

(vii) In [E5] was described how, from an action $(b, \mathfrak{a})$ of a measured quantum groupoid $\mathfrak{G}$, it is possible to construct another measured quantum groupoid $\mathfrak{G}(\mathfrak{a})$; as a particular case, this allows to canonically associate to any action $\mathfrak{a}$ of a locally compact quantum group $\mathbf{G}$ on a von Neumann algebra A, a measured quantum groupoid $\mathfrak{G}(\mathfrak{a})$.

(viii) In [VV] was given a specific procedure to construct locally compact quantum groups, starting from a locally compact group G, whose almost all elements belong to the product G_1G_2 (where G_1 and G_2 are closed subgroups of G such that $G_1 \cap G_2 = \{e\}$, where e is the neutral element of G); such (G_1, G_2) is called a "matched pair" of locally compact groups. Then, G_1 acts naturally on $L^\infty(G_2)$ (and vice versa), and the two crossed products obtained bear the structure of two locally compact quantum groups in duality. In [Val5], J.-M. Vallin generalizes this constructions up to groupoids, and then obtains examples of measured quantum groupoids; more specific examples are then given by the action of a matched pair of groups on a locally compact space, and also more exotic examples.

(ix) In [L], 9.5.5, was given the following exemple, called "quantum space quantum groupoid"; let N be a von Neumann algebra; let us consider $M = N^o \otimes_{Z(N)} N$, the representation α of N into M given by $(n \in N)$ $\alpha(n) = 1 \otimes_{Z(N)} n$, and the antirepresentation β given by $\beta(n) = n^o \otimes_{Z(N)} 1$. Then if τ is a normal semi-finite faithful trace on $Z(N)$, ν a normal faithful semi-finite weight on N, let T_ν be the normal faithful semi-finite operator-valued weight from N onto $Z(N)$ such that $\nu = \tau \circ T_\nu$, we can easily get that the relative tensor product $(H_\nu \otimes_\tau H_\nu) \ {}_\beta\!\underset{\nu}{\otimes}_\alpha \ (H_\nu \otimes_\tau H_\nu)$ is canonically isomorphic to $H_\nu \otimes_\tau H_\nu \otimes_\tau H_\nu$ and this isomorphism sends $M \ {}_\beta\!\underset{N}{*}_\alpha \ M$ onto $N^o \otimes_\tau Z(N) \otimes_\tau N$; we can therefore identify $M \ {}_\beta\!\underset{N}{*}_\alpha \ M$ with M, and verify that $(N, M, \alpha, \beta, id)$ is a Hopf bimodule.

Moreover, we can get that $\mathfrak{G}(N) = (N, M, \alpha, \beta, id, T^o_\nu \otimes_{Z(N)} id, id \otimes_{Z(N)} T_\nu, \nu)$ is a measured quantum groupoid. We shall call it the N-measured quantum groupoid.

The dual $\widehat{\mathfrak{G}(N)} = (N, Z(N)', \alpha, \hat{\beta}, id, (T^o_\nu)^{-1}, T_\nu^{-1}, \nu)$, where $\hat{\beta}(n) = J_\nu n^* J_\nu$, T_ν^{-1} is the canonical operator-valued weight from $Z(N)'$ to N' given from T_ν, and $(T^o_\nu)^{-1}$ is the canonical operator-valued weight from $Z(N)'$ to N given from T^o_ν. This measured quantum groupoid will be called the dual N-measured quantum groupoid.

(x) If $\mathfrak{G}_i=(N_i,M_i,\alpha_i,\beta_i,\Gamma_i,T_i,T'_i,\nu_i)$ $(i=1,2)$ are two measured quantum groupoids, then we can define another measured quantum groupoid $\mathfrak{G}_1\otimes\mathfrak{G}_2$:

$$(N_1\otimes N_2, M_1\otimes M_2, \alpha_1\otimes\alpha_2, \beta_1\otimes\beta_2, (id\otimes\varsigma\otimes id)(\Gamma_1\otimes\Gamma_2), T_1\otimes T_2, T'_1\otimes T'_2, \nu_1\otimes\nu_2).$$

Moreover, it easy to get that $\widehat{\mathfrak{G}_1\otimes\mathfrak{G}_2}=\widehat{\mathfrak{G}_1}\otimes\widehat{\mathfrak{G}_2}$.

(xi) The $SU(2)$ dynamical quantum group, as studied in particular by E. Koelink and H. Rosengren ([KR]) can be lifted, thanks to [Ti], to the level of operator algebras, and give another example of a measured quantum groupoid.

(xii) Last but not least, De Commer studied Morita equivalence between the quantum group $SU_q(2)$, and various quantum groups ([DC2], [DC3]). In a new work ([DC4]), he obtains an integrable Galois action of $SU_q(2)$ which is not ergodic. Therefore, this leads to a measured quantum groupoid (6.12.4).

2.4. Action of a measured quantum groupoid ([E5]). An action ([E5], 6.1) of $\mathfrak{G}$ on a von Neumann algebra A is a couple $(b,\mathfrak{a})$, where:

(i) b is an injective $*$-antihomomorphism from N into A;

(ii) $\mathfrak{a}$ is an injective $*$-homomorphism from A into $A\underset{N}{{}_b*_\alpha} M$;

(iii) b and $\mathfrak{a}$ are such that, for all n in N:

$$\mathfrak{a}(b(n))=1\underset{N}{{}_b\otimes_\alpha}\beta(n)$$

(which allow us to define $\mathfrak{a}\underset{N}{{}_b*_\alpha} id$ from $A\underset{N}{{}_b*_\alpha} M$ into $A\underset{N}{{}_b*_\alpha} M\underset{N}{{}_\beta*_\alpha} M$) and such that

$$(\mathfrak{a}\underset{N}{{}_b*_\alpha} id)\mathfrak{a}=(id\underset{N}{{}_b*_\alpha}\Gamma)\mathfrak{a}.$$

If we start from a measured groupoid, we get the usual notion of action of a groupoid ([E5], 6.3).

The invariant subalgebra $A^{\mathfrak{a}}$ is defined by

$$A^{\mathfrak{a}}=\{x\in A\cap b(N)'; \mathfrak{a}(x)=x\underset{N}{{}_b\otimes_\alpha}1\}.$$

As $A^{\mathfrak{a}}\subset b(N)'$, A (and $L^2(A)$) is a $A^{\mathfrak{a}}-N^o$-bimodule.

Let us write, for any $x\in A^+$, $T_{\mathfrak{a}}(x)=(id\underset{\nu}{{}_b*_\alpha}\Phi)\mathfrak{a}(x)$; this formula defines a normal faithful operator-valued weight from A onto $A^{\mathfrak{a}}$; the action $\mathfrak{a}$ will be called integrable if $T_{\mathfrak{a}}$ is semi-finite ([E5], 6.11, 12, 13 and 14).

If the von Neumann algebra A acts on a Hilbert space $\mathfrak{H}$, and if there exists a representation a of N on $\mathfrak{H}$ such that $b(N)\subset A\subset a(N)'$, a corepresentation V of $\mathfrak{G}$ on the bimodule ${}_a\mathfrak{H}_b$ will be called an implementation of $\mathfrak{a}$ if we have $\mathfrak{a}(x)=V(x\underset{N^o}{{}_a\otimes_b}1)V^*$, for all $x\in A$ ([E5], 6.6); moreover, if ψ is a normal semi-finite faithful weight on A, we shall define a representation a of N on H_ψ by $a(n)=J_\psi b(n^*)J_\psi$, for all $n\in N$, and we shall look after an implementation V of $\mathfrak{a}$ on ${}_a(H_\psi)_b$ such that ([E5], 6.9):

$$V^*=(J_\psi\underset{\nu^o}{{}_\alpha\otimes_\beta}J_{\widehat{\Phi}})V(J_\psi\underset{\nu}{{}_b\otimes_\alpha}J_{\widehat{\Phi}}).$$

If the weight ψ is δ-invariant, which means that, for all $\eta \in D({}_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$ such that $\delta^{1/2}\eta$ belongs to $D(H_\beta, \nu^o)$, and $x \in \mathfrak{N}_\psi$, we have

$$\psi[(id \underset{N}{{}_b*_\alpha} \omega_\eta)\mathfrak{a}(x^*x)] = \|\Lambda_\psi(x) \underset{\nu^o}{{}_a\otimes_\beta} \delta^{1/2}\eta\|^2$$

and if, moreover, ψ has the density property, (i.e. $D((H_\psi)_b, \nu^o) \cap D({}_a H_\psi, \nu)$ is dense in H_ψ), then such an implementation V_ψ was constructed in [E5], 8.8); more precisely ([E5], 8.4), if $x \in \mathfrak{N}_\psi$, $\xi \in D({}_\alpha H, \nu)$ and η is as above, we get that $(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\mathfrak{a}(x)$ belongs to $\mathfrak{N}_\psi$ and that

$$\Lambda_\psi[(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\mathfrak{a}(x)] = (id * \omega_{\delta^{1/2}\eta,\xi})(V_\psi)\Lambda_\psi(x).$$

In ([E6], 7.6) was introduced the notion of invariant weight by an action; a normal faithful semi-finite weight ϕ on A will be called invariant by $\mathfrak{a}$ if, for all $\eta \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$, and $x \in \mathfrak{N}_\phi$, we have

$$\phi[(id \underset{N}{{}_b*_\alpha} \omega_\eta)\mathfrak{a}(x^*x) = \|\Lambda_\phi(x) \underset{\nu^o}{{}_a\otimes_\beta} \eta\|^2.$$

If, moreover, ϕ has the density property, a similar implementation V'_ϕ was constructed also in ([E6], 7.7). Moreover, with these hypothesis, it is possible to prove that there exists a normal semi-finite operator-valued weight $\mathfrak{T}$ from A onto $b(N)$ (we shall say that the action is "weighted"), such that $\phi = \nu^o \circ b^{-1} \circ \mathfrak{T}$. This operator-valued weight $\mathfrak{T}$ satisfies, for all positive x in A:

$$(\mathfrak{T} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x) = \mathfrak{a}(\mathfrak{T}(x)) = 1 \underset{N}{{}_b\otimes_\alpha} \beta \circ b^{-1}\mathfrak{T}(x).$$

Note that, if we define, for $n \in N$, $b^o(n) = b(n)^o$, we obtain a $*$-homomorphism from N into A^o; moreover, for $x \in A$, let us write $\mathfrak{a}^o(x^o) = (.^o \underset{N}{{}_b*_\alpha} R) \circ a(x)$; it is straightforward to get that $(b^o, \mathfrak{a}^o)$ is an action of $\mathfrak{G}^o$ on A^o.

Of course, one should write in this section "right action" instead of simply action. At some stage of this paper, we shall need left actions. A left action of $\mathfrak{G}$ on a von Neumann algebra A is a couple $(a, \mathfrak{b})$, where:

(i) a is an injective *-homomorphism from N into A;
(ii) $\mathfrak{b}$ is an injective $*$-homomorphism from A into $M \underset{N}{{}_\beta*_a} A$;
(iii) a and $\mathfrak{b}$ are such that, for all n in N:

$$\mathfrak{b}(a(n)) = \alpha(n) \underset{N}{{}_\beta\otimes_a} 1, \quad (id \underset{N}{{}_\beta*_a} \mathfrak{b})\mathfrak{b} = (\Gamma \underset{N}{{}_\beta*_a} id)\mathfrak{b}.$$

Then, it is clear that $(a, \varsigma_N\mathfrak{b})$ is an action (a right action) of $\mathfrak{G}^o$ on A, and $(a^o, (\sigma_N\mathfrak{b})^o)$ is an action (a right action) of $\mathfrak{G}$ on A^o. Conversely, if $(b, \mathfrak{a})$ is an action of $\mathfrak{G}$ on A, then, $(b^o, \sigma_{N^o}\mathfrak{a}^o)$ is a left action of $\mathfrak{G}$ on A^o.

The invariant subalgebra $A^\mathfrak{b}$ is defined by

$$A^\mathfrak{b} = \{x \in A \cap a(N)'; \mathfrak{b}(x) = 1 \underset{N}{{}_\beta\otimes_a} x\}$$

and $T_\mathfrak{b} = (\Phi \underset{\nu}{{}_\beta*_a} id)\mathfrak{b}$ is a normal faithful operator-valued weight from A onto $A^\mathfrak{b}$; the action $\mathfrak{b}$ will be called integrable if $T_\mathfrak{b}$ is semi-finite. It is clear that $\mathfrak{b}$ is integrable if and only if $(\sigma_N\mathfrak{b})^o$ is integrable.

If $(b,\mathfrak{a})$ is an action of $\mathfrak{G}_1=(N_1,M_1,\alpha_1,\beta_1,\Gamma_1,T_1,T_1',\nu_1)$ on a von Neumann algebra A, and $(a,\mathfrak{b})$ a left action of $\mathfrak{G}_2=(N_2,M_2,\alpha_2,\beta_2,\Gamma_2,T_2,T_2',\nu_2)$ on A, such that $a(N_2)\subset b(N_1)'$, then, we shall say that the actions $\mathfrak{a}$ and $\mathfrak{b}$ commute if we have

$$b(N_1)\subset A^{\mathfrak{b}},\quad a(N_2)\subset A^{\mathfrak{a}},\quad (\mathfrak{b}\ {}_b{*_{\alpha_1}}_{N_1}\ id)\mathfrak{a}=(id\ {}_{\beta_2}{*_a}_{N_2}\ \mathfrak{a})\mathfrak{b}.$$

Let us remark that the first two properties allow us to write the fiber products $\mathfrak{b}\ {}_b{*_{\alpha_1}}_{N_1}\ id$ and $id\ {}_{\beta_2}{*_a}_{N_2}\ \mathfrak{a}$.

2.5. Crossed product ([E5]). The crossed product of A by $\mathfrak{G}$ via the action $\mathfrak{a}$ is the von Neumann algebra generated by $\mathfrak{a}(A)$ and $1\ {}_b{\otimes_\alpha}_N\ \widehat{M}'$ ([E5], 9.1) and is denoted $A\rtimes_{\mathfrak{a}}\mathfrak{G}$; then there exists ([E5], 9.3) an integrable action $(1\ {}_b{\otimes_\alpha}_N\ \hat{\alpha},\tilde{\mathfrak{a}})$ of $(\widehat{\mathfrak{G}})^c$ on $A\rtimes_{\mathfrak{a}}\mathfrak{G}$.

The biduality theorem ([E5], 11.6) says that the bicrossed product $(A\rtimes_{\mathfrak{a}}\mathfrak{G})\rtimes_{\tilde{\mathfrak{a}}}\widehat{\mathfrak{G}}^o$ is canonically isomorphic to $A\ {}_b{*_\alpha}_N\ \mathcal{L}(H)$; more precisely, this isomorphism is given by

$$\Theta(\mathfrak{a}\ {}_b{*_\alpha}_N\ id)(A\ {}_b{*_\alpha}_N\ \mathcal{L}(H))=(A\rtimes_{\mathfrak{a}}\mathfrak{G})\rtimes_{\tilde{\mathfrak{a}}}\widehat{\mathfrak{G}}^o$$

where Θ is the spatial isomorphism between $\mathcal{L}(\mathfrak{H}\ {}_b{\otimes_\alpha}_\nu\ H\ {}_\beta{\otimes_\alpha}_\nu\ H)$ and $\mathcal{L}(\mathfrak{H}\ {}_b{\otimes_\alpha}_\nu\ H\ {}_{\hat{\alpha}}{\otimes_\beta}_{\nu^o}\ H)$ implemented by $1_{\mathfrak{H}}\ {}_b{\otimes_\alpha}_\nu\ \sigma_\nu W^o\sigma_\nu$; the biduality theorem says also that this isomorphism sends the action $(1\ {}_b{\otimes_\alpha}_N\ \hat{\beta},\underline{\mathfrak{a}})$ of $\mathfrak{G}$ on $A\ {}_b{*_\alpha}_N\ \mathcal{L}(H)$, defined, for any $X\in A\ {}_b{*_\alpha}_N\ \mathcal{L}(H)$, by

$$\underline{\mathfrak{a}}(X)=(1\ {}_b{\otimes_\alpha}_N\ \sigma_{\nu^o}W\sigma_{\nu^o})(id\ {}_b{*_\alpha}_N\ \varsigma_N)(\mathfrak{a}\ {}_b{*_\alpha}_N\ id)(X)(1\ {}_b{\otimes_\alpha}_N\ \sigma_{\nu^o}W\sigma_{\nu^o})^*$$

to the bidual action (of $\mathfrak{G}^{co}$) on $(A\rtimes_{\mathfrak{a}}\mathfrak{G})\rtimes_{\tilde{\mathfrak{a}}}\widehat{\mathfrak{G}}^o$.

We have $(A\rtimes_{\mathfrak{a}}\mathfrak{G})^{\tilde{\mathfrak{a}}}=\mathfrak{a}(A)$ ([E5], 11.5), and therefore the normal faithful semi-finite operator-valued weight $T_{\tilde{\mathfrak{a}}}$ sends $A\rtimes_{\mathfrak{a}}\mathfrak{G}$ onto $\mathfrak{a}(A)$; therefore, starting with a normal semi-finite weight ψ on A, we can construct a dual weight $\tilde{\psi}$ on $A\rtimes_{\mathfrak{a}}\mathfrak{G}$ by the formula $\tilde{\psi}=\psi\circ\mathfrak{a}^{-1}\circ T_{\tilde{\mathfrak{a}}}$ ([E5], 13.2).

Moreover ([E5], 13.3), the linear set generated by all the elements $(1\ {}_b{\otimes_\alpha}_N\ a)\mathfrak{a}(x)$, for all $x\in\mathfrak{N}_\psi$, $a\in\mathfrak{N}_{\widehat{\Phi}^c}\cap\mathfrak{N}_{\hat{T}^c}$, is a core for $\Lambda_{\tilde{\psi}}$, and it is possible to identify the GNS representation of $A\rtimes_{\mathfrak{a}}\mathfrak{G}$ associated to the weight $\tilde{\psi}$ with the natural representation on $H_\psi\ {}_b{\otimes_\alpha}_\nu\ H$ by writing

$$\Lambda_\psi(x)\ {}_b{\otimes_\alpha}_\nu\ \Lambda_{\widehat{\Phi}^c}(a)=\Lambda_{\tilde{\psi}}[(1\ {}_b{\otimes_\alpha}_N\ a)\mathfrak{a}(x)]$$

which leads to the identification of $H_{\tilde{\psi}}$ with $H_\psi\ {}_b{\otimes_\alpha}_\nu\ H$.

If the weight ψ is δ-invariant (resp. invariant) and has the density property, then the implementation V_ψ (resp. V'_ψ) recalled in 2.4 is equal to $J_{\tilde{\psi}}(J_\psi\ {}_a{\otimes_\beta}_{N^o}\ J_{\widehat{\Phi}})$ ([E6], 3.2). More generally, if we write $V=J_{\tilde{\psi}}(J_\psi\ {}_a{\otimes_\beta}_{N^o}\ J_{\widehat{\Phi}})$, we have

$$V^*=(J_\psi\ {}_\alpha{\otimes_\beta}_{\nu^o}\ J_{\widehat{\Phi}})V(J_\psi\ {}_b{\otimes_\alpha}_\nu\ J_{\widehat{\Phi}})$$

and if it is an implementation of $\mathfrak{a}$, we shall call it a standard implementation of $\mathfrak{a}$. It has been proved that it is the case, for any normal semi-finite faithful weight ψ on A, whenever the action is weighted (i.e. if there exists a normal semi-finite faithful operator-valued weight from A onto $b(N)$).

If $(a, \mathfrak{b})$ is a left action of $\mathfrak{G}$ on A, we shall define the crossed product $\mathfrak{G} \ltimes_{\mathfrak{b}} A$ as the von Neumann algebra generated by $\widehat{M} \underset{N}{{}_\beta\otimes_a} 1$ and $b(A)$; therefore, it is the image under σ_N of the crossed product $A \rtimes_{\sigma_N \mathfrak{b}} \mathfrak{G}^o$.

2.6. Basic Construction. Let $M_0 \subset M_1$ be an inclusion of σ-finite von Neumann algebras, equipped with a normal faithful semi-finite operator-valued weight T_1 from M_1 to M_0. Let ψ_0 be a normal faithful semi-finite weight on M_0, and $\psi_1 = \psi_0 \circ T_1$.

Following ([J], 3.1.5(i)), the von Neumann algebra $M_2 = J_{\psi_1} M_0' J_{\psi_1}$ defined on the Hilbert space H_{ψ_1} will be called the basic construction made from the inclusion $M_0 \subset M_1$. We have $M_1 \subset M_2$, and we shall say that the inclusion $M_0 \subset M_1 \subset M_2$ is standard.

Let us write r for the inclusion of M_0 into M_1 (or the representation of M_0 on H_{ψ_1} given by the restriction of π_{ψ_1} to M_0), and let us define s, for any $x \in M_0$, by $s(x) = J_{\psi_1} r(x)^* J_{\psi_1}$; s is a normal faithful antirepresentation of M_0 on H_{ψ_1}, and $M_2 = s(M_0)'$. Therefore (2.1), the operators $\theta^{s,\psi_0^o}(\xi, \eta)$, for all ξ, η in $D((H_{\psi_1})_s, \psi_0^o)$ generate a dense ideal in M_2.

Following ([EN], 10.6), for x in $\mathfrak{N}_{T_1}$, we shall define $\Lambda_{T_1}(x)$ by the following formula, for all z in $\mathfrak{N}_{\psi_0}$:

$$\Lambda_{T_1}(x)\Lambda_{\psi_0}(z) = \Lambda_{\psi_1}(xz).$$

This operator belongs to $Hom_{M_0^o}(H_{\psi_0}, H_{\psi_1})$; if x, y belong to $\mathfrak{N}_{T_1}$, then $\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*$ belongs to M_2, and $\Lambda_{T_1}(x)^*\Lambda_{T_1}(y) = T_1(x^*y) \in M_0$.

Using then Haagerup's construction ([T], IX.4.24), it is possible to construct a normal semi-finite faithful operator-valued weight T_2 from M_2 to M_1 ([EN], 10.7), which will be called the basic construction made from T_1. If x, y belong to $\mathfrak{N}_{T_1}$, then the operators $\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*$ form a dense sub-$*$algebra of M_2, included into $\mathfrak{M}_{T_2}$, and we have $T_2(\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*) = xy^*$. The operator-valued weight T_2 is characterized by the equality ([EN], 10.3):

$$\frac{d\psi_1 \circ T_2}{d\psi_0^o} = \frac{d\psi_1}{d(\psi_0 \circ T_1)^o} = \Delta_{\psi_1}$$

from which, writing $\psi_2 = \psi_1 \circ T_2$, we get that

$$\sigma_t^{\psi_2}(\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*) = \Lambda_{T_1}(\sigma_t^{\psi_1}(x))\Lambda_{T_1}(\sigma_t^{\psi_1}(y^*))^*.$$

The operator-valued weight T_2 from M_2 to M_1 will be called the basic construction made from the operator-valued weight T_1 from M_1 to M_0. Using ([EN], 3.7 and 10.6 (v)), we easily get that, for any x, y in $\mathfrak{N}_{T_1} \cap \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_1}^* \cap \mathfrak{N}_{\psi_1}^*$, we have $T_2(\Lambda_{T_1}(x)\Lambda_{T_1}(y^*)^*) = xy$, and

$$\|\Lambda_{\psi_2}(\Lambda_{T_1}(x)\Lambda_{T_1}(y^*)^*)\| = \|\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \Lambda_{\psi_1}(y)\|$$

where r is the inclusion of M_0 into M_1, and for $a \in M_0$, $s(a) = J_{\psi_1} a^* J_{\psi_1}$; so, we can identify H_{ψ_2} with $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$ by writing $\Lambda_{\psi_2}(\Lambda_{T_1}(x)\Lambda_{T_1}(y^*)^*) = \Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \Lambda_{\psi_1}(y)$;

then, we identify $\Delta^{it}_{\psi_2}$ with $\Delta^{it}_{\psi_1} {}_s\otimes_r{}_{\psi_0} \Delta^{it}_{\psi_1}$ (here, this relative tensor product of operators means that there exists a bounded operator with natural values on elementary tensors) and J_{ψ_2} with $\sigma_{M_0^o}(J_{\psi_1} {}_s\otimes_r{}_{M_0} J_{\psi_1})$.

Then, for any $\xi \in D((H_{\psi_1})_s, \psi_0^o)$ and $\eta \in D((H_{\psi_1})_s, \psi_0^o) \cap \mathcal{D}(\Delta^{1/2}_{\psi_1})$ such that $\Delta^{-1/2}_{\psi_1}\eta$ belongs to $D((H_{\psi_1})_s, \psi_0^o)$, we have $\Lambda_{\psi_2}(\theta^{s,\psi_0^o}(\xi, \eta)) = \xi {}_s\otimes_r{}_{\psi_0} J_{\psi_1}\Delta^{1/2}_{\psi_1}\eta$.

Using similar arguments as in ([E6], 4.7(ii)), we can prove that there exists a family $(e_i)_{i\in I}$, which is an orthogonal (s, ψ_0^o)-basis of H_{ψ_1}, such that each vector e_i belongs to $\mathcal{D}(\Delta^{1/2}_{\psi_1})$; we can prove then, as in ([E6], 4.7(iii)), that $\psi_2 = \sum_i \omega_{\Delta^{1/2}_{\psi_1} e_i}$.

Let $\mathcal{T}_{\psi_1,T_1}$ be the Tomita algebra associated to the operator-valued weight T_1 and the weight ψ_1 ([EN], 10.12, and [E5], 2.2.1), which is made of elements x in $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}^*_{\psi_1} \cap \mathfrak{N}_{T_1} \cap \mathfrak{N}^*_{T_1}$, which are analytic with respect to $\sigma^{\psi_1}_t$, and such that, for any $z \in \mathbb{C}$, $\sigma^{\psi_1}_z(x)$ belongs to $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}^*_{\psi_1} \cap \mathfrak{N}_{T_1} \cap \mathfrak{N}^*_{T_1}$; such elements are a dense $*$ subalgebra of M_1. Moreover, it is possible to prove ([DC1], 1.4) that an element $X \in M_2$ belongs to $\mathfrak{N}_{\psi_2}$ if and only if there exists $\Xi \in H_{\psi_2}$ such that, for any x, y in $\mathcal{T}_{\psi_1,T_1}$, we have

$$(\Lambda_{\psi_1}(x) {}_s\otimes_r{}_{\psi_0} \Lambda_{\psi_1}(y)|\Xi) = (\Lambda_{\psi_1}(x)|X\Lambda_{\psi_1}(\sigma^{\psi_1}_{-i}(y^*)))$$

and then we have $\Xi = \Lambda_{\psi_2}(X)$.

3. Integrable actions of a measured quantum groupoid. In that chapter are generalized, up to measured quantum groupoids, results about integrable actions (([V2], 5.3, [DC1], 2.1); namely, if $(b, \mathfrak{a})$ is an integrable action of $\mathfrak{G}$ on a von Neumann algebra A (the definition has been given in 2.4), we construct then a representation $\pi_{\mathfrak{a}}$ of the crossed product on the Hilbert space $L^2(A)$(3.6), whose image is the von Neumann algebra $s(A^{\mathfrak{a}})'$ given by the standard construction made from the inclusion $A^{\mathfrak{a}} \subset A$; moreover is constructed an isometry G from $L^2(s(A^{\mathfrak{a}})')$ into $L^2(A \rtimes_{\mathfrak{a}} \mathfrak{G})$ (3.8), which is a unitary if and only if the representation $\pi_{\mathfrak{a}}$ is faithful; following ([DC1], 2.7), we say that the integrable action $(b, \mathfrak{a})$ is then Galois (3.11).

3.1. Lemma. *Let $(b, \mathfrak{a})$ be an integrable action of a measured quantum groupoid on a von Neumann algebra A; let ψ_0 be a normal faithful semi-finite weight on $A^{\mathfrak{a}}$, and $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$ be the lifted normal semi-finite faithful weight on A; let $(\xi_i)_{i\in I}$ be a family of vectors in $D((H_{\psi_1})_b, \nu^o)$ such that $\psi_0(x) = \sum_i \omega_{\xi_i}(x)$, for all positive $x \in A^{\mathfrak{a}}$. Then there exists an isometry $\mathcal{V}$ from H_{ψ_1} into $\oplus_i(H_{\psi_1} {}_b\otimes_\alpha{}_\nu H) = (H_{\psi_1} {}_b\otimes_\alpha{}_\nu H) \otimes l^2(I)$ such that:*

(i) For all $y \in A$, $(\mathfrak{a}(y) \otimes 1_{l^2(I)})\mathcal{V} = \mathcal{V}y$;
(ii) For all $n \in N$, $(1 {}_b\otimes_\alpha{}_N \hat{\alpha}(n) \otimes 1_{l^2(I)})\mathcal{V} = \mathcal{V}a(n)$.

Proof. Let $(\eta_j)_{j\in J}$ be an orthogonal (b, ν^o) basis of H_{ψ_1}; we have, for any i, j and $x \in \mathfrak{N}_{\psi_1}$:

$$[(\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)]^*[(\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)] = (\omega_{\xi_i} {}_b*_\alpha{}_N id)[\mathfrak{a}(x^*)(\theta^{b,\nu^o}(\eta_j, \eta_j) {}_b\otimes_\alpha{}_N 1)\mathfrak{a}(x)]$$

and therefore

$$\Phi([(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]^*[(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]) \leq \Phi[(\omega_{\xi_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x^*x)] = \omega_{\xi_i} \circ T_{\mathfrak{a}}(x^*x) \leq \psi_1(x^*x).$$

So, for any i, j and $x \in \mathfrak{N}_{\psi_1}$, $(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)$ belongs to $\mathfrak{N}_\Phi$; moreover, we have

$$\Phi([\sum_j(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]^*[\sum_j(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]) = \omega_{\xi_i} \circ T_{\mathfrak{a}}(x^*x)$$

and therefore

$$\sum_i \Phi([\sum_j(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]^*[\sum_j(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]) = \psi_1(x^*x)$$

which proves that we can define now $\mathcal{V}$, for all $x \in \mathfrak{N}_{\psi_1}$ by

$$\mathcal{V}\Lambda_{\psi_1}(x) = \oplus_i \sum_j \eta_j \underset{\nu}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)).$$

As, for $x \in \mathfrak{N}_{\psi_1}$, we have $\|\mathcal{V}\Lambda_{\psi_1}(x)\|^2 = \psi_1(x^*x)$, we can extend $\mathcal{V}$ to an isometry from H_{ψ_1} into $\oplus_{i\in I}(H_{\psi_1} \underset{\nu}{{}_b\otimes_\alpha} H) = (H_{\psi_1} \underset{\nu}{{}_b\otimes_\alpha} H) \otimes l^2(I)$. Let now y be in A; we have

$$\begin{aligned}(\mathfrak{a}(y) \otimes 1_{l^2(I)})\mathcal{V}\Lambda_{\psi_1}(x) &= \oplus_i \sum_j \mathfrak{a}(y)(\eta_j \underset{\nu}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)))\\ &= \oplus_i \sum_j \sum_k \eta_k \underset{N}{{}_b\otimes_\alpha} (\omega_{\eta_k,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(y)\Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))\\ &= \oplus_i \sum_j \sum_k \eta_k \underset{N}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\eta_k,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(y)(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))\\ &= \oplus_i \sum_k \eta_k \underset{N}{{}_b\otimes_\alpha} \Lambda_\Phi(\omega_{\xi_i,\eta_k} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(yx)) = \mathcal{V}\Lambda_{\psi_1}(yx)\end{aligned}$$

and therefore $(\mathfrak{a}(y) \otimes 1_{l^2(I)})\mathcal{V} = \mathcal{V}y$.

Let n be in the Tomita algebra of the weight ν; we have

$$\begin{aligned}(1 \underset{N}{{}_b\otimes_\alpha} \hat{\alpha}(n)) \otimes 1_{l^2(I)})\mathcal{V}\Lambda_{\psi_1}(x) &= \oplus_i \sum_j \eta_j \underset{\nu}{{}_b\otimes_\alpha} \hat{\alpha}(n)\Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))\\ &= \oplus_i \sum_j \eta_j \underset{\nu}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)\beta(\sigma^\nu_{-i/2}(n))\\ &= \oplus_i \sum_j \eta_j \underset{\nu}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(xb(\sigma^\nu_{-i/2}(n))))\\ &= \mathcal{V}\Lambda_{\psi_1}(xb(\sigma^\nu_{-i/2}(n)) = \mathcal{V}a(n)\Lambda_{\psi_1}(x)\end{aligned}$$

which, by continuity, remains true for all $n \in N$. ■

3.2. Theorem. *Let $(b, \mathfrak{a})$ be an integrable action of a measured quantum groupoid on a von Neumann algebra A; let ψ_0 be a normal faithful semi-finite weight on $A^{\mathfrak{a}}$, and $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$ be the lifted normal semi-finite faithful weight on A. Then the weight ψ_1 is δ-invariant, and has the density property, in the sense of 2.4.*

Proof. Let's use the notations of 3.1; let x be in $\mathfrak{N}_{\psi_1}$, and $\eta \in D(_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$, such that $\delta^{1/2}\eta$ belongs to $D(H_\beta, \nu^o)$; we have, using the isometry $\mathcal{V}$ and ([E5], 8.2):

$$\begin{aligned}\|\Lambda_{\psi_1}(x) {}_b\otimes_\alpha{}_{\nu} \delta^{1/2}\eta\|^2 &= \oplus_i \| \sum_j \eta_j {}_b\otimes_\alpha{}_\nu \Lambda_\Phi((\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)) {}_{\hat\alpha}\otimes_\beta{}_{\nu^o} \delta^{1/2}\eta\|^2 \\ &= \oplus \|\Lambda_\Phi(\alpha(< \eta_j, \eta_j >_{b,\nu^o})(\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)) {}_{\hat\alpha}\otimes_\beta{}_{\nu^o} \delta^{1/2}\eta\|^2 \\ &= \oplus \|\Lambda_\Phi((\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)) {}_{\hat\alpha}\otimes_\beta{}_{\nu^o} \delta^{1/2}\eta\|^2 \\ &= \sum_i \Phi(\sum_j (id {}_b*_\alpha{}_N \omega_\eta)\Gamma[(\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)^*(\omega_{\xi_i,\eta_j} {}_b*_\alpha{}_N id)\mathfrak{a}(x)] \\ &= \sum_i \Phi[(id {}_b*_\alpha{}_N \omega_\eta)\Gamma(\omega_{\xi_i} {}_b*_\alpha{}_N *id)\mathfrak{a}(x^*] \\ &= \sum_i \omega_{\xi_i} \circ T_\mathfrak{a}(id {}_b*_\alpha{}_N \omega_\eta)\mathfrak{a}(x^*x)) = \psi_1[(id {}_b*_\alpha{}_N \omega_\eta)\mathfrak{a}(x^*x)]\end{aligned}$$

which proves that ψ_1 is δ-invariant; moreover, if we take the Tomita algebra relative to the weight ψ_1 and the operator-valued weight $T_\mathfrak{a}$, we get that the weight ψ_1 has the density property. ∎

3.3. Proposition. *Let $\mathfrak{G}_i = (N_i, M_i, \alpha_i, \beta_i, \Gamma_i, T_i, T_i', \nu_i)$ $(i = 1, 2)$ be two measured quantum groupoids, $(b, \mathfrak{a})$ an action of $\mathfrak{G}_1$ on a von Neumann algebra A, and $(a, \mathfrak{b})$ a left action of $\mathfrak{G}_2$ on A; let us suppose that the actions $\mathfrak{a}$ and $\mathfrak{b}$ commute. Then:*

(i) The operator-valued weight $T_\mathfrak{b}$ from A onto $A^\mathfrak{b}$ satisfies:

$$(T_\mathfrak{b} {}_b*_{\alpha_1}{}_{\nu_1} id)\mathfrak{a} = \mathfrak{a} \circ T_\mathfrak{b}.$$

(ii) If $\mathfrak{b}$ is integrable and if $A^\mathfrak{b} = b(N_1)$, the weight $\phi_1 = \nu_1 \circ b^{-1} \circ T_\mathfrak{b}$ is a normal semi-finite faithful weight on A, invariant under the action $\mathfrak{a}$, $\delta_{\mathfrak{G}_2}$-invariant under the action $\mathfrak{b}$, and has the density property.

Proof. Result (i) is straightforward, using the definition of commuting actions. With the hypothesis of (ii), we get that $T_\mathfrak{b}$ is a normal semi-finite faithful operator-valued weight from A onto $b(N_1)$, that ϕ_1 is a normal semi-finite faithful weight on A which satisfies, for all $x \in A^+$:

$$(T_b {}_b*_{\alpha_1}{}_{\nu_1} id)\mathfrak{a}(x) = \mathfrak{a} \circ T_\mathfrak{b}(x) = 1 {}_b\otimes_{\alpha_1}{}_{N_1} \beta_1 \circ b^{-1} \circ T_\mathfrak{b}(x), \quad (\phi_1 {}_b*_{\alpha_1}{}_{\nu_1} id)\mathfrak{a}(x) = \beta_1 \circ b^{-1} \circ T_\mathfrak{b}(x),$$

from which we get that ϕ_1 is invariant by $\mathfrak{a}$. On the other hand, ϕ_1 is $\delta_{\mathfrak{G}_2}$-invariant under the action $\mathfrak{b}$, and has the density property by 3.2. ∎

3.4. Lemma. *Let $\mathfrak{G}$ be a measured quantum groupoid; let V be a corepresentation of $\mathfrak{G}$ on an N-N bimodule ${}_a\mathfrak{H}_b$ ([E5], 5.1), and let $(b, \mathfrak{a})$ the canonical action implemented by V on $a(N)'$ by $\mathfrak{a}(x) = V(x {}_a\otimes_\beta{}_{N^o} 1)V^*$ ([E5], 6.6). Then*

$$(a(N)')^\mathfrak{a} = a(N)' \cap b(N)' \cap \{(id * \omega_{\xi,\eta})(V), \xi \in D(_\alpha H, \nu), \eta \in D((H_\phi)_\beta, \nu^o)\}'.$$

Proof. Clear. ∎

3.5. Lemma. *Let $(b, \mathfrak{a})$ be an integrable action of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A, and let ψ_0 be a normal semi-finite faithful weight on $A^{\mathfrak{a}}$, and $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$ be the normal semi-finite faithful lifted weight on A; let V_{ψ_1} be the standard implementation of $\mathfrak{a}$ defined in 2.4: let $s(A^{\mathfrak{a}})' = J_{\psi_1}(A^{\mathfrak{a}})'J_{\psi_1}$ the basic construction made from the inclusion $A^{\mathfrak{a}} \subset A$ (cf. 2.6). Then*

$$s(A^{\mathfrak{a}})' = (A \cup a(N) \cup \{(id * \omega_{\eta,\xi})(V_{\psi_1}^*), \xi \in D(_\alpha H, \nu), \eta \in D(H_\beta, \nu^o)\})''.$$

Proof. Using 3.4, we get $A^{\mathfrak{a}} = A \cap b(N)' \cap \{(id*\omega_{\xi,\eta})(V_{\psi_1}), \xi \in D(_\alpha H, \nu), \eta \in D((H_\phi)_\beta, \nu^o)\}'$, and therefore $J_{\psi_1} A^{\mathfrak{a}} J_{\psi_1}$ is equal to

$$A' \cap a(N)' \cap J_{\psi_1}\{(id * \omega_{\xi,\eta})(V_{\psi_1}), \xi \in D(_\alpha H, \nu), \eta \in D((H_\phi)_\beta, \nu^o)\}' J_{\psi_1}.$$

As $V_{\psi_1}(J_{\psi_1} \underset{N}{{}_b\otimes_\alpha} \hat{J}) = (J_{\psi_1} \underset{N}{{}_b\otimes_\alpha} \hat{J})V_{\psi_1}^*$, we have

$$J_{\psi_1}(id * \omega_{\xi,\eta})(V_{\psi_1})J_{\psi_1} = (id * \omega_{\hat{J}\xi,\hat{J}\eta})(V_{\psi_1}^*)$$

and we get

$$J_{\psi_1} A^{\mathfrak{a}} J_{\psi_1} = A' \cap a(N)' \cap \{(id * \omega_{\eta,\xi})(V_{\psi_1}^*), \xi \in D(_\alpha H, \nu), \eta \in D(H_\beta, \nu^o)\}'$$

from which we get the result. ∎

3.6. Theorem. *Let $(b, \mathfrak{a})$ be an integrable action of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A, let V_{ψ_1} be the standard implementation of $\mathfrak{a}$, as defined in 2.5; let us denote by r the injection of $A^{\mathfrak{a}}$ into A, and let us write $s(x) = J_{\psi_1} r(x)^* J_{\psi_1}$ for any $x \in A^{\mathfrak{a}}$. Then $s(A^{\mathfrak{a}})'$ is the basic construction made from the inclusion $A^{\mathfrak{a}} \subset A$ (cf. 2.6). Moreover, there exists a normal surjective $*$-homomorphism $\pi_{\mathfrak{a}}$ from the crossed product $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ onto $s(A^{\mathfrak{a}})'$, called the Galois homomorphism associated to the integrable action $(b, \mathfrak{a})$, such that, for all $x \in A$, $n \in N$, $\xi \in D(_\alpha H, \nu)$, $\eta \in D(H_\beta, \nu^o)$:*

$$\pi_{\mathfrak{a}}(\mathfrak{a}(x)) = x, \quad \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{\alpha}(n)) = a(n),$$

$$\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} (\omega_{\eta,\xi} * id)[(W^o)^*]) = (id * \omega_{\eta,\xi})(V_{\psi_1}).$$

For simplification, we shall write $\mu(m) = \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} m)$, for any $m \in \widehat{M}'$, and we obtain this way a representation of $\widehat{M}'$ on $\mathcal{L}(H_{\psi_1})$.

Proof. Let us use the notations of 3.1 and 3.2; let's suppose that η belongs also to $\mathcal{D}(\delta^{-1/2})$ and that $\delta^{-1/2}\eta$ belongs to $D(_\alpha H, \nu)$. Then

$$(1 \underset{N}{{}_b\otimes_\alpha} (\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})]) \otimes 1_{l^2(I)})\mathcal{V}\Lambda_{\psi_1}(x) =$$

$$\oplus_i \sum_j \eta_j \underset{\nu}{{}_b\otimes_\alpha} (\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})])\Lambda_\Phi((\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))$$

which, using ([E5], 3.10(ii) applied to $\mathfrak{G}^o$, 3.8(vi)), and the identification of $H_{\Phi\circ R}$ with H made in 2.2) is equal to

$$\oplus_i \sum_j \eta_j \underset{\nu}{{}_b\otimes_\alpha} \Lambda_\Phi((id \underset{N}{{}_\beta*_\alpha} \omega_{\delta^{-1/2}\eta,\xi})\Gamma[(\omega_{\xi_i,\eta_j} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)])$$

or, to

$$\oplus_i \sum_j \eta_j \ {}_b\underset{\nu}{\otimes}_\alpha \ \Lambda_\Phi((\omega_{\xi_i,\eta_j} \ {}_b\underset{N}{*}_\alpha \ id)\mathfrak{a}[(id \ {}_\beta\underset{N}{*}_\alpha \ \omega_{\delta^{-1/2}\eta,\xi})\mathfrak{a}(x)])$$

which is, using ([E5], 8.4), equal to

$$\mathcal{V}\Lambda_{\psi_1}[(id \ {}_\beta\underset{N}{*}_\alpha \ \omega_{\delta^{-1/2}\eta,\xi})\mathfrak{a}(x)]) = \mathcal{V}(id * \omega_{\eta,\xi})(V_{\psi_1})\Lambda_{\psi_1}(x)$$

from which, by density, we get that

$$(1 \ {}_b\underset{N}{\otimes}_\alpha \ (\omega_{\eta,\xi} * id)[(\hat{J} \ {}_\beta\underset{N}{\otimes}_\alpha \ \hat{J})W^*(\hat{J} \ {}_\beta\underset{N}{\otimes}_{\hat{\alpha}} \ \hat{J})]) \otimes 1_{l^2(I)})\mathcal{V} = \mathcal{V}(id * \omega_{\eta,\xi})(V_{\psi_1})$$

which, by density and continuity, remains true for any η in $D(H_\beta, \nu^o)$. Using now 2.2, we get that the weak closure of the linear span of all operators of the form

$$(\omega_{\eta,\xi} * id)[(\hat{J} \ {}_\beta\underset{N}{\otimes}_\alpha \ \hat{J})W^*(\hat{J} \ {}_\beta\underset{N}{\otimes}_{\hat{\alpha}} \ \hat{J})]$$

for all $\xi \in D({}_\alpha H, \nu)$, $\eta \in D(H_\beta, \nu^o)$, is equal to the von Neumann algebra $\widehat{M}'$; therefore, we get that, for any $y \in \widehat{M}'$, the image of $(1 \ {}_b\underset{N}{\otimes}_\alpha \ y \otimes 1_{l^2(I)})\mathcal{V}$ is included in the image of $\mathcal{V}$, which means that $\mathcal{V}\mathcal{V}^*(1 \ {}_b\underset{N}{\otimes}_\alpha \ y \otimes 1_{l^2(I)})\mathcal{V} = (1 \ {}_b\underset{N}{\otimes}_\alpha \ y \otimes 1_{l^2(I)})\mathcal{V}$; therefore, we have, for any $y \in \widehat{M}'$, $\mathcal{V}\mathcal{V}^*(1 \ {}_b\underset{N}{\otimes}_\alpha \ y \otimes 1_{l^2(I)})\mathcal{V}\mathcal{V}^* = (1 \ {}_b\underset{N}{\otimes}_\alpha \ y \otimes 1_{l^2(I)})\mathcal{V}\mathcal{V}^*$, which proves that $\mathcal{V}\mathcal{V}^*$ commutes with $1 \ {}_b\underset{N}{\otimes}_\alpha \ \widehat{M}' \otimes 1_{l^2(I)}$. Using 3.1, we easily get that $\mathcal{V}\mathcal{V}^*$ commutes also with $\mathfrak{a}(A) \otimes 1_{l^2(I)}$, and therefore that it commutes with $A \rtimes_\mathfrak{a} \mathfrak{G} \otimes 1_{l^2(I)}$. Let us write now, for any $z \in A \rtimes_\mathfrak{a} \mathfrak{G}$:

$$\pi_\mathfrak{a}(z) = \mathcal{V}^*(z \otimes 1_{l^2(I)})\mathcal{V}.$$

Thanks to this commutation property, $\pi_\mathfrak{a}$ is a $*$-homomorphism from $A \rtimes_\mathfrak{a} \mathfrak{G}$ into $\mathcal{L}(H_{\psi_1})$. Using now 3.5, we get that the image of $\pi_\mathfrak{a}$ is $s(A^\mathfrak{a})'$. ∎

3.7. Lemma. *With the notations of 3.6, we have, for all $m \in \widehat{M}'$:*

$$\pi_\mathfrak{a}(1 \ {}_b\underset{N}{\otimes}_\alpha \ \hat{R}^c(m)) = J_{\psi_1}\pi_\mathfrak{a}(1 \ {}_b\underset{N}{\otimes}_\alpha \ m^*)J_{\psi_1}.$$

Proof. Let $\xi \in D({}_\alpha H, \nu)$, $\eta \in D(H_\beta, \nu^o)$; using 3.6, we get that

$$J_{\psi_1}\pi_\mathfrak{a}(1 \ {}_b\underset{N}{\otimes}_\alpha \ (\omega_{\eta,\xi} * id)[(\hat{J} \ {}_\beta\underset{N}{\otimes}_\alpha \ \hat{J})W^*(\hat{J} \ {}_\beta\underset{N}{\otimes}_{\hat{\alpha}} \ \hat{J})])^*J_{\psi_1}$$

is equal to $J_{\psi_1}(id * \omega_{\eta,\xi})(V_{\psi_1})^*J_{\psi_1}$, which (2.4) is equal to $(i * \omega_{\hat{J}\xi,\hat{J}\eta})(V_{\psi_1})$, and using 3.6 again, is equal to

$$\pi_\mathfrak{a}(1 \ {}_b\underset{N}{\otimes}_\alpha \ (\omega_{\hat{J}\xi,\hat{J}\eta} * id)[(\hat{J} \ {}_\beta\underset{N}{\otimes}_\alpha \ \hat{J})W^*(\hat{J} \ {}_\beta\underset{N}{\otimes}_{\hat{\alpha}} \ \hat{J})])$$

which is $\pi_\mathfrak{a}((1 \ {}_b\underset{N}{\otimes}_\alpha \ \hat{J}(\omega_{\xi,\eta} * id)(W^*)\hat{J})$, and using ([E5], 3.11(iii)), is equal to

$$\pi_\mathfrak{a}(1 \ {}_b\underset{N}{\otimes}_\alpha \ J(\omega_{\xi,\eta} * id)[(\hat{J} \ {}_\alpha\underset{N^o}{\otimes}_{\hat{\beta}} \ \hat{J})W(\hat{J} \ {}_\alpha\underset{N^o}{\otimes}_{\hat{\beta}} \ \hat{J})]J)$$

which is

$$\pi_\mathfrak{a}(1 \ {}_b\underset{N}{\otimes}_\alpha \ J(\omega_{\eta,\xi} * id)[(\hat{J} \ {}_\beta\underset{N}{\otimes}_\alpha \ \hat{J})W^*(\hat{J} \ {}_\beta\underset{N}{\otimes}_{\hat{\alpha}} \ \hat{J})])^*J)$$

which is $\pi_{\mathfrak{a}}(1 \,{}_b\underset{N}{\otimes}{}_\alpha\, \hat{R}^c[(\omega_{\eta,\xi} * id)[(\hat{J} \,{}_\beta\underset{N}{\otimes}{}_\alpha\, \hat{J})W^*(\hat{J} \,{}_\beta\underset{N}{\otimes}{}_{\hat{\alpha}}\, \hat{J})]])$; we get then the result by density. ∎

3.8. Theorem. *Let $(b, \mathfrak{a})$ an integrable action of $\mathfrak{G}$ on a von Neumann algebra A, $\pi_{\mathfrak{a}}$ the Galois homomorphism associated by 3.6; let ψ_0 be a normal semi-finite faithful weight on $A^{\mathfrak{a}}$, and $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$; let a be the representation of N on H_{ψ_1} defined, for $n \in N$, by*

$$a(n) = J_{\psi_1} b(n^*) J_{\psi_1}.$$

Let us write r for the injection of $A^{\mathfrak{a}}$ into A, and s for the antirepresentation of $A^{\mathfrak{a}}$ on H_{ψ_1} given, for $a \in A^{\mathfrak{a}}$, by $s(a) = J_{\psi_1} r(a^*) J_{\psi_1}$. Then:

(i) There exists an isometry G from $H_{\psi_1} \,{}_s\underset{\psi_0}{\otimes}{}_r\, H_{\psi_1}$ into $H_{\psi_1} \,{}_b\underset{\nu}{\otimes}{}_\alpha\, H$, such that

$$G(\Lambda_{\psi_1}(x) \,{}_s\underset{\psi_0}{\otimes}{}_r\, \zeta) = \sum_i e_i \,{}_b\underset{\nu}{\otimes}{}_\alpha\, \Lambda_\Phi[(\omega_{\zeta,e_i} \,{}_b\underset{N}{*}{}_\alpha\, id)\mathfrak{a}(x)]$$

for all x in $\mathfrak{N}_{T_{\mathfrak{a}}} \cap \mathfrak{N}_{\psi_1}$, $\zeta \in D((H_{\psi_1})_b, \nu^o)$, and for all (b, ν^o)-orthogonal basis $(e_i)_{i\in I}$ of H_{ψ_1}. Moreover, for any $n \in N$, $a \in A^{\mathfrak{a}}$, we have

$$G(b(n) \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, 1) = (1 \,{}_b\underset{N}{\otimes}{}_\alpha\, \beta(n))G,$$

$$G(1 \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, b(n)) = (1 \,{}_b\underset{N}{\otimes}{}_\alpha\, \hat{\beta}(n))G,$$

$$G(1 \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, a(n)) = (a(n) \,{}_b\underset{A^{\mathfrak{a}}}{\otimes}{}_\alpha\, 1)G,$$

$$G(r(a) \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, 1) = (r(a) \,{}_b\underset{A^{\mathfrak{a}}}{\otimes}{}_\alpha\, 1)G,$$

$$G(1 \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, s(a)) = (s(a) \,{}_b\underset{A^{\mathfrak{a}}}{\otimes}{}_\alpha\, 1)G.$$

(ii) For any $e \in \mathfrak{N}_\Phi$, we have

$$(1 \,{}_b\underset{N}{\otimes}{}_\alpha\, J_\Phi e J_\Phi)G(\Lambda_{\psi_1}(x) \,{}_s\underset{\psi_0}{\otimes}{}_r\, \zeta) = \mathfrak{a}(x)(\zeta \,{}_b\underset{\nu}{\otimes}{}_\alpha\, J_\phi \Lambda_\Phi(e)).$$

(iii) For all ζ' in $D((H_{\psi_1})_b, \nu^o)$, $(\omega_{\zeta,\zeta'} \,{}_b\underset{N}{*}{}_\alpha\, id)\mathfrak{a}(x)$ belongs to $\mathfrak{N}_\Phi$, and we have

$$\Lambda_\Phi[(\omega_{\zeta,\zeta'} \,{}_b\underset{N}{*}{}_\alpha\, id)\mathfrak{a}(x)] = (\omega_{\Lambda_{\psi_1}(x),\zeta'} * id)(G)\zeta.$$

(iv) For any $a' \in A$, $Y \in \widehat{M}'$, we have

$$\mathfrak{a}(a')G = G(a' \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, 1), \qquad (1 \,{}_b\underset{N}{\otimes}{}_\alpha\, Y)G = G(\pi_{\mathfrak{a}}(1 \,{}_b\underset{N}{\otimes}{}_\alpha\, Y)_s\otimes_r 1).$$

(v) The projection GG^* commutes with $A \rtimes_{\mathfrak{a}} \mathfrak{G}$, and for any $X \in A \rtimes_{\mathfrak{a}} \mathfrak{G}$, we have

$$\pi_{\mathfrak{a}}(X) \,{}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r\, 1_{H_{\psi_1}} = G^* X G.$$

(vi) For any $t \in \mathbb{R}$, we have $\Delta^{it}_{\tilde{\psi}_1} G = G \Delta^{it}_{\psi_2}$.
(vii) For all $t \in \mathbb{R}$, we have

$$\tilde{G}(\Delta^{it}_{\psi_1} \,{}_s\underset{\psi_0}{\otimes}{}_r\, \Delta^{it}_{\psi_1}) = ((\delta \Delta_{\widehat{\Phi}})^{-it} \,{}_\alpha\underset{\nu^o}{\otimes}{}_b\, \Delta^{it}_{\psi_1})\tilde{G}.$$

Proof. As
$$[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]^*[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]\leq(\omega_\zeta\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x^*x)$$
we get that
$$\Phi([(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]^*[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)])\leq\Phi[(\omega_\zeta\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x^*x)]=\omega_\zeta\circ T_\mathfrak{a}(x^*x)$$
and we get that $[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]$ belongs to $\mathfrak{N}_\Phi$; defining G by the formula given in (i), we obtain, for x, x', in $\mathfrak{N}_{T_\mathfrak{a}}\cap\mathfrak{N}_{\psi_1}$, ζ, ζ' in $D(H_{\psi_1})_b,\nu^o)$, that
$$(G(\Lambda_{\psi_1}(x)\ {}_s\otimes_r{}_{\psi_0}\ \zeta)|G(\Lambda_{\psi_1}(x')\ {}_s\otimes_r{}_{\psi_0}\ \zeta'))$$
is equal to
$$\sum_i(\Lambda_\Phi[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]|\Lambda_\Phi[(\omega_{\zeta',e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x')])=(T_\mathfrak{a}(x'^*x)\zeta|\zeta')$$
or, to $(\Lambda_{\psi_1}(x)\ {}_s\otimes_r{}_{\psi_0}\ \zeta|\Lambda_{\psi_1}(x')\ {}_s\otimes_r{}_{\psi_0}\ \zeta')$ which implies that this formula defines an isometry which can be extended by continuity to $H_{\psi_1}\ {}_b\otimes_a{}_{\psi_0}\ H_{\psi_1}$ and does not depend upon the choice of the basis, which is the first result of (i). If n is a unitary in N, $(a(n)e_i)_{i\in I}$ is another orthogonal (b,ν^o)-basis of H_{ψ_1}, and the independence of G from the basis gives the second and the third formula of (i); let us remark that, for all $n\in N$, $b(n)$ belongs to A and therefore commutes with s, and that it commutes with $A^\mathfrak{a}$, and therefore to r; moreover, as $\mathfrak{a}(b(n))=1\ {}_b\otimes_\alpha{}_N\ \beta(n)$, we easily get the first formula linking G with $b(n)$. If we suppose now that n is analytic with respect to ν, we obtain
$$\begin{aligned}G(1\ {}_s\otimes_r{}_{A^\mathfrak{a}}\ b(n))(\Lambda_{\psi_1}(x)\ {}_s\otimes_r{}_{\psi_0}\ \zeta)&=\sum_i e_i\ {}_b\otimes_\alpha{}_\nu\ \Lambda_\Phi[(\omega_{b(n)\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]\\&=\sum_i e_i\ {}_b\otimes_\alpha{}_\nu\ \Lambda_\Phi[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)\alpha(\sigma^\nu_{-i/2}(n))]\\&=\sum_i e_i\ {}_b\otimes_\alpha{}_\nu\ J_\Phi\alpha(n^*)J_\Phi\Lambda_\Phi[(\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)]\\&=(1\ {}_b\otimes_\alpha{}_N\ \hat\beta(n))G(\Lambda_{\psi_1}(x)\ {}_s\otimes_r{}_{\psi_0}\ \zeta)\end{aligned}$$
which, by continuity, finishes the proof of (i). We then obtain
$$\begin{aligned}(1\ {}_b\otimes_\alpha{}_N\ J_\Phi eJ_\Phi)G(\Lambda_{\psi_1}(x)\ {}_s\otimes_r{}_{\psi_0}\ \zeta)&=\sum_i e_i\ {}_b\otimes_\alpha{}_\nu\ J_\Phi eJ_\Phi\Lambda_\Phi((\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ \mathfrak{a}(x))\\&=\sum_i e_i\ {}_b\otimes_\alpha{}_\nu\ (\omega_{\zeta,e_i}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)J_\Phi\Lambda_\Phi(e)\\&=\mathfrak{a}(x)(\zeta\ {}_b\otimes_\alpha{}_\nu\ J_\Phi\Lambda_\Phi(e))\end{aligned}$$
which is (ii). We then get
$$(\omega_{\zeta,\zeta'}\ {}_b*_\alpha{}_N\ id)\mathfrak{a}(x)J_\Phi\Lambda_\Phi(e)=J_\Phi eJ_\Phi(id*\omega_{\Lambda_{\psi_1}(x),\zeta'})(G)\zeta$$
from which we deduce (iii).

On the other hand, using again (ii), we get

$$\begin{aligned}(1 \underset{N}{{}_b\otimes_\alpha} J_\Phi e J_\Phi)\mathfrak{a}(a')G(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta) &= \mathfrak{a}(a')(1 \underset{N}{{}_b\otimes_\alpha} J_\Phi e J_\Phi)G(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta)\\ &= \mathfrak{a}(a')\mathfrak{a}(x)(\zeta \underset{\nu}{{}_b\otimes_\alpha} J_\phi \Lambda_\Phi(e))\\ &= \mathfrak{a}(a'x)(\zeta \underset{\nu}{{}_b\otimes_\alpha} J_\phi \Lambda_\Phi(e))\\ &= (1 \underset{N}{{}_b\otimes_\alpha} J_\Phi e J_\Phi)G(\Lambda_{\psi_1}(a'x) \underset{\psi_0}{{}_s\otimes_r} \zeta)\\ &= (1 \underset{N}{{}_b\otimes_\alpha} J_\Phi e J_\Phi)G(a' \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta)\end{aligned}$$

from which we get, by continuity:

$$(1 \underset{N}{{}_b\otimes_\alpha} J_\Phi e J_\Phi)\mathfrak{a}(a')(G) = (1 \underset{N}{{}_b\otimes_\alpha} J_\Phi e J_\Phi)G(a' \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)$$

and making e go weakly to 1, we get the first result of (iv).

Let $\xi \in D({}_\alpha H, \nu)$, $\eta \in D(H_\beta, \nu^o) \cap \mathcal{D}(\delta^{-1/2})$, such that $\delta^{-1/2}\eta$ belongs to $D({}_\alpha H, \nu)$; using (iii), we get that

$$(\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})](\omega_{\Lambda_{\psi_1}(x),\zeta'} * id)(G)\zeta$$

is equal to

$$(\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})]\Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]$$

which, using ([E5], 4.3), is equal to

$$\Lambda_\Phi[(id \underset{N}{{}_\beta*_\alpha} \omega_{\delta^{-1/2}\eta,\xi})\Gamma((\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))] = \Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}((id \underset{N}{{}_\beta*_\alpha} \omega_{\delta^{-1/2}\eta,\xi})\mathfrak{a}(x)]$$

which, using (iii) again, and ([E5], 8.4), is equal to

$$(\omega_{\Lambda_{\psi_1}[(id \underset{N}{{}_\beta*_\alpha}\omega_{\delta^{-1/2}\eta,\xi})\mathfrak{a}(x)],\zeta'} * id)(G)\zeta = (\omega_{(id*\omega_{\eta,\xi})(V_\psi)\Lambda_{\psi_1}(x),\zeta'} * id)(G)\zeta$$

from which we get, by continuity:

$$(\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})](\omega_{\Lambda_{\psi_1}(x),\zeta'} * id)(G) = (\omega_{(id*\omega_{\eta,\xi})(V_\psi)\Lambda_{\psi_1}(y),\zeta'} * id)(G)$$

which, by continuity and density, remains true for any $\eta \in D(H_\beta, \nu^o)$.

Using 3.6, we get, by continuity and density:

$$\begin{aligned}(1 \underset{N}{{}_b\otimes_\alpha} (\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})])G\\ = G(\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} (\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})]) \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1).\end{aligned}$$

Using now 2.2, we get that the weak closure of the linear span of all operators of the form $(\omega_{\eta,\xi} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})]$, for all $\xi \in D({}_\alpha H, \nu)$, $\eta \in D(H_\beta, \nu^o)$, is equal to the von Neumann algebra $\widehat{M}'$; therefore, we get, for all $Y \in \widehat{M}'$, that

$$(1 \underset{N}{{}_b\otimes_\alpha} Y)G = G(\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} Y) \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)$$

which finishes the proof of (iv).

From (iv), we get that

$$(1 \underset{N}{{}_b\otimes_\alpha} Y)GG^* = G(\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} Y) \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)G^*$$

and that the projection GG^* commutes with $1 \underset{N}{{}_b\otimes_\alpha} \widehat{M}'$; using same arguments, we get that GG^* commutes with $\mathfrak{a}(A)$, and therefore it commutes with $A \rtimes_{\mathfrak{a}} \mathfrak{G}$. So, we get that the map which sends $Z \in A \rtimes_{\mathfrak{a}} \mathfrak{G}$ on G^*ZG is a $*$-homomorphism, which is equal to $\pi_{\mathfrak{a}}(Z)$ for any $Z = 1 \underset{N}{{}_b\otimes_\alpha} Y$, with $Y \in \widehat{M}'$; using 3.6, we get that the same property holds if $Z = \mathfrak{a}(a')$; therefore, it is true for any $Z \in A \rtimes_{\mathfrak{a}} \mathfrak{G}$, which is (v).

Let us remark that, because ψ_1 is δ-invariant (3.2), we have ([E6], 3.2(ii)):

$$\Delta^{it}_{\tilde{\psi}_1} = \Delta^{it}_{\psi_1} \underset{N}{{}_b\otimes_\alpha} (\delta\Delta_{\widehat{\Phi}})^{-it}$$

where this relative tensor product of operators means that it is possible to define a bounded operator with natural values on elementary tensors. With the same definition of relative tensors of operators, we have (2.6) $\Delta^{it}_{\psi_2} = \Delta^{it}_{\psi_1} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \Delta^{it}_{\psi_1}$. Using these remaks, we get, using (iii), for any x in $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_{\mathfrak{a}}}$ and ζ, ζ' in $D((H_{\psi_1})_b, \nu^o)$, that

$$\begin{aligned}(\omega_{\Lambda_{\psi_1}(x),\zeta'} * id)(\Delta^{it}_{\tilde{\psi}_1} G \Delta^{-it}_{\psi_2})\zeta &= (\delta\Delta_{\widehat{\Phi}})^{-it}(\omega_{\Delta^{-it}_{\psi_1}\Lambda_{\psi_1}(x),\Delta^{-it}_{\psi_1}\zeta'} * id)(G)\Delta^{-it}_{\psi_1}\zeta \\ &= (\delta\Delta_{\widehat{\Phi}})^{-it}\Lambda_\Phi((\omega_{\Delta^{-it}_{\psi_1}\zeta,\Delta^{-it}_{\psi_1}\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(\sigma^{\psi_1}_{-t}(x))).\end{aligned}$$

As ψ_1 is δ-invariant, we have ([E5], 88(iii)), for all $t \in \mathbb{R}$ and $x \in A$:

$$\mathfrak{a}(\sigma^{\psi_1}_t(x)) = (\sigma^{\psi_1}_t \underset{N}{{}_b*_\alpha} \tau_{-t}\sigma^{\Phi\circ R}_{-t}\sigma^{\Phi}_t)\mathfrak{a}(x)$$

and therefore we have, using (iii):

$$(\omega_{\Lambda_{\psi_1}(x),\zeta'} * id)(\Delta^{it}_{\tilde{\psi}_1} G \Delta^{-it}_{\psi_2})\zeta = (\delta\Delta_{\widehat{\Phi}})^{-it}\Lambda_\Phi(\tau_t\sigma^{\Phi\circ R}_t\sigma^{\Phi}_{-t}[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)])$$

which, using using ([E5], 3.8(vii) and (vi)) is equal to

$$\begin{aligned}&(\delta\Delta_{\widehat{\Phi}})^{-it}\lambda^{-t/2}P^{it}\Lambda_\Phi(\sigma^{\Phi\circ R}_t\sigma^{\Phi}_{-t}[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]) \\ &\qquad = (\delta\Delta_{\widehat{\Phi}})^{-it}\lambda^{-t/2}P^{it}\lambda^{t/2}\delta^{it}J_\Phi\delta^{it}J_\Phi\Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)])\end{aligned}$$

which, using ([E5], 3.10 (vii)) and again (iii), is equal to

$$\Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]) = (\omega_{\Lambda_{\psi_1}(x),\zeta'} * id)(G)\zeta$$

which gives (vi). As (vii) has been proved as well, this finishes the proof. ∎

3.9. Theorem. *Let* $(b, \mathfrak{a})$ *an integrable action of* $\mathfrak{G}$ *on a von Neumann algebra* A, $\pi_{\mathfrak{a}}$ *the Galois homomorphism associated by 3.6 from the crossed product* $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ *onto the von Neumann algebra* $s(A^{\mathfrak{a}})'$ *obtained by the basic construction made from the inclusion* $A^{\mathfrak{a}} \subset A$*; let* ψ_0 *be a normal semi-finite faithful weight on* $A^{\mathfrak{a}}$*, and* $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$*; let us define the representation* a *of* N *on* H_{ψ_1} *by, for* $n \in N$*:*

$$a(n) = J_{\psi_1} b(n^*) J_{\psi_1}.$$

Let us write r *for the injection of* $A^{\mathfrak{a}}$ *into* A*, and* s *for the antirepresentation of* $A^{\mathfrak{a}}$ *on* H_{ψ_1} *given, for* $a \in A^{\mathfrak{a}}$*, by* $s(a) = J_{\psi_1} r(a^*) J_{\psi_1}$*, and let* G *be the isometry from*

$H_{\psi_1} \,{}_s\underset{\psi_0}{\otimes}{}_r\, H_{\psi_1}$ into $H_{\psi_1} \,{}_b\underset{\nu}{\otimes}{}_\alpha\, H$ constructed in 3.8; let ψ_2 be the weight $\psi_1 \circ T_2$ where T_2 is the operator-valued weight from $s(A^{\mathfrak{a}})'$ onto A obtained by the basic construction (2.6). Then:

(i) For any $\xi \in D({}_\alpha H, \nu)$, $\eta \in D(H_\beta, \nu^o)$ such that $(\omega_{\xi,\eta} * id)(W^o)$ belongs to $\mathfrak{N}_{\widehat{\Phi}^c}$, and for any z in $\mathfrak{N}_{\psi_1}$, we have

$$G^*(\Lambda_{\psi_1}(z) \,{}_b\underset{N}{\otimes}{}_\alpha\, \Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)(W^o)] = \Lambda_{\psi_2}[\pi_{\mathfrak{a}}((1 \,{}_b\underset{N}{\otimes}{}_\alpha\, (\omega_{\xi,\eta} * id)(W^o))\mathfrak{a}(z)].$$

(ii) For any $X \in \mathfrak{N}_{\tilde{\psi_1}}$, $\pi_{\mathfrak{a}}(X)$ belongs to $\mathfrak{N}_{\psi_2}$, and

$$G^*\Lambda_{\tilde{\psi_1}}(X) = \Lambda_{\psi_2}(\pi_{\mathfrak{a}}(X)).$$

(iii) $G^* J_{\tilde{\psi_1}} = J_{\psi_2} G^*$.

(iv) The projection GG^* is equal to the support p of $\pi_{\mathfrak{a}}$; let us consider $\pi_{\mathfrak{a}}$ as an isomorphism between $(A \rtimes_{\mathfrak{a}} \mathfrak{G})_p$ and $s(A^{\mathfrak{a}})'$; then, this isomorphism sends the weight $\tilde{\psi}_{1p}$ to ψ_2.

Proof. Let x, y be in the Tomita algebra $\mathcal{T}_{\psi_1,T_1}$; we get that the scalar product

$$(G^*(\Lambda_{\psi_1}(z) \,{}_b\underset{N}{\otimes}{}_\alpha\, \Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}}\, \hat{J})W(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, \hat{J})]])|\Lambda_{\psi_1}(x) \,{}_s\underset{\psi_0}{\otimes}{}_r\, \Lambda_{\psi_1}(y))$$

is equal to

$$(\Lambda_{\psi_1}(z) \,{}_b\underset{N}{\otimes}{}_\alpha\, \Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}}\, \hat{J})W(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, \hat{J})]]|G(\Lambda_{\psi_1}(x) \,{}_s\underset{\psi_0}{\otimes}{}_r\, \Lambda_{\psi_1}(y)))$$

or, to

$$(\Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}}\, \hat{J})W(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, \hat{J})]]|(\omega_{\Lambda_{\psi_1}(x),\Lambda_{\psi_1}(z)} * id)(G)\Lambda_{\psi_1}(y))$$

which, using 3.8(iii), is equal to

$$(\Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}}\, \hat{J})W(\hat{J} \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, \hat{J})]]|\Lambda_{\Phi}((\omega_{\Lambda_{\psi_1}(y),\Lambda_{\psi_1}(z)} \,{}_b\underset{N}{*}{}_\alpha\, id)\mathfrak{a}(x)).$$

If, moreover, η belongs to $\mathcal{D}(\delta^{-1/2})$, we get, using 2.2, that it is equal to

$$(\xi|(\omega_{\Lambda_{\psi_1}(y),\Lambda_{\psi_1}(z)} \,{}_b\underset{N}{*}{}_\alpha\, id)\mathfrak{a}(x)\delta^{-1/2}\eta)$$

and if moreover $\delta^{-1/2}\eta$ belongs to $D({}_\alpha H, \nu)$, this is equal to

$$\begin{aligned}(\Lambda_{\psi_1}(z)|(id \,{}_b\underset{N}{*}{}_\alpha\, \omega_{\delta^{-1/2}\eta,\xi})\mathfrak{a}(x)\Lambda_{\psi_1}(y) &= \psi_1(y^*(id \,{}_b\underset{N}{*}{}_\alpha\, \omega_{\xi,\delta^{-1/2}\eta})\mathfrak{a}(x^*)z) \\ &= \psi_1((id \,{}_b\underset{N}{*}{}_\alpha\, \omega_{\xi,\delta^{-1/2}\eta})\mathfrak{a}(x^*)z\sigma^{\psi_1}_{-i}(y^*)) \\ &= (z\Lambda_{\psi_1}(\sigma^{\psi_1}_{-i}(y^*))|\Lambda_{\psi_1}((id \,{}_b\underset{N}{*}{}_\alpha\, \omega_{\delta^{-1/2}\eta,\xi})\mathfrak{a}(x))\end{aligned}$$

which, using 2.4 and the standard implementation associated to the weight ψ_1, thanks to 3.2, is equal to

$$(z\Lambda_{\psi_1}(\sigma^{\psi_1}_{-i}(y^*))|(id * \omega_{\eta,\xi})(V_{\psi_1})\Lambda_{\psi_1}(x))$$

and, by continuity, we get that the equality

$$(G^*(\Lambda_{\psi_1}(z) \underset{N}{{}_b\otimes_\alpha} \Lambda_{\widehat{\Phi}^c}[(\omega_{\xi,\eta} * id)[(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})]])|\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \Lambda_{\psi_1}(y))$$
$$= (z\Lambda_{\psi_1}(\sigma^{\psi_1}_{-i}(y^*))|(id * \omega_{\eta,\xi})(V_{\psi_1})\Lambda_{\psi_1}(x))$$
$$= ((id * \omega_{\xi,\eta})(V^*_{\psi_1})z\Lambda_{\psi_1}(\sigma^{\psi_1}_{-i}(y^*))|\Lambda_{\psi_1}(x))$$

remains true for the initial hypothesis on ξ and η. Therefore, we get, using 3.6, that this scalar product is equal to

$$(\pi_{\mathfrak{a}}[(1 \underset{N}{{}_b\otimes_\alpha} [(\omega_{\xi,\eta} * id)[(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})]]\mathfrak{a}(z)]\Lambda_{\psi_1}(\sigma^{\psi_1}_{-i}(y^*))|\Lambda_{\psi_1}(x))$$

which, using 2.6, is equal to

$$\Lambda_{\psi_2}(\pi_{\mathfrak{a}}[(1 \underset{N}{{}_b\otimes_\alpha} [(\omega_{\xi,\eta} * id)[(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \hat{J})W(\hat{J} \underset{N^o}{{}_\alpha\otimes_\beta} \hat{J})]]\mathfrak{a}(z)])|\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \Lambda_{\psi_1}(y))$$

which, by continuity and density, gives (i). By density, we get, using 2.5, that, for any $z \in \mathfrak{N}_{\psi_1}$ and $a \in \mathfrak{N}_{\widehat{\Phi}^c} \cap \mathfrak{N}_{\hat{T}^c}$, we have

$$G^*\Lambda_{\tilde{\psi_1}}((1 \underset{N}{{}_b\otimes_\alpha} a)\mathfrak{a}(z)) = G^*(\Lambda_{\psi_1}(z) \underset{N}{{}_b\otimes_\alpha} \Lambda_{\widehat{\Phi}^c}(a)) = \Lambda_{\psi_2}(\pi_{\mathfrak{a}}[(1 \underset{N}{{}_b\otimes_\alpha} a)\mathfrak{a}(z)]).$$

The linear set generated by elements of the form $(1 \underset{N}{{}_b\otimes_\alpha} a)\mathfrak{a}(z)$, with $a \in \mathfrak{N}_{\widehat{\Phi}^c} \cap \mathfrak{N}_{\hat{T}^c}$ and $z \in \mathfrak{N}_{\psi_1}$, is a core for $\Lambda_{\tilde{\psi_1}}$ ([E5], 10.8(ii)). So, if $X \in \mathfrak{N}_{\tilde{\psi_1}}$, there exists elements $a_i \in \mathfrak{N}_{\widehat{\Phi}^c} \cap \mathfrak{N}_{\hat{T}^c}$ and $z_i \in \mathfrak{N}_{\psi_1}$ such that the finite sums $\sum_i(1 \underset{N}{{}_b\otimes_\alpha} a_i)\mathfrak{a}(z_i)$ are weakly converging to X, and $\sum_i \Lambda_{\tilde{\psi_1}}[(1 \underset{N}{{}_b\otimes_\alpha} a_i)\mathfrak{a}(z_i)]$ is converging to $\Lambda_{\tilde{\psi_1}}(X)$. But then, on one hand, $\pi_{\mathfrak{a}}(\sum_i(1 \underset{N}{{}_b\otimes_\alpha} a_i)\mathfrak{a}(z_i))$ is converging to $\pi_{\mathfrak{a}}(X)$, and on the other hand, the finite sums $\sum_i \Lambda_{\psi_2}(\pi_{\mathfrak{a}}[(1 \underset{N}{{}_b\otimes_\alpha} a_i)\mathfrak{a}(z_i)]) = \sum_i G^*(\Lambda_{\tilde{\psi_1}}[(1 \underset{N}{{}_b\otimes_\alpha} a_i)\mathfrak{a}(z_i)]$ are converging. So, applying the closed graph theorem to the closed map Λ_{ψ_2}, we get (ii). If now $X \in \mathfrak{N}_{\tilde{\psi_1}} \cap \mathfrak{N}^*_{\tilde{\psi_1}}$, we get that $\pi_{\mathfrak{a}}(X)$ belongs to $\mathfrak{N}_{\psi_2} \cap \mathfrak{N}^*_{\psi_2}$, and that $G^*S_{\tilde{\psi_1}}\Lambda_{\tilde{\psi_1}}(X) = S_{\psi_2}G^*\Lambda_{\tilde{\psi_1}}(X)$. So, we have $G^*S_{\tilde{\psi_1}} \subset S_{\psi_2}G^*$; using now 3.8(vii), we get (iii).

Using (iii), we get that $J_{\tilde{\psi_1}}GG^*J_{\tilde{\psi_1}} = GG^*$; as, in 3.8(v), we have obtained that GG^* belongs to $(A \rtimes_{\mathfrak{a}} \mathfrak{G})'$, we get that $GG^* \in Z(A \rtimes_{\mathfrak{a}} \mathfrak{G})$. Using then 3.8(v) again, we see that, for any $X \in A \rtimes_{\mathfrak{a}} \mathfrak{G}$, we have $XGG^* = G\pi_{\mathfrak{a}}(X)G^*$, and therefore that $\pi_{\mathfrak{a}}(X) = 0$ if and only if $XGG^* = 0$, from which we get that GG^* is equal to the support of $\pi_{\mathfrak{a}}$. Using then (ii), we finish the proof. ∎

3.10. Lemma. *With the hypothesis and notations of 3.9, we get, for any $m \in \widehat{M}'$:*

$$1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} m) = G^*V_{\psi_1}[1 \underset{N}{{}_a\otimes_\beta} \hat{J}\hat{R}^c(m^*)\hat{J}]V^*_{\psi_1}G.$$

Proof. Using 3.7, 3.9(iii), 3.8(iv) and 2.5, we get

$$G[1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} m)] = G[1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} J_{\psi_1}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{R}^c(m^*))J_{\psi_1}]$$
$$= GJ_{\psi_2}[\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{R}^c(m^*)) \underset{N}{{}_s\otimes_r} 1]J_{\psi_2}$$
$$= J_{\tilde{\psi_1}}G[\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{R}^c(m^*)) \underset{N}{{}_s\otimes_r} 1]J_{\psi_2}$$

$$= J_{\tilde{\psi_1}}[1 \underset{N}{{}_b\otimes_\alpha} \hat{R}^c(m^*)]GJ_{\psi_2} = J_{\tilde{\psi_1}}[1 \underset{N}{{}_b\otimes_\alpha} \hat{R}^c(m^*)]J_{\tilde{\psi_1}}G$$
$$= V_{\psi_1}[1 \underset{N^o}{{}_a\otimes_\beta} \hat{J}\hat{R}^c(m^*)\hat{J}]V^*_{\psi_1}G$$

from which, G being an isometry, we get the result. ■

3.11. Definitions. Let $(b, \mathfrak{a})$ be an integrable action of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A, $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ be the crossed product, $\pi_{\mathfrak{a}}$ be the Galois homomorphism from $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ onto the algebra $s(A^{\mathfrak{a}})'$ obtained by the standard construction made from the inclusion $A^{\mathfrak{a}} \subset A$ (3.6), and G be the isometry constructed in 3.8; then, using 3.9(iv), we get that the following properties are equivalent:

(i) $\pi_{\mathfrak{a}}$ is an isomorphism between $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ and $s(A^{\mathfrak{a}})'$;

(ii) the isometry G is a unitary;

(iii) the inclusion $A^{\mathfrak{a}} \underset{N}{{}_b\otimes_\alpha} 1_H \subset \mathfrak{a}(A) \subset A \rtimes_{\mathfrak{a}} \mathfrak{G}$ is standard, and the operator-valued weight $T_{\tilde{\mathfrak{a}}}$ is obtained from $T_{\mathfrak{a}}$ by this standard construction.

In that situation, following [DC1], we shall say that $(b, \mathfrak{a})$ is a Galois action of $\mathfrak{G}$, and that the $A^{\mathfrak{a}} - N^o$-bimodule A will be called a Galois bimodule for $\mathfrak{G}$, and the unitary $\tilde{G} = \sigma_\nu G$ from $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$ onto $H \underset{\nu^o}{{}_\alpha\otimes_b} H_{\psi_1}$ will be called its Galois unitary. Then, it is clear that the representation μ of $\widehat{M}'$ on H_{ψ_1}, defined in 3.6, is faithful.

Moreover, a normal semi-finite faithful weight ψ_0 on $A^{\mathfrak{a}}$ will be called $\mathfrak{a}$-relatively invariant, if there exists a normal semi-finite faithful weight ϕ on A, invariant by $\mathfrak{a}$, and having the density property, such that the two automorphism groups σ^ϕ and σ^{ψ_1} on A commute (where $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$). In that situation, we shall say that the 5-uple $(A, b, \mathfrak{a}, \phi, \psi_0)$ is a Galois system for $\mathfrak{G}$. Then, thanks to [V1], we know that there exists a positive operator δ_A affiliated to A, and a positive operator λ_A affiliated to $Z(A)$ such that
$$(D\phi : D\psi_1)_t = \lambda_A^{it^2/2}\delta_A^{it}$$
We shall call δ_A the modulus of the action $(b, \mathfrak{a})$, and λ_A the scaling operator of the action.

Starting from a left action $(a, \mathfrak{b})$, we get the notion of left Galois system and that $(A, a, \mathfrak{b}, \phi, \psi_0)$ is left Galois if and only if $(A^o, a^o, (\sigma_N\mathfrak{b})^o, \phi^o, \psi_0^o)$ is Galois, and $(A, b, \mathfrak{a}, \phi, \psi_0)$ is Galois if and only if $(A^o, b^o, \sigma_N\mathfrak{a}^o, \phi^o, \psi_0^o)$ is left-Galois.

3.12. Examples. (i) Let $(b, \mathfrak{a})$ be any action of $\mathfrak{G}$ on a von Neumann algebra A; then (2.5), there exists an action $(1 \underset{N}{{}_b\otimes_\alpha} \hat{\alpha}, \tilde{\mathfrak{a}})$ of $\widehat{\mathfrak{G}}^c$ on the crossed product $A \rtimes_{\mathfrak{a}} \mathfrak{G}$. This action is integrable ([E5], 9.8); we have $(A \rtimes_{\mathfrak{a}} \mathfrak{G})^{\tilde{\mathfrak{a}}} = \mathfrak{a}(A)$ ([E5], 11.5), and as the inclusion $\mathfrak{a}(A) \subset A \rtimes_{\mathfrak{a}} \mathfrak{G} \subset A \underset{N}{{}_b*_\alpha} \mathcal{L}(H)$ is depth 2 ([E5], 13.8), we obtain by ([E5], 13.9) that the dual action $(1 \underset{N}{{}_b\otimes_\alpha} \hat{\alpha}, \tilde{\mathfrak{a}})$ is a Galois action of $\widehat{\mathfrak{G}}^c$, with $\alpha(A) \subset A \rtimes_{\mathfrak{a}} \mathfrak{G}$ and $1 \underset{N}{{}_b\otimes_\alpha} \hat{\alpha}(N) \subset A \rtimes_{\mathfrak{a}} \mathfrak{G}$ as Galois bimodule.

(ii) In particular ([E5], 9.5) we get that (β, Γ) is a Galois action of $\mathfrak{G}$, with $\alpha(N) \subset M$ and $\beta(N) \subset M$ as Galois bimodule. Moreover, we get that $(M, \beta, \Gamma, \Phi \circ R, \nu)$ is a Galois system for $\mathfrak{G}$. Then, we can easily check that $M^\Gamma = \alpha(N)$, that the operator-valued weight T_Γ is equal to the left-invariant weight T_L, and therefore that $\psi_1 = \Phi$, $a = \hat{\alpha}$, $r = \alpha$, $s = \hat{\beta}$, $V_{\psi_1} = \sigma W^{o*}\sigma$, $\pi_\Gamma = id$, and $\tilde{G} = \widehat{W}$.

(iii) If $(b, \mathfrak{a})$ is an integrable outer action of $\mathfrak{G}$ on A, then, $(b, \mathfrak{a})$ is Galois: let G be the isometry constructed in 3.8; as, by definition ([E7]), we have $A \rtimes_{\mathfrak{a}} \mathfrak{G} \cap \mathfrak{a}(A)' = 1 {}_b\underset{N}{\otimes}_\alpha \hat{\alpha}(N)$, we get, by 3.9, that there exists a projection $p \in Z(N)$ such that $GG^* = 1 {}_b\underset{N}{\otimes}_\alpha \hat{\alpha}(p)$ is the support of $\pi_{\mathfrak{a}}$; using 3.8, we get that $p = 1$, which gives the injectivity of $\pi_{\mathfrak{a}}$.

3.13. Lemma. *(i) Let $(b, \mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on the von Neumann algebra A; let $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, V_{ψ_1} the standard implementation of $\mathfrak{a}$ associated to ψ_1 (2.5), and $\tilde{G}$ the Galois unitary of the Galois system. Then*

$$\tilde{G}\sigma_{\psi_0^o}(J_{\psi_1} {}_s\underset{A^{\mathfrak{a}}}{\otimes}_r J_{\psi_1}) = \sigma_\nu V_{\psi_1}\sigma_{\nu^o}(\hat{J} {}_\alpha\underset{N^o}{\otimes}_b J_{\psi_1})\tilde{G}.$$

(ii) Let W be the pseudo-multiplicative unitary of $\mathfrak{G}$, W^o be the pseudo-multiplicative unitary of $\mathfrak{G}^o$, $\widehat{W}$ be the pseudo-multiplicative unitary of $\widehat{\mathfrak{G}}$. We have

$$\widehat{W}\sigma_{\nu^o}(J {}_{\hat{\beta}}\underset{N}{\otimes}_\alpha J) = W^{o*}(\hat{J} {}_\alpha\underset{N^o}{\otimes}_\beta J)\widehat{W}.$$

Proof. Using 3.9(iii) and 2.5, we have

$$\sigma_\nu\tilde{G}\sigma_{\psi_0^o}(J_{\psi_1} {}_s\underset{A^{\mathfrak{a}}}{\otimes}_r J_{\psi_1}) = GJ_{\psi_2} = J_{\tilde{\psi_1}}G = V_{\psi_1}(J_{\psi_1} {}_b\underset{N}{\otimes}_\alpha \hat{J})G = V_{\psi_1}\sigma_{\nu^o}(\hat{J} {}_\alpha\underset{N^o}{\otimes}_b J_{\psi_1})\tilde{G}$$

from which we get (i). Using 3.12(ii), we obtain (ii). ∎

4. From Galois actions to Galois systems and back. In this chapter, we suppose that we have a Galois action $(b, \mathfrak{a})$ of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A, and a normal semi-finite faithful weight ψ_0 on $A^{\mathfrak{a}}$, such that the subspace $D((H_{\psi_1})_b, \nu^o) \cap D({}_rH_{\psi_1}, \psi_0)$ is dense in H_{ψ_1}, where $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$. We then prove that right leg of the Galois unitary introduced in 3.11 generates A and that this unitary satisfies a pentagonal relation (4.2). This allows us to prove, in some particular cases (4.6, 4.7) that there exists then a normal semi-finite faithful weight ϕ on A such that $(A, b, \mathfrak{a}, \phi, \psi_0)$ is a Galois system for $\mathfrak{G}$. Conversely, if there exists a Galois system $(A, b, \mathfrak{a}, \phi, \psi_0)$ for $\mathfrak{G}$, then the weight ψ_0 satisfies this density property (4.11).

4.1. Definition. Let $(b, \mathfrak{a})$ be a Galois action of the quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 a normal semi-finite faithful weight on $A^{\mathfrak{a}}$; let us write $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, r for the injection of $A^{\mathfrak{a}}$ into A. We shall say that the weight ψ_0 has the Galois density property if the subspace $D((H_{\psi_1})_b, \nu^o) \cap D({}_rH_{\psi_1}, \psi_0)$ is dense in H_{ψ_1}.

4.2. Theorem. *Let $(b, \mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 a normal semi-finite faithful weight on $A^{\mathfrak{a}}$, having the Galois density property, in the sense of 4.1; let $\tilde{G}$ be the Galois unitary of $(b, \mathfrak{a})$, from $H_{\psi_1} {}_s\underset{\psi_0}{\otimes}_r H_{\psi_1}$ onto $H {}_\alpha\underset{\nu^o}{\otimes}_b H_{\psi_1}$, as defined in 3.11. We have:*

(i) For any $\zeta \in D((H_{\psi_1})_b, \nu^o) \cap D({}_rH_{\psi_1}, \psi_0)$, ζ' in $D((H_{\psi_1})_b, \nu^o)$, $\eta \in H$, x in $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_{\mathfrak{a}}}$, we have

$$(\tilde{G}(\Lambda_{\psi_1}(x) {}_s\underset{\psi_0}{\otimes}_r \zeta)|\eta {}_\alpha\underset{\nu^o}{\otimes}_b \zeta') = (\Lambda_\Phi[(\omega_{\zeta,\zeta'} {}_b\underset{N}{*}_\alpha id)\mathfrak{a}(x)]|\eta)$$

and therefore

$$(id * \omega_{\zeta,\zeta'})(\tilde{G})\Lambda_{\psi_1}(x) = \Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)].$$

(ii) For any $x \in \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_\mathfrak{a}}$, $y \in \mathfrak{N}_\Phi \cap \mathfrak{N}_T$, $\xi \in D({}_\alpha H, \nu)$, *we have*

$$(\omega_{\Lambda_{\psi_1}(x), J_\Phi y^* J_\Phi \xi} * id)(\tilde{G}) = (id \underset{N}{{}_b*_\alpha} \omega_{J_\Phi \Lambda_\Phi(y), \xi})\mathfrak{a}(x).$$

(iii) For any $x \in \mathfrak{N}_{\psi_1} \cap \mathfrak{N}^*_{\psi_1}$, y, z *in* $\mathfrak{N}_\Phi \cap \mathfrak{N}_T$, *we have*

$$(\omega_{\Lambda_{\psi_1}(x), J_\Phi \Lambda_\Phi(y^*z)} * id)(\tilde{G})^* = (\omega_{\Lambda_{\psi_1}(x^*), J_\Phi \Lambda_\Phi(z^*y)} * id)(\tilde{G}).$$

(iv) The two unitaries $(1 \underset{N^o}{{}_\alpha\otimes_b} \tilde{G})(\tilde{G} \underset{A^\mathfrak{a}}{{}_s\otimes_r} 1)$ *and*

$$(\widehat{W} \underset{N^o}{{}_\alpha\otimes_b} 1)\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^\mathfrak{a}}{{}_s\otimes_r} \sigma_{\nu^o})(1 \underset{A^\mathfrak{a}}{{}_s\otimes_r} \tilde{G})$$

from $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$ *to* $H \underset{\nu^o}{{}_\alpha\otimes_\beta} H \underset{\nu^o}{{}_\alpha\otimes_b} H_{\psi_1}$, *are equal.*

Proof. Using 3.8(iii) and the definition of $\tilde{G}$, we get (i) by a direct calculation. Using (i), we then get

$$\begin{aligned}((\omega_{\Lambda_{\psi_1}(x), J_\Phi y^* J_\Phi \xi} * id)(\tilde{G})\zeta|\zeta') &= (\tilde{G}(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta)|J_\Phi y^* J_\Phi \xi \underset{\nu^o}{{}_\alpha\otimes_b} \zeta') \\ &= (\Lambda_\Phi(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))|J_\Phi y^* J_\Phi \xi) \\ &= (J_\Phi y J_\Phi \Lambda_\Phi(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x))|\xi) \\ &= ((\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x) J_\Phi \Lambda_\Phi(y)|\xi) \\ &= ((id \underset{N}{{}_b*_\alpha} \omega_{J_\Phi\Lambda_\Phi(y),\xi})\mathfrak{a}(x)\zeta|\zeta')\end{aligned}$$

from which we get (ii). Using (ii), we get

$$\begin{aligned}(\omega_{\Lambda_{\psi_1}(x), J_\Phi \Lambda_\Phi(y^*z)} * id)(\tilde{G})^* &= (id \underset{N}{{}_b*_\alpha} \omega_{J_\Phi\Lambda_\Phi(y), J_\Phi\Lambda_\Phi(z)})\mathfrak{a}(x)^* \\ &= (id \underset{N}{{}_b*_\alpha} \omega_{J_\Phi\Lambda_\Phi(z), J_\Phi\Lambda_\Phi(y)})\mathfrak{a}(x^*) \\ &= (\omega_{\Lambda_{\psi_1}(x^*), J_\Phi\Lambda_\Phi(z^*y)} * id)(\tilde{G})\end{aligned}$$

from which we get (iii).

Let $v \in D(H_\beta, \nu^o)$, $w \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$, $\zeta \in D((H_{\psi_1})_b, \nu^o) \cap D({}_r H_{\psi_1}, \psi_0)$, $x \in \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_\mathfrak{a}}$; we have, using (i) and ([E5], 3.10 (i)):

$$\begin{aligned}(i * \omega_{v,w})(\widehat{W})(id * \omega_{\zeta,\zeta'})(\tilde{G})\Lambda_{\psi_1}(x) &= (i * \omega_{v,w})(\widehat{W})\Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)] \\ &= \Lambda_\Phi[(\omega_{v,w} \underset{N}{{}_\beta*_\alpha} id)\Gamma[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)]] \\ &= \Lambda_\Phi[(\omega_{\zeta,\zeta'} \underset{N}{{}_b*_\alpha} \omega_{v,w} \underset{N}{{}_\beta*_\alpha} id)(\mathfrak{a} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)] \\ &= \Lambda_\Phi[((\omega_{v,w} \underset{N^o}{{}_\alpha*_b} \omega_{\zeta,\zeta'}) \circ \varsigma_N \mathfrak{a})\mathfrak{a}(x)].\end{aligned}$$

Using 3.8(iv), we get that, for all $y \in A$:

$$(\omega_{v,w} \underset{N^o}{{}_\alpha*_b} \omega_{\zeta,\zeta'}) \circ \varsigma_N \mathfrak{a})(y) = (\tilde{G}(y \underset{A^\mathfrak{a}}{{}_s\otimes_r} 1)\tilde{G}^*(v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)|w \underset{\nu^o}{{}_\alpha\otimes_b} \zeta')$$

Let $(e_i)_{i\in I}$ be an orthogonal (r,ψ_0)-basis for H_{ψ_1}; there exists $(v_i)_{i\in I}$ and $(w_i)_{i\in I}$ in H_{ψ_1}, such that

$$\tilde{G}^*(v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta) = \sum_i v_i \underset{\psi_0}{{}_s\otimes_r} e_i, \qquad \tilde{G}^*(w \underset{\nu^o}{{}_\alpha\otimes_b} \zeta') = \sum_i w_i \underset{\psi_0}{{}_s\otimes_r} e_i.$$

Using the intertwining properties given in 3.8(i), we get, for all $n \in N$:

$$\begin{aligned}\sum_i \|b(n)v_i\|^2 &= \|\sum_i b(n)v_i \underset{\psi_0}{{}_s\otimes_r} e_i\|^2 = \|(b(n) \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)\tilde{G}^*(v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)\|^2 \\ &= \|\tilde{G}^*(\beta(n)v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)\|^2 = \|\beta(n)v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta\|^2\end{aligned}$$

and therefore as ζ is in $D((H_{\psi_1})_b,\nu^o)$ and v is in $D(H_\beta,\nu^o)$, we get that each v_i is in $D((H_{\psi_1})_b,\nu^o)$, and the same result holds for the w_is.

On the other hand, for any $z \in A^{\mathfrak{a}}$, we get

$$\begin{aligned}\sum_i \|r(z)v_i\|^2 &= \|\sum_i r(z)v_i \underset{\psi_0}{{}_s\otimes_r} e_i\|^2 \\ &= \|(r(z) \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)\tilde{G}^*(v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)\|^2 \\ &= \|\tilde{G}^*(v \underset{\nu^o}{{}_\alpha\otimes_b} r(z)\zeta)\|^2 = \|v \underset{\nu^o}{{}_\alpha\otimes_b} r(z)\zeta\|^2\end{aligned}$$

and therefore as v is in $D({}_\alpha H,\nu)$ and ζ is in $D({}_r H_{\psi_1},\psi_0)$, we get that each v_i is in $D({}_r H_{\psi_1},\psi_0)$.

So, we get that there exists v_i in $D((H_{\psi_1})_b,\nu^o)\cap D({}_r H_{\psi_1},\psi_0)$, and w_i in $D((H_{\psi_1})_b,\nu^o)$, such that, for all $y \in A$, we have

$$(\omega_{v,w} \underset{N^o}{{}_\alpha *_b} \omega_{\zeta,\zeta'}) \circ \varsigma_N \mathfrak{a}(y) = \sum_i (yv_i|w_i)$$

and therefore $(\omega_{v,w} \underset{N^o}{{}_\alpha *_b} \omega_{\zeta,\zeta'}) \circ \varsigma_N \mathfrak{a} = \sum_i \omega_{v_i,w_i}$. So, we get

$$\begin{aligned}(i * \omega_{v,w})(\widehat{W})(id * \omega_{\zeta,\zeta'})(\tilde{G})\Lambda_{\psi_1}(x) &= \Lambda_\Phi[((\omega_{v,w} \underset{N^o}{{}_\alpha *_b} \omega_{\zeta,\zeta'}) \circ \varsigma_N \mathfrak{a})\mathfrak{a}(x)] \\ &= \sum_i \Lambda_\Phi[(\omega_{v_i,w_i} \underset{N}{{}_b *_\alpha} id)\mathfrak{a}(x)] \\ &= \sum_i (i * \omega_{v_i,w_i})(\tilde{G})\Lambda_{\psi_1}(x)\end{aligned}$$

and therefore for all $\eta \in H$, we have

$$\begin{aligned}((i * \omega_{v,w})(\widehat{W})(id * \omega_{\zeta,\zeta'})(\tilde{G})\Lambda_{\psi_1}(x)|\eta) &= \sum_i ((i * \omega_{v_i,w_i})(\tilde{G})\Lambda_{\psi_1}(x)|\eta) \\ &= \sum_i (\tilde{G}(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} v_i)|\eta \underset{\nu^o}{{}_\alpha\otimes_b} w_i)\end{aligned}$$

which is equal to

$$\begin{aligned}&\sum_i ((\tilde{G} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} v_i \underset{\psi_o}{{}_s\otimes_r} e_i)|\eta \underset{\nu^o}{{}_\alpha\otimes_b} w_i \underset{\psi_0}{{}_s\otimes_r} e_i) \\ &\quad = ((\tilde{G} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \tilde{G}^*)(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} (v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta))|(1 \underset{N^o}{{}_\alpha\otimes_b} \tilde{G}^*)(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} w \underset{\nu^o}{{}_\alpha\otimes_b} \zeta')).\end{aligned}$$

On the other hand, we get that

$$\begin{aligned}((i*\omega_{v,w})(\widehat{W})(id*\omega_{\zeta,\zeta'})(\tilde{G})\Lambda_{\psi_1}(x)|\eta) &= ((id*\omega_{\zeta,\zeta'})(\tilde{G})\Lambda_{\psi_1}(x)|(i*\omega_{v,w})(\widehat{W})^*\eta)\\ &= (\tilde{G}(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta)|(i*\omega_{v,w})(\widehat{W})^*\eta \underset{\nu^o}{{}_\alpha\otimes_b} \zeta')\end{aligned}$$

is, using again 3.8(i), equal to

$$(\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})[\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} (v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)]|(\widehat{W}^* \underset{N^o}{{}_\alpha\otimes_b} 1)(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} w \underset{\nu^o}{{}_\alpha\otimes_b} \zeta')$$

where $\sigma^{2,3}_{\alpha,\hat{\beta}}$ is the flip from $(H \underset{N^o}{{}_\alpha\otimes_b} H_{\psi_1}) \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} H$ onto $(H \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} H) \underset{N^o}{{}_\alpha\otimes_b} H_{\psi_1}$, exchanging the second and the third leg, and α with $\hat{\beta}$. ∎

From which we get that

$$((1 \underset{N^o}{{}_\alpha\otimes_b} \tilde{G})(\tilde{G} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \tilde{G}^*)([\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} (v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)]|\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} w \underset{\nu^o}{{}_\alpha\otimes_b} \zeta')$$

is equal to

$$(\widehat{W} \underset{N^o}{{}_\alpha\otimes_b} 1)\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})[\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} (v \underset{\nu^o}{{}_\alpha\otimes_b} \zeta)]|\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} w \underset{\nu^o}{{}_\alpha\otimes_b} \zeta')$$

and therefore that

$$(\widehat{W} \underset{N^o}{{}_\alpha\otimes_b} 1)\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \tilde{G}) = (1 \underset{N^o}{{}_\alpha\otimes_b} \tilde{G})(\tilde{G} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1).$$

4.3. Corollary. *Let $(b,\mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 a normal semi-finite faithful weight on $A^{\mathfrak{a}}$, having the Galois density property defined in 4.1; let us write $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, and let us write r for the injection of $A^{\mathfrak{a}}$ in A, and s for the antirepresentation $s(x) = J_{\psi_1} r(x^*) J_{\psi_1}$ of $A^{\mathfrak{a}}$ on H_{ψ_1}; let $\tilde{G}$ be the Galois unitary of $(b,\mathfrak{a})$, from $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$ onto $H \underset{\nu^o}{{}_\alpha\otimes_b} H_{\psi_1}$, as defined in 3.11. Then, the linear space generated by the elements of the form $(\omega_{\zeta,\zeta'} * id)(\tilde{G})$, for all ζ in $D((H_{\psi_1})_s, \psi_0^o)$, and $\zeta' \in D({}_\alpha H, \nu)$ is weakly dense in A.*

Proof. Let us first look at the product of two elements of that form. Let $\zeta_1 \in D((H_{\psi_1})_s, \psi_0^o)$ and $\zeta_1' \in D({}_\alpha H, \nu)$; let ξ, η be in H_{ψ_1}. Then

$$\begin{aligned}&((\omega_{\zeta,\zeta'} * id)(\tilde{G})(\omega_{\zeta_1,\zeta_1'} * id)(\tilde{G})\xi|\eta)\\ &\qquad = ([\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \tilde{G})](\zeta \underset{\psi_0}{{}_s\otimes_r} \zeta_1 \underset{\psi_0}{{}_s\otimes_r} \xi)|(\zeta' \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} \zeta_1') \underset{\psi_0^o}{{}_\alpha\otimes_b} \eta)\end{aligned}$$

which, using 4.2(iv), is equal to

$$((1 \underset{N^o}{{}_\alpha\otimes_b} \tilde{G})(\tilde{G} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)(\zeta \underset{\psi_0}{{}_s\otimes_r} \zeta_1 \underset{\psi_0}{{}_s\otimes_r} \xi)|(\widehat{W} \underset{(A^{\mathfrak{a}})^o}{{}_\alpha\otimes_b} 1)[(\zeta' \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} \zeta_1') \underset{\psi_0^o}{{}_\alpha\otimes_b} \eta]).$$

Let $(e_i)_{i\in I}$ be an orthogonal (α,ν)-basis. As in ([E3], 3.4), we can prove that there exist $(\zeta_i)_{i\in I} \in D((H_{\psi_1})_s, \psi_0^o)$ and $(\zeta_i')_{i\in I} \in D({}_\alpha H, \nu)$ such that

$$\tilde{G}(\zeta \underset{\psi_0}{{}_s\otimes_r} \zeta_1) = \sum_i e_i \underset{\nu^o}{{}_\alpha\otimes_b} \zeta_1, \qquad \widehat{W}(\zeta' \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} \zeta_1') = \sum_i e_i \underset{\nu^o}{{}_\alpha\otimes_\beta} \zeta_i',$$

and therefore we get that

$$((\omega_{\zeta,\zeta'} * id)(\tilde{G})(\omega_{\zeta_1,\zeta'_1} * id)(\tilde{G})\xi|\eta) = \sum_i (\omega_{\zeta_i,\zeta'_i} * id)(\tilde{G})\xi|\eta)$$

which proves that the product $(\omega_{\zeta,\zeta'} * id)(\tilde{G})(\omega_{\zeta_1,\zeta'_1} * id)(\tilde{G})$ is the weak limit of the finite sums $(\omega_{\zeta_i,\zeta'_i} * id)(\tilde{G})$. So, by continuity, we get that the weak closure of the linear space generated by the elements of the form $(\omega_{\zeta,\zeta'} * id)(\tilde{G})$, for all ζ in $D((H_{\psi_1})_s, \psi_0^o)$, and $\zeta' \in D({}_\alpha H, \nu)$ is an algebra.

Using 4.2(ii), we get, on one hand, that all the operators of the form $(\omega_{\zeta_2,\zeta'} * id)(\tilde{G})$ (with $\zeta_2 \in D((H_{\psi_1})_s, \psi_0^o))$ belong to A, and on the other hand, that the closure of the linear set generated by these operators is the closure of the set of all operators $(id \underset{N}{{}_b*_\alpha} \omega_{\zeta'',\zeta'})\mathfrak{a}(x)$, for all ζ', ζ'' in $D({}_\alpha H, \nu)$, and $x \in A$, and is therefore invariant by taking the adjoint, and that it contains all operators $b(< \zeta'', \zeta'' >_{b,\nu^o}) = (id \underset{N}{{}_b*_\alpha} \omega_{\zeta'',\zeta'})\mathfrak{a}(1)$. Therefore, it is a sub-von Neumann algebra B of A which contains $b(N)$. If now $X \in B'$, we get that $X \underset{N}{{}_b\otimes_\alpha} 1$ belongs to $\mathfrak{a}(A)' \cap b(N)' \underset{N}{{}_b\otimes_\alpha} 1 = \mathfrak{a}(A)' \cap \mathcal{L}(H_{\psi_1}) \underset{N}{{}_b*_\alpha} \alpha(N)$, which is equal to the commutant $(\mathfrak{a}(A) \cup 1_{H_{\psi_1}} \underset{N}{{}_b\otimes_\alpha} \alpha(N)')'$. Thanks to ([E5], 11.5(ii)), we have

$$(\mathfrak{a}(A) \cup 1_{H_{\psi_1}} \underset{N}{{}_b\otimes_\alpha} \alpha(N)')'' = A \underset{N}{{}_b*_\alpha} \mathcal{L}(H)$$

and therefore we get that $(\mathfrak{a}(A) \cup 1_{H_{\psi_1}} \underset{N}{{}_b\otimes_\alpha} \alpha(N))' = A' \underset{N}{{}_b\otimes_\alpha} 1$. So, any $X \in B'$ belongs to A', and we finally get that $B = A$, which finishes the proof. ∎

4.4. Proposition. *With the assumptions of 4.3, let us write*

$$K^{it} = \tilde{G}^*(J_\Phi \delta^{it} J_\Phi \underset{N^o}{{}_\alpha\otimes_b} 1)\tilde{G}.$$

Let T_2 be the canonical operator-valued weight from $s(A^{\mathfrak{a}})'$ onto A obtained by the basic construction from $T_{\mathfrak{a}}$, and $\psi_2 = \psi_1 \circ T_2$. Then:

(i) The one-parameter group of unitaries K^{it} on $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$ belongs to $A' \underset{A^{\mathfrak{a}}}{{}_s*_r} A$.

(ii) There exists a one-parameter group of automorphisms ρ_t of $s(A^{\mathfrak{a}})'$ such that, for all $y \in s(A^{\mathfrak{a}})'$, we have

$$K^{it}(y \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)K^{-it} = \rho_t(y) \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1$$

with $\rho_t(x) = x$, for all $x \in A$.

(iii) For any $y \in s(A^{\mathfrak{a}})'^+$, we have $\psi_2 \circ \rho_t(y) = \psi_2(b(q)^{-t}y)$, where q belongs to $Z(N)$ and is such that the scaling operator of $\mathfrak{G}$ is $\lambda = \alpha(q) = \beta(q)$, and $b(q) \in Z(A)$, and $T_2(\rho_t(y)) = b(q)^{-t}T_2(y)$.

(iv) Let us identify H_{ψ_2} and $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$ (2.6); then K^{it} is the standard implementation of ρ_t, and therefore

$$K^{it}\Lambda_{\psi_2}(X) = \Lambda_{\psi_2}(b(q)^{t/2}\rho_t(X))$$

for any $X \in \mathfrak{N}_{\psi_2}$, and

$$\sigma_{\psi_0^o}(J_{\psi_1} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} J_{\psi_1})K^{it}(J_{\psi_1} \underset{(A^{\mathfrak{a}})^o}{{}_r\otimes_s} J_{\psi_1})\sigma_{\psi_0} = K^{it}$$

(v) It is possible to define a one-parameter group of unitaries $K^{it} {}_{1_s \underset{A^{\mathfrak{a}}}{\otimes}_r b} \underset{\nu}{\otimes}_\alpha \delta^{it}$ *on* $H_{\psi_1} {}_s\underset{\psi_0}{\otimes}_r H_{\psi_1} {}_b\underset{\nu}{\otimes}_\alpha H$*, with natural values on elementary tensors; moreover, we have*

$$(id {}_s\underset{A^{\mathfrak{a}}}{*}_r \mathfrak{a})(K^{it}) = K^{it} {}_{1_s \underset{A^{\mathfrak{a}}}{\otimes}_r b} \underset{\nu}{\otimes}_\alpha \delta^{it}.$$

(vi) For any s,t in $\mathbb{R}$*, we have* $(\Delta_{\psi_1}^{it} {}_s\underset{\psi_0}{\otimes}_r \Delta_{\psi_1}^{it})(K^{is})(\Delta_{\psi_1}^{-it} {}_s\underset{\psi_0}{\otimes}_r \Delta_{\psi_1}^{-it}) = K^{is}$.

Proof. By a straightforward application of 4.2(iv), we get that K^{it} belongs to $\mathcal{L}(H_{\psi_1}) {}_s\underset{A^{\mathfrak{a}}}{*}_r A$; moreover, using 3.8(v), we get, for any $X \in A \rtimes_{\mathfrak{a}} \mathfrak{G}$, that

$$K^{it}(\pi_{\mathfrak{a}}(X) {}_s\underset{A^{\mathfrak{a}}}{\otimes}_r 1)K^{-it} = K^{it}G^*XGK^{-it}$$

and therefore if $X = \mathfrak{a}(x)$, with $x \in A$, we have, using 3.8(iv):

$$K^{it}(x {}_s\underset{A^{\mathfrak{a}}}{\otimes}_r 1)K^{-it} = G^*(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{it}J_\Phi)\mathfrak{a}(x)(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{-it}J_\Phi)G = G^*\mathfrak{a}(x)G = x {}_s\underset{A^{\mathfrak{a}}}{\otimes}_r 1$$

from which we finish the proof of (i).

Using ([E5], 3.11(ii)), we get that, for any $a \in M$, that $\hat{\delta}^{it}a\hat{\delta}^{-it} = \tau_{-t}\sigma_{-t}^{\Phi\circ R}(a)$, and applying this result to $\widehat{\mathfrak{G}}$, we get that $\delta^{it}b\delta^{-it} = \hat{\tau}_{-t}\sigma_{-t}^{\widehat{\Phi}\circ\hat{R}}(b)$, for any b in $\widehat{M}$; moreover, using now ([E5], 3.10(iv)), we get that

$$J_\Phi\delta^{it}J_\Phi bJ_\Phi\delta^{-it}J_\Phi = \hat{\tau}_{-t}\sigma_t^{\widehat{\Phi}}(b)$$

and that, for any c in $\widehat{M}'$, $J_\Phi\delta^{it}J_\Phi cJ_\Phi\delta^{-it}J_\Phi$ belongs to $\widehat{M}'$, and more precisely, that

$$J_\Phi\delta^{it}J_\Phi cJ_\Phi\delta^{-it}J_\Phi = \hat{\tau}_{-t}^c\sigma_t^{\widehat{\Phi}^c}(c)$$

from which we infer that the one-parameter group of unitaries $1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{it}J_\Phi$ implements a one-parameter group of automorphisms of $A \rtimes_{\mathfrak{a}} \mathfrak{G}$; which gives (ii), thanks to 3.8(iv).

Then, using ([E5], 13.4, and 3.8(vii) applied to $\widehat{\mathfrak{G}}^c$), we get that, for any $x \in \mathfrak{N}_{\psi_1}$ and $c \in \mathfrak{N}_{\widehat{\Phi}^c}$, we get that

$$\begin{aligned}\widetilde{\psi_1}[(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{it}J_\Phi)\mathfrak{a}(x^*)(1 {}_b\underset{N}{\otimes}_\alpha c^*c)\mathfrak{a}(x)(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{-it}J_\Phi) = \\ \|\Lambda_{\psi_1}(x) {}_b\underset{\nu}{\otimes}_\alpha \Lambda_{\widehat{\Phi}^c}(\hat{\tau}_{-t}^c\sigma_t^{\widehat{\Phi}^c}(c))\|^2 = \|\Lambda_{\psi_1}(x) {}_b\underset{\nu}{\otimes}_\alpha \Lambda_{\widehat{\Phi}^c}(\lambda^{-t/2}c)\|^2 \\ = \widetilde{\psi_1}[\mathfrak{a}(x^*)(1 {}_b\underset{N}{\otimes}_\alpha \lambda^{-t}c^*c)\mathfrak{a}(x)]\end{aligned}$$

from which we get, using again ([E5], 13.4), that

$$\widetilde{\psi_1}[(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{it}J_\Phi)X(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{-it}J_\Phi)] = \widetilde{\psi_1}((1 {}_b\underset{N}{\otimes}_\alpha \lambda^{-t})X)$$

for any $X \in \mathfrak{M}_{\tilde{\psi}_1}^+$.

As $\lambda = \alpha(q) = \beta(q)$ is affiliated to $Z(M)$ ([E5], 3.8(vi)), we get that $1 {}_b\underset{N}{\otimes}_\alpha \lambda = \mathfrak{a}(b(q))$, and that $b(q)$ is affiliated to $Z(A)$. Using now 3.9, we get the result.

Using now ([E5], 3.10(vii) and again 13.4), we get that $(1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{it}J_\Phi)$ is the standard implementation of $Ad((1 {}_b\underset{N}{\otimes}_\alpha J_\Phi\delta^{it}J_\Phi)_{|A\rtimes_{\mathfrak{a}}\mathfrak{G}}$ on $H_{\psi_1} {}_b\underset{\nu}{\otimes}_\alpha H$ (which is identified with $H_{\tilde{\psi}_1}$

by ([E5], 13.4). Therefore, using again 3.9, we get that K^{it} is the standard implementation of ρ_t; thanks to (iii), we finish the proof of (iv).

Similarly, using again ([E5], 3.8 (i) and (v)), we get $\alpha(n)\delta^{it} = \delta^{it}\alpha(\gamma_t\sigma_t^\nu(n))$; as ν is invariant under γ_t ([E5], 3.8 (v)), there exists a one-parameter group of unitaries h^{it} on H_ν, such that, for all $n \in \mathfrak{N}_\nu$, we gave $\Lambda_\nu(\gamma_t\sigma_t^\nu(n)) = h^{it}\Lambda_\nu(n)$, and $h^{it}mh^{-it} = \gamma_t\sigma_t^\nu(m)$, for all $m \in N$; therefore, if η is in $D({}_\alpha H, \nu)$, it is straightforward to get that $\delta^{it}\eta$ belongs also to $D({}_\alpha H, \nu)$; more precisely, we have then:

$$R^{\alpha,\nu}(\delta^{it}\eta)\Lambda_\nu(n) = \alpha(n)\delta^{it}\eta = \delta^{it}\alpha(\gamma_t\sigma_t^\nu(n))\eta = \delta^{it}R^{\alpha,\nu}(\eta)h^{it}\Lambda_\nu(n)$$

from which we infer that $R^{\alpha,\nu}(\delta^{it}\eta) = \delta^{it}R^{\alpha,\nu}(\eta)h^{it}$, and if η, η' belong to $D({}_\alpha H, \nu)$, we get that $< \delta^{it}\eta, \delta^{it}\eta' >^o_{\alpha,\nu} = \gamma_{-t}\sigma^\nu_{-t}(< \eta, \eta' >^o_{\alpha,\nu})$. From which we get, for all ξ, ξ' in $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$, using (ii), that

$$\begin{aligned}(K^{it}\xi \underset{\nu}{{}_{1_s\underset{A^{\mathfrak{a}}}{\otimes}_r b}\otimes_\alpha} \delta^{it}\eta | K^{it}\xi' \underset{\nu}{{}_{1_s\underset{A^{\mathfrak{a}}}{\otimes}_r b}\otimes_\alpha} \delta^{it}\eta') &= ((1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} b(\gamma_{-t}\sigma^\nu_{-t}(< \eta, \eta' >^o_{\alpha,\nu})))K^{it}\xi|\xi') \\ &= (K^{it}(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} b(< \eta, \eta' >^o_{\alpha,\nu}))\xi|K^{it}\xi') \\ &= (\xi \underset{\nu}{{}_{1_s\underset{A^{\mathfrak{a}}}{\otimes}_r b}\otimes_\alpha} \eta|\xi' \underset{\nu}{{}_{1_s\underset{A^{\mathfrak{a}}}{\otimes}_r b}\otimes_\alpha} \eta')\end{aligned}$$

from which we get the first result of (v).

Using now 3.8(iv) and 4.2(iv), we get that $(id \underset{A^{\mathfrak{a}}}{{}_s*_r} \varsigma_N\mathfrak{a})(K^{it})$ is equal to

$$[(\widehat{W} \underset{N^o}{{}_\alpha\otimes_b} 1)\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})]^*(1 \underset{N}{{}_\alpha\otimes_b} \tilde{G})(J_\Phi\delta^{it}J_\Phi \underset{N}{{}_\alpha\otimes_b} 1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} 1)(1 \underset{N}{{}_\alpha\otimes_b} \tilde{G})^* \dots$$

$$\dots(\widehat{W} \underset{N^o}{{}_\alpha\otimes_b} 1)\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})$$

and is therefore equal to

$$[\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})]^*(\widehat{W}^*(J_\Phi\delta^{it}J_\Phi \underset{N^o}{{}_\alpha\otimes_\beta} 1)\widehat{W} \underset{N^o}{{}_\alpha\otimes_b} 1)\dots$$

$$\dots[\sigma^{2,3}_{\alpha,\hat{\beta}}(\tilde{G} \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \sigma_{\nu^o})].$$

Using ([E5], successively 3.10 (vii), 3.11 (iii), 3.6 and 3.8 (vi)) we get

$$\begin{aligned}\widehat{W}^*(J_\Phi\delta^{it}J_\Phi \underset{N^o}{{}_\alpha\otimes_\beta} 1)\widehat{W} &= \sigma_{\nu^o}W(1 \underset{N}{{}_\beta\otimes_\alpha} J_\Phi\delta^{it}J_\Phi)W^*\sigma_{\nu^o} \\ &= \sigma_{\nu^o}(\hat{J} \underset{\nu}{{}_\beta\otimes_\alpha} J_\Phi)W^*(1 \underset{N}{{}_\beta\otimes_\alpha} \delta^{it})W(\hat{J} \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} J_\Phi)\sigma_{\nu^o} \\ &= \sigma_{\nu^o}(\hat{J} \underset{\nu}{{}_\beta\otimes_\alpha} J_\Phi)\Gamma(\delta^{it})(\hat{J} \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} J_\Phi)\sigma_{\nu^o} \\ &= \sigma_{\nu^o}(\hat{J} \underset{\nu}{{}_\beta\otimes_\alpha} J_\Phi)(\delta^{it} \underset{\nu}{{}_\beta\otimes_\alpha} \delta^{it})(\hat{J} \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} J_\Phi)\sigma_{\nu^o} \\ &= J_\Phi\delta^{it}J_\Phi \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} \hat{J}\delta^{it}\hat{J} = J_\Phi\delta^{it}J_\Phi \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} \delta^{it}\end{aligned}$$

from which we get the second result of (v).

Using (iv) and ([E5], 3.11(ii)), we get that $(\Delta_{\psi_1}^{it} {}_s\underset{\psi_0}{\otimes}{}_r \Delta_{\psi_1}^{it})(K^{is})(\Delta_{\psi_1}^{-it} {}_s\underset{\psi_0}{\otimes}{}_r \Delta_{\psi_1}^{-it})$ is equal to

$$(\Delta_{\psi_1}^{it} {}_s\underset{\psi_0}{\otimes}{}_r \Delta_{\psi_1}^{it})\tilde{G}^*(J_\Phi\delta^{is}J_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_b 1)\tilde{G}(\Delta_{\psi_1}^{-it} {}_s\underset{\psi_0}{\otimes}{}_r \Delta_{\psi_1}^{-it})$$
$$= \tilde{G}^*((\delta\Delta_{\widehat{\Phi}})^{-it}J_\Phi\delta^{is}J_\Phi(\delta\Delta_{\widehat{\Phi}})^{it} {}_\alpha\underset{\nu^o}{\otimes}{}_b 1)\tilde{G} = \tilde{G}^*(J_\Phi\delta^{is}J_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_b 1)\tilde{G}$$

which is (vi). ■

4.5. Theorem. *Let's suppose again the assumptions of 4.3 and 4.4; let K^{it} be the one-parameter group of unitaries on $H_{\psi_1} {}_s\underset{\psi_0}{\otimes}{}_r H_{\psi_1}$ defined in 4.4; let us suppose that there exists a positive non-singular operator δ_A affiliated to A such that we have, for all $t \in \mathbb{R}$:*

$$K^{it} = J_{\psi_1}\delta_A^{it}J_{\psi_1} {}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r \delta_A^{it}.$$

Then:

(i) It is possible to define a one-parameter group of unitaries $\delta_A^{it} {}_b\underset{\nu}{\otimes}{}_\alpha \delta^{it}$ on $H_{\psi_1} {}_b\underset{\nu}{\otimes}{}_\alpha H$, with natural values on elementary tensors; moreover,

$$\mathfrak{a}(\delta_A^{it}) = \delta_A^{it} {}_b\underset{\nu}{\otimes}{}_\alpha \delta^{it}.$$

(ii) For all s, t in $\mathbb{R}$, we have $\sigma_s^{\psi_1}(\delta_A^{it}) = b(q)^{ist}\delta_A^{it}$.

(iii) There exists a normal semi-finite faithful weight ϕ on A such that $(A, b, \mathfrak{a}, \phi, \psi_0)$ is a Galois system. Moreover, the modulus of this Galois action is the operator δ_A, and the scaling operator is equal to $b(q)$, where $q \in Z(N)$ is such that $\alpha(q) = \beta(q) = \lambda$, the scaling operator of $\mathfrak{G}$.

Proof. Using 4.4(v), we easily get (i). Using now [E5], 8.8(iii), we get that $\mathfrak{a}(\sigma_s^{\psi_1}(\delta_A)^{it}) = \sigma_s^{\psi_1}(\delta_A)^{it} {}_b\underset{\nu}{\otimes}{}_\alpha \delta^{it}$, and therefore that $\sigma_s^{\psi_1}(\delta_A)^{it}\delta_A^{-it}$ belongs to $r(A^{\mathfrak{a}})$.

So, there exists $k\eta r(A^\alpha)$ such that $\sigma_s^{\psi_1}(\delta_A^{it}) = k^{ist}\delta_A^{it} = \delta_A^{it}k^{ist}$.

Let us write $k = \int_0^\infty \lambda de_\lambda$, and let us put $f_n = \int_{1/n}^n de_\lambda$; then using ([E5], 2.2.2), we get that, for any $x \in \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_{\mathfrak{a}}}$, $xf_nk^{-t/2}$ is bounded and belongs to $\mathfrak{N}_{T_{\mathfrak{a}}} \cap \mathfrak{N}_{\psi_1}$, and with same arguments, we get that $xf_nk^{-t/2}\delta_A^{-it}$ belongs also to $\mathfrak{N}_{T\mathfrak{a}} \cap \mathfrak{N}_{\psi_1}$. We then get that

$$J_{\psi_1}\delta_A^{it}J_{\psi_1}\Lambda_{T_{\mathfrak{a}}}(xf_n) = \Lambda_{T_{\mathfrak{a}}}(xf_nk^{-t/2}\delta_A^{-it})$$

and therefore with the notations of 4.4(ii):

$$\rho_t(\Lambda_{T_{\mathfrak{a}}}(xf_n)\Lambda_{T_{\mathfrak{a}}}(xf_n)^*) = \Lambda_{T_{\mathfrak{a}}}(xf_nk^{-t/2}\delta_A^{-it})\Lambda_{T_{\mathfrak{a}}}(xf_nk^{-t/2}\delta_A^{-it})^*$$

from which we get that

$$T_2\rho_t(\Lambda_{T_{\mathfrak{a}}}(xf_n)\Lambda_{T_{\mathfrak{a}}}(xf_n)^*) = xf_nk^{-t}x^*$$

and, on the other hand, using 4.4(iii), we have, using the fact that $b(q)$ is affiliated to $Z(A)$:

$$T_2\rho_t(\Lambda_{T_{\mathfrak{a}}}(xf_n)\Lambda_{T_{\mathfrak{a}}}(xf_n)^*) = b(q)^{-t}xf_nx^* = xf_nb(q)^{-t}x^*$$

from which we easily deduce that $k = b(q)$, which finishes the proof of (ii).

Using [V1], we get that there is a normal semi-finite faithful weight ϕ on A, such that $(D\phi : D\psi_1)_t = b(q)^{it^2/2}\delta_A^{it}$, and that the modular groups σ^ϕ and σ^{ψ_1} commute. If $x \in N_\phi$ is such that $x\delta_A^{1/2}$ is bounded, then this last operator belongs to $\mathfrak{N}_{\psi_1}$ and we can identify $\Lambda_\phi(x)$ with $\Lambda_{\psi_1}(x\delta_A^{1/2})$ and J_ϕ with $b(q)^{i/4}J_{\psi_1}$; we shall denote, for $n \in N$, $a(n) = J_{\psi_1}b(n^*)J_{\psi_1} = J_\phi b(n^*)J_\phi$.

For $x \in \mathfrak{N}_\phi$ and $\eta \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o) \cap \mathcal{D}(\delta^{1/2})$, such that $\delta^{-1/2}\eta$ belongs to $D({}_\alpha H, \nu)$, we have, using these remarks, and the fact that ψ_1 is δ-invariant, 4.6(iii):

$$\begin{aligned}\|\Lambda_\phi(x) {}_a\underset{\nu^o}{\otimes}{}_\beta\, \eta\|^2 &= \|\Lambda_{\psi_1}(x\delta_A^{1/2}) {}_a\underset{\nu^o}{\otimes}{}_\beta\, \eta\|^2 \\ &= (\psi_1 {}_b\underset{N}{*}{}_\alpha\, \omega_{\delta^{-1/2}\eta})(\mathfrak{a}(\delta_A^{1/2}x^*x\delta_A^{1/2})] \\ &= (\psi_1 {}_b\underset{N}{*}{}_\alpha\, \omega_{\delta^{-1/2}\eta})[(\delta_A^{1/2} {}_b\underset{\nu}{\otimes}{}_\alpha\, \delta^{1/2})\mathfrak{a}(x^*x)(\delta_A^{1/2} {}_b\underset{\nu}{\otimes}{}_\alpha\, \delta^{1/2})] \\ &= (\phi {}_b\underset{N}{*}{}_\alpha\, \omega_\eta)[\mathfrak{a}(x^*x)]\end{aligned}$$

which remains true for any $\eta \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$, and gives then, using ([E6], 7.6) that the weight ϕ is invariant under $\mathfrak{a}$, which finishes the proof. ∎

4.6. Corollary. *Let $(b, \mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 be a normal semi-finite faithful weight on $A^\mathfrak{a}$, having the Galois density property defined in 4.1; let ρ_t be the one-parameter group of automorphisms of $s(A^\mathfrak{a})'$ defined in 4.4. Let us suppose that this one-parameter group is inner. Then:*

(i) there exists a non-singular positive operator δ_A affiliated to $A \cap r(A^\mathfrak{a})'$ such that

$$K^{it} = J_{\psi_1}\delta_A^{it}J_{\psi_1} {}_s\underset{A^\mathfrak{a}}{\otimes}{}_r\, \delta_A^{it}.$$

(ii) There exists a normal semi-finite faithful weight ϕ on A such that $(A, b, \mathfrak{a}, \phi, \psi_0)$ is a Galois system.

Proof. As $\rho_t(x) = x$ for all $x \in A$, we get that there exists a positive non-singular operator δ_A affiliated to $A \cap r(A^\mathfrak{a})'$ such that, for all $x \in s(A^\mathfrak{a})'$, we have

$$\rho_t(x) = J_{\psi_1}\delta_A^{it}J_{\psi_1}xJ_{\psi_1}\delta_A^{-it}J_{\psi_1}$$

and then, using 4.4(iv), we have (i). Result (ii) is then a direct corollary of 4.5(iii). ∎

4.7. Corollary. *Let $(b, \mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let us suppose that the invariant subalgebra $A^\mathfrak{a}$ is a finite sum of factors (in particular, if $A^\mathfrak{a}$ is finite dimensional); let ψ_0 be a normal semi-finite faithful weight on $A^\mathfrak{a}$, having the Galois density property defined in 4.1; then, there exists a normal semi-finte faithful weight ϕ on A such that $(A, b, \mathfrak{a}, \phi, \psi_0)$ is a Galois system.*

Proof. The center $Z(s(A^\mathfrak{a})')$ is equal to $s(Z(A^\mathfrak{a}))$; if $A^\mathfrak{a} = \oplus_{i\in I}F_i$ (with a finite set I), we get that $Z(A^\mathfrak{a}) = \oplus_{i\in I}\mathbb{C}$, and that any automorphism of $Z(s(A^\mathfrak{a})')$ gives a permutation in the set I; therefore, the restriction of the one-parameter group ρ_t defined in 4.4 to this center gives a continuous function ρ from $\mathbb{R}$ to the set $\mathfrak{S}(I)$ (with the pointwise topology); as $\rho^{-1}(id)$ is open, closed, non-empty, we get that ρ_t acts identically on the center $Z(s(A^\mathfrak{a})')$; therefore, ρ_t is inner, by ([StZ], 8.11), and we get the result by 4.6. ∎

4.8. Theorem. *Let $(b, \mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 be a normal semi-finite faithful weight on $A^{\mathfrak{a}}$, having the Galois density property defined in 4.1; let $\tilde{G}$ be the Galois unitary of $(b, \mathfrak{a})$, as defined in 3.11; let $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, and $\tilde{\psi_1}$ its dual weight on the crossed product $A \rtimes_{\mathfrak{a}} \mathfrak{G}$. Then:*

(i) For all s, t in $\mathbb{R}$, we have

$$\sigma_t^{\tilde{\psi_1}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J}) = 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J}.$$

(ii) There exists a one-parameter group of unitaries $P_A^{it} = \Delta_{\psi_1}^{it}\pi_{\mathfrak{a}}((1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})$ on H_{ψ_1}, which defines a one-parameter group τ_t^A of automorphism of A defined, for all $X \in A$, by $\tau_t^A(X) = P_A^{it} A P_A^{-it}$. For any $x \in A^{\mathfrak{a}}$, we have $\tau_t^A(x) = \sigma_t^{\psi_0}(x)$, and for all $n \in N$, we have $\tau_t^A(b(n)) = b(\sigma_t^\nu(n))$.

(iii) For all $t \in \mathbb{R}$, we have

$$\mathfrak{a}(\tau_t^A(X)) = (\sigma_t^{\psi_1} \underset{N}{{}_b*_\alpha} \sigma_t^{\Phi\circ R})\mathfrak{a}(X) = (\tau_t^A \underset{N}{{}_b*_\alpha} \tau_t)\mathfrak{a}(X),$$

$$\mathfrak{a}(\sigma_t^{\psi_1}(X)) = (\tau_t^A \underset{N}{{}_b*_\alpha} \sigma_t^\Phi)\mathfrak{a}(X).$$

(iv) We have, for any positive $X \in A$:

$$\psi_1 \circ \tau_t^A(X) = \psi_1(b(q)^{-t}X)$$

where q is the positive non-singular operator affiliated to N such that the scaling operator λ of $\mathfrak{G}$ satisfies $\lambda = \alpha(q) = \beta(q)$ ([E5], 3.8(vi)).

(v) For any $X \in \mathfrak{N}_{\psi_1}$, we have $P_A^{it}\Lambda_{\psi_1}(X) = b(q)^{t/2}\Lambda_{\psi_1}(\tau_t^A(X))$. So, P_A^{it} is the standard implementation of τ_t^A, and $J_{\psi_1}P_A^{it} = P_A^{it}J_{\psi_1}$.

(vi) There exists a one-parameter group of unitaries $P_A^{it} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} P_A^{it}$ on $H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}$, with natural values on elementary tensors, and a one-parameter group of unitaries $P^{it} \underset{N^o}{{}_\alpha\otimes_b} P_A^{it}$ on $H \underset{\nu^o}{{}_\alpha\otimes_b} H_{\psi_1}$, with natural values on elementary tensors, and we have, for all $t \in \mathbb{R}$:

$$\tilde{G}(P_A^{it} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} P_A^{it}) = (P^{it} \underset{N^o}{{}_\alpha\otimes_b} P_A^{it})\tilde{G}.$$

Proof. We know ([E6], 3.2) that $\Delta_{\tilde{\psi_1}}^{it} = \Delta_{\psi_1}^{it} \underset{N}{{}_b\otimes_\alpha} (\delta\Delta_{\widehat{\Phi}}^{-it})$; from which, using ([E5], 3.11 (ii), (vii) and (iv)) we get that

$$\begin{aligned}
\sigma_t^{\tilde{\psi_1}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J}) &= (\Delta_{\psi_1}^{it} \underset{N}{{}_b\otimes_\alpha} (\delta\Delta_{\widehat{\Phi}}^{-it}))(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J})(\Delta_{\psi_1}^{-it} \underset{N}{{}_b\otimes_\alpha} (\delta\Delta_{\widehat{\Phi}}^{it})) \\
&= 1 \underset{N}{{}_b\otimes_\alpha} (\hat{\delta}\Delta_\Phi)^{it}(\hat{J}\hat{\delta}^{is}\hat{J})(\hat{\delta}\Delta_\Phi)^{-it} \\
&= 1 \underset{N}{{}_b\otimes_\alpha} \Delta_\Phi^{it}(\hat{J}\hat{\delta}^{is}\hat{J})\Delta_\Phi^{-it} \\
&= 1 \underset{N}{{}_b\otimes_\alpha} P^{it}(\hat{J}\hat{\delta}^{it}\hat{J})(\hat{J}\hat{\delta}^{is}\hat{J})(\hat{J}\hat{\delta}^{-it}\hat{J})P^{-it} \\
&= 1 \underset{N}{{}_b\otimes_\alpha} P^{it}(\hat{J}\hat{\delta}^{is}\hat{J})P^{-it} = 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\tau}_t(\hat{\delta}^{is})\hat{J} = 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J}
\end{aligned}$$

which gives (i). From (i), and 3.9(iv) and 3.11, we get that

$$\sigma_t^{\psi_2}(\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J})) = \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J})$$

from which we get

$$\Delta_{\psi_1}^{it}(\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J})\Delta_{\psi_1}^{-it} = \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J})$$

which gives the commutation of the two one-parameter groups of unitaries $\Delta_{\psi_1}^{it}$ and $\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{is}\hat{J})$, and the existence of the one-parameter group of unitaries P_A^{it}.

We easily get that $\tilde{\mathfrak{a}}[(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\mathfrak{a}(X)(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})]$ is equal to

$$\begin{aligned}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J} \underset{N^o}{{}_{\hat{\alpha}}\otimes_\beta} \hat{J}\hat{\delta}^{-it}\hat{J})(\mathfrak{a}(X)_{\hat{\alpha}}\otimes_\beta 1)(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J} \underset{N^o}{{}_{\hat{\alpha}}\otimes_\beta} \hat{J}\hat{\delta}^{it}\hat{J})& \\ = [(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\mathfrak{a}(X)(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})] \underset{N^o}{{}_{\hat{\alpha}}\otimes_\beta} 1&\end{aligned}$$

from which we get, using ([E5], 10.12), that $(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\mathfrak{a}(X)(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})$ belongs to $\mathfrak{a}(A)$, and therefore, that $\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})X\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})$ belongs to A, from which it is straightforward to get that $P_A^{it}XP_A^{-it}$ belongs to A, and gives the existence of τ^A.

We have

$$\begin{aligned}\mathfrak{a}(\tau_t^A(x)) &= \mathfrak{a}[\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\sigma_t^{\psi_0}(x)\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})] \\ &= (1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})(\sigma_t^{\psi_0}(x) \underset{N}{{}_b\otimes_\alpha} 1)(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J}) = \sigma_t^{\psi_0}(x) \underset{N}{{}_b\otimes_\alpha} 1 = \mathfrak{a}(\sigma_t^{\psi_0}(x))\end{aligned}$$

from which we get that $\tau_t^A(x) = \sigma_t^{\psi_0}(x)$.

We have, for all $n \in N$:

$$\begin{aligned}\mathfrak{a}(\tau_t^A(b(n))) &= \mathfrak{a}[\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\sigma_t^{\psi_1}(b(n))\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})] \\ &= \mathfrak{a}[\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\delta_A^{-it}\sigma_t^{\phi}(b(n))\delta_A^{it}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})] \\ &= \mathfrak{a}[\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\delta_A^{-it}b(\sigma_{-t}^{\nu}(n))\delta_A^{it}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})] \\ &= (1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})(\delta_A^{-it} \underset{N}{{}_b\otimes_\alpha} \delta^{-it})(1 \underset{N}{{}_b\otimes_\alpha} \beta((\sigma_{-t}^{\nu}(n)))(\delta_A^{it} \underset{N}{{}_b\otimes_\alpha} \delta^{it})(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J}) \\ &= 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J}\delta^{-it}\beta((\sigma_{-t}^{\nu}(n)))\delta^{it}\hat{J}\hat{\delta}^{it}\hat{J} \\ &= 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J}\sigma_{-t}^{\Phi\circ R}\sigma_t^{\Phi}(\beta((\sigma_{-t}^{\nu}(n))))\hat{J}\hat{\delta}^{it}\hat{J} \\ &= 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J}\beta(\gamma_t(n))\hat{J}\hat{\delta}^{it}\hat{J}\end{aligned}$$

and therefore we get

$$\begin{aligned}\mathfrak{a}(\tau_t^A(b(n))) = 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\alpha(\gamma_t(n^*))\hat{\delta}^{it}\hat{J} = 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\sigma_{-t}^{\widehat{\Phi}\circ\hat{R}}\sigma_t^{\widehat{\Phi}}(\alpha(\gamma_t(n^*)))\hat{J}& \\ = 1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\alpha(\sigma_t^{\nu}(n^*))\hat{J} = 1 \underset{N}{{}_b\otimes_\alpha} \beta(\sigma_t^{\nu}(n)) = \mathfrak{a}(b(\sigma_t^{\nu}(n)))&\end{aligned}$$

which finishes the proof of (ii).

As, using ([E5], 3.10(vi)), we get that $\hat{J}\hat{\delta}^{-it}\hat{J}$ is equal to $P^{it}\Delta_{\Phi}^{-it}$, and therefore implements $\tau_t\sigma_{-t}^{\Phi}$, we get, using ([E5], 8.8), that

$$\mathfrak{a}(\tau_t^A(X)) = (\sigma_t^{\psi_1} \underset{N}{{}_b*_\alpha} \sigma_{-t}^{\Phi\circ R})\mathfrak{a}(X).$$

Using ([E5], 3.8 (i) and (ii)), we get that $\Gamma\circ\sigma_{-t}^{\Phi\circ R} = (\sigma_{-t}^{\Phi\circ R} \underset{N}{{}_b*_\alpha} \tau_t)\Gamma$, from which we infer

$$\begin{aligned}(\mathfrak{a} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(\tau_t^A(x)) &= (id \underset{N}{{}_b*_\alpha} \Gamma)\mathfrak{a}(\tau_t^A(X)) \\ &= (id \underset{N}{{}_b*_\alpha} \Gamma)(\sigma_t^{\psi_1} \underset{N}{{}_b*_\alpha} \sigma_{-t}^{\Phi\circ R})\mathfrak{a}(X) \\ &= (\sigma_t^{\psi_1} \underset{N}{{}_b*_\alpha} \sigma_{-t}^{\Phi\circ R} \underset{N}{{}_b*_\alpha} \tau_t)(id \underset{N}{{}_b*_\alpha} \Gamma)\mathfrak{a}(X) \\ &= (\sigma_t^{\psi_1} \underset{N}{{}_b*_\alpha} \sigma_{-t}^{\Phi\circ R} \underset{N}{{}_b*_\alpha} \tau_t)(\mathfrak{a} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(X) \\ &= (\mathfrak{a} \underset{N}{{}_b*_\alpha} id)(\tau_t^A \underset{N}{{}_b*_\alpha} \tau_t)\mathfrak{a}(X)\end{aligned}$$

from which we get that $\mathfrak{a}(\tau_t^A(X)) = (\tau_t^A \underset{N}{{}_b*_\alpha} \tau_t)\mathfrak{a}(X)$.

Finally, we have

$$\begin{aligned}\mathfrak{a}(\sigma_t^{\psi_1}(X)) &= (1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{it}\hat{J})\mathfrak{a}(\tau_t^A(X))(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J}) \\ &= (id \underset{N}{{}_b*_\alpha} \tau_{-t}\sigma_t^{\Phi})(\tau_t^A \underset{N}{{}_b*_\alpha} \tau_t)\mathfrak{a}(X) = (\tau_t^A \underset{N}{{}_b*_\alpha} \sigma_t^{\Phi})\mathfrak{a}(X)\end{aligned}$$

which finishes the proof of (iii).

We have, for any positive $X \in A$:

$$\begin{aligned}T_{\mathfrak{a}}(\tau_t^A(X)) &= (id \underset{N}{{}_b*_\alpha} \Phi)\mathfrak{a}(\tau_t^A(X)) \\ &= (id \underset{N}{{}_b*_\alpha} \Phi)(\tau_t^A \underset{N}{{}_b*_\alpha} \tau_t)\mathfrak{a}(X) = \tau_t^A(id \underset{N}{{}_b*_\alpha} \Phi\circ\tau_t)\mathfrak{a}(X).\end{aligned}$$

As $\Phi\circ\tau_t(Y) = \Phi(\lambda^{-t}Y)$ for any positive $Y \in M$ ([E5], 3.8 (vii)), we get that it is equal to $\tau_t^A[T_{\mathfrak{a}}(b(q)^{-t}X)]$, and using (ii), to $\sigma_t^{\psi_0}[T_{\mathfrak{a}}(b(q)^{-t}X)]$; from which we get (iv).

If ζ is in $D({}_rH_{\psi_1}, \psi_0)$, we have, using (3.8(i) and (iv)), and then (iii):

$$\begin{aligned}G(\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})(\Lambda_{\psi_1}(X) \underset{\psi_0}{{}_s\otimes_r} \zeta) &= (1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\sum_i e_i \underset{N}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\zeta,e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(X)) \\ &= \sum_i e_i \underset{N}{{}_b\otimes_\alpha} P^{it}\Delta_\Phi^{-it}\Lambda_\Phi((\omega_{\zeta,e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(X)) \\ &= \sum_i e_i \underset{N}{{}_b\otimes_\alpha} \lambda^{-t/2}\Lambda_\Phi[\tau_t\sigma_{-t}^{\Phi}(\omega_{\zeta,e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(X)] \\ &= \sum_i e_i \underset{N}{{}_b\otimes_\alpha} \Lambda_\Phi((\omega_{\zeta,e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(\tau_t^A\sigma_t^{\psi_1}X)) \\ &= (1 \underset{N}{{}_b\otimes_\alpha} \lambda^{-t/2})G(\Lambda_{\psi_1}(\tau_t^A\sigma_{-t}^{\psi_1}(X)) \underset{\psi_0}{{}_s\otimes_r} \zeta)\end{aligned}$$

from which we get that

$$\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\Lambda_{\psi_1}(X) = b(q)^{-t/2}\Lambda_{\psi_1}(\tau_t^A\sigma_{-t}^{\psi_1}(X))$$

which gives (v).

Thanks to (ii), one can easily get that if ζ belongs to $D({}_rH_{\psi_1},\psi_0)$, so does, for all $t\in\mathbb{R}$, $P_A^{it}\zeta$, and that $R^{r,\psi_0}(P_A^{it}\zeta)=P_A^{it}R^{r,\psi_0}(\zeta)\Delta_{\psi_0}^{-it}$. Using then (v), we obtain easily the existence of the first one-parameter group of unitaries. Using again (ii), we get that, if ζ' belongs to $D((H_{\psi_1})_b,\nu^o)$, so does $P_A^{it}\zeta'$, and that $R^{b,\nu^o}(P_A^{it}\zeta')=P_A^{it}R^{b,\nu^o}(\zeta')\Delta_\nu^{-it}$, from which one gets the existence of the second one-parameter group of unitaries. Moreover, using successively (v), 4.2(i), (iii), [E6], 3.8(vii) and (vi), and again 4.2(i), we get, for all $\zeta\in D({}_rH_{\psi_1},\psi_0)\cap D((H_{\psi_1})_b,\nu^o)$, $\zeta'\in D((H_{\psi_1})_b,\nu^o)$, $x\in\mathfrak{N}_{T_\mathfrak{a}}\cap\mathfrak{N}_{\psi_1}$:

$$\begin{aligned}(id*\omega_{P_A^{it}\zeta,P_A^{it}\zeta'}(\tilde G)P_A^{it}\Lambda_{\psi_1}(x)&=(id*\omega_{P_A^{it}\zeta,P_A^{it}\zeta'}(\tilde G)b(q)^{t/2}\Lambda_{\psi_1}(\tau_t^A(x))\\&=\Lambda_\Phi((\omega_{P_A^{it}\zeta,P_A^{it}\zeta'}\ {}_b\underset{N}{*}{}_\alpha\ id)\mathfrak{a}(b(q)^{t/2}\tau_t^A(x))\\&=\Lambda_\Phi(\beta(q)^{t/2}\tau_t^A(\omega_{\zeta,\zeta'}\ {}_b\underset{N}{*}{}_\alpha\ id)\mathfrak{a}(x))\\&=P^{it}\Lambda_\Phi(\omega_{\zeta,\zeta'}\ {}_b\underset{N}{*}{}_\alpha\ id)\mathfrak{a}(x))=P^{it}(id*\omega_{\zeta,\zeta'})(\tilde G)\Lambda_{\psi_1}(x)\end{aligned}$$

from which we get the formula we were looking for, and which finishes the proof. ∎

4.9. Theorem. *Let $(b,\mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 be a normal semi-finite faithful weight on $A^\mathfrak{a}$, having the Galois density property defined in 4.1; let $\psi_1=\psi_0\circ T_\mathfrak{a}$, and $\tilde G$ be its Galois unitary. Let us suppose that there exist two strongly commuting positive non-singular operators δ_A and λ_A, affiliated to A, such that the normal semi-finite faithful weight ϕ on A defined by $(D\phi:D\psi_1)_t=\lambda_A^{it^2/2}\delta_A^{it}$ (by [V1], 5.1) is invariant under $\mathfrak{a}$. Then:*

(i) There exists a one-parameter group of unitaries $\delta_A^{it}\ {}_b\underset{N}{\otimes}{}_\alpha\ \delta^{it}$ on $H_{\psi_1}\ {}_b\underset{\nu}{\otimes}{}_\alpha\ H$, having natural values on elementary tensors, such that, for all $t\in\mathbb{R}$:

$$\mathfrak{a}(\delta_A^{it})=\delta_A^{it}\ {}_b\underset{N}{\otimes}{}_\alpha\ \delta^{it}.$$

(ii) There exists a one-parameter group of unitaries $J_{\psi_1}\delta_A^{it}J_{\psi_1}\ {}_s\underset{A^\mathfrak{a}}{\otimes}{}_r\ \delta_A^{it}$ on $H_{\psi_1}\ {}_s\underset{\psi_0}{\otimes}{}_r\ H_{\psi_1}$, having natural values on elementary tensors, such that, for all $t\in\mathbb{R}$:

$$J_{\psi_1}\delta_A^{it}J_{\psi_1}\ {}_s\underset{A^\mathfrak{a}}{\otimes}{}_r\ \delta_A^{it}=K^{it}=\tilde G^*(J\delta^{it}J\ {}_\alpha\underset{N^o}{\otimes}{}_b\ 1)\tilde G.$$

(iii) We have $\lambda_A=b(q)$, where q is the positive non-singular operator affiliated to $Z(N)$, such that $\lambda=\alpha(q)=\beta(q)$ ([E5], 3.8(vi)); the operator λ_A is affiliated to $Z(A)$, and $(A,b,\mathfrak{a},\phi,\psi_0)$ is a Galois system for $\mathfrak{G}$.

(iv) We have $\tau_t^A(\delta_A^{is})=\delta_A^{is}$.

Proof. By definition, $(D\phi:D\psi_1)_t=\lambda_A^{it^2/2}\delta_A^{it}$, and therefore, $\mathfrak{a}(\lambda_A^{it^2/2})\mathfrak{a}(\delta_A^{it})=(D\overline{\phi}:D\overline{\psi_1})_t$ where $\overline{\phi}$ (resp. $\overline{\psi_1}$) is the weight on $A\ {}_b\underset{N}{*}{}_\alpha\ \mathcal{L}(H)$ given by the bidual weight on the bicrossed product (which is isomorphic to $A\ {}_b\underset{N}{*}{}_\alpha\ \mathcal{L}(H)$) ([E5], 11.6). As the weight ϕ is invariant with respect to $\mathfrak{a}$, the weight $\overline{\phi}$ is equal to another weight $\underline{\phi}$ ([E6], 7.7(x)), which is defined by the formula $\frac{d\underline{\phi}}{d\underline{\phi}^o}=\Delta_\phi^{1/2}\ {}_b\underset{N}{\otimes}{}_\alpha\ \Delta_{\widehat\Phi}^{-1/2}$ ([E6], 4.4). On the other hand, using ([E5], 13.7), and ([E6], 3.2), we get that $\frac{d\overline{\psi_1}}{d\psi_1^o}=\Delta_{\psi_1}^{1/2}\ {}_b\underset{N}{\otimes}{}_\alpha\ (\delta\Delta_{\widehat\Phi})^{-1/2}$.

Finally, we get

$$\mathfrak{a}(\lambda_A^{it^2/2})\mathfrak{a}(\delta_A^{it}) = (\Delta_\phi^{it} \ {}_b\underset{N}{\otimes}{}_\alpha \ \Delta_{\widehat{\Phi}}^{-it})[(D\phi^o : D\psi_1^o)_t \ {}_b\underset{N}{\otimes}{}_\alpha \ 1](\Delta_{\psi_1}^{-it} \ {}_b\underset{N}{\otimes}{}_\alpha \ (\delta\Delta_{\widehat{\Phi}})^{it})$$
$$= (D\phi : D\psi_1)_t \ {}_b\underset{N}{\otimes}{}_\alpha \ \delta^{it} = \lambda_A^{it^2/2}\delta_A^{it} \ {}_b\underset{N}{\otimes}{}_\alpha \ \delta^{it}$$

from which we get (i), and that λ_A is affiliated to $A^{\mathfrak{a}}$.

It is straightforward to get that there exists on $H_{\psi_1} \ {}_s\underset{\psi_0}{\otimes}{}_r \ H_{\psi_1}$ a one-parameter group of unitaries $J_{\psi_1}\delta_A^{it}J_{\psi_1} \ {}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r \ \delta_A^{it}$, having natural values on elementary tensors; using 3.8(i), we get, for any $x \in \mathfrak{N}_{T_{\mathfrak{a}}} \cap \mathfrak{N}_{\psi_1}$, $\zeta \in D((H_{\psi_1})_b, \nu^o)$ and $(e_i)_{i\in I}$ an orthogonal (b, ν^o)-basis of H_{ψ_1}:

$$\tilde{G}(J_{\psi_1}\delta_A^{it}J_{\psi_1} \ {}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r \ \delta_A^{it})(\Lambda_{\psi_1}(x) \ {}_s\underset{\psi_0}{\otimes}{}_r \ \zeta) = \tilde{G}(\Lambda_{\psi_1}(x\lambda_A^{-t/2}\delta_A^{-it}) \ {}_s\underset{\psi_0}{\otimes}{}_r \ \delta_A^{it}\zeta)$$
$$= \sum_i \Lambda_\Phi(\omega_{\delta_A^{it}\zeta, e_i} \ {}_b\underset{N}{*}{}_\alpha \ id)\mathfrak{a}(x\lambda_A^{-t/2}\delta_A^{-it}) \ {}_\alpha\underset{\nu^o}{\otimes}{}_b \ e_i$$

which, using (i) and the fact that λ_A is affiliated to $A^{\mathfrak{a}}$, is equal to

$$\sum_i \Lambda_\Phi((\omega_{\lambda_A^{-t/2}\zeta, e_i} \ {}_b\underset{N}{*}{}_\alpha \ id)\mathfrak{a}(x)\delta^{-it}) \ {}_\alpha\underset{\nu^o}{\otimes}{}_b \ e_i =$$
$$\sum_i J\delta^{it}J\lambda^{t/2}\Lambda_\Phi((\omega_{\lambda_A^{-t/2}\zeta, e_i} \ {}_b\underset{N}{*}{}_\alpha \ \mathfrak{a}(x)) \ {}_\alpha\underset{\nu^o}{\otimes}{}_b \ e_i =$$
$$(J\delta^{it}J\lambda^{t/2} \ {}_\alpha\underset{N^o}{\otimes}{}_b \ 1)\tilde{G}(\Lambda_{\psi_1}(x) \ {}_s\underset{\psi_0}{\otimes}{}_r \ \lambda_A^{-t/2}\zeta) =$$
$$(J\delta^{it}J \ {}_\alpha\underset{N^o}{\otimes}{}_b \ 1)\tilde{G}(\pi_{\mathfrak{a}}(1 \ {}_b\underset{N}{\otimes}{}_\alpha \ \lambda^{t/2})\Lambda_{\psi_1}(x) \ {}_s\underset{\psi_0}{\otimes}{}_r \ \lambda_A^{-t/2}\zeta)$$

from which we get that $(J_{\psi_1}\delta_A^{it}J_{\psi_1} \ {}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r \ \delta_A^{it})(\Lambda_{\psi_1}(x) \ {}_s\underset{\psi_0}{\otimes}{}_r \ \zeta)$ is equal to

$$\tilde{G}^*(J\delta^{it}J \ {}_\alpha\underset{N^o}{\otimes}{}_b \ 1)\tilde{G}(\pi_{\mathfrak{a}}(1 \ {}_b\underset{N}{\otimes}{}_\alpha \ \lambda^{t/2})\Lambda_{\psi_1}(x) \ {}_s\underset{\psi_0}{\otimes}{}_r \ \lambda_A^{-t/2}\zeta).$$

So, the map Q which sends $\Lambda_{\psi_1}(x){}_s\underset{\psi_0}{\otimes}{}_r\zeta$ on $\pi_{\mathfrak{a}}(1{}_b\underset{N}{\otimes}{}_\alpha\lambda^{t/2})\Lambda_{\psi_1}(x){}_s\underset{\psi_0}{\otimes}{}_r\lambda_A^{-t/2}\zeta$ is bounded; as it is clearly positive, by the unicity of polar decomposition, we get (ii), and the fact that $Q = 1$; from which one gets that λ_A is affiliated to $Z(A^{\mathfrak{a}})$, and that $\lambda_A^{t/2} = \pi_{\mathfrak{a}}(1{}_b\underset{N}{\otimes}{}_\alpha\lambda^{t/2})$, and therefore that

$$\lambda_A^{t/2} \ {}_b\underset{N}{\otimes}{}_\alpha \ 1 = \mathfrak{a}(\lambda_A^{t/2}) = 1 \ {}_b\underset{N}{\otimes}{}_\alpha \ \lambda^{t/2} = 1 \ {}_b\underset{N}{\otimes}{}_\alpha \ \beta(q^{t/2}) = \mathfrak{a}(b(q^{t/2}))$$

from which we get that $\lambda_A = b(q)$. Then, we get, for all $x \in A$:

$$\mathfrak{a}(\lambda_A^{it}x\lambda_A^{-it}) = (1 \ {}_b\underset{N}{\otimes}{}_\alpha \ \lambda^{it})\mathfrak{a}(x)(1 \ {}_b\underset{N}{\otimes}{}_\alpha \ \lambda^{-it}) = \mathfrak{a}(x)$$

because λ is affiliated to $Z(M)$; so we get that λ_A is affiliated to $Z(A)$, and by [V1] 5.2, that the modular groups of σ^ϕ and σ^{ψ_1} commute; which, thanks to 4.5, gives (iii).

We have

$$\tau_t^A(\delta_A^{is}) = \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\sigma_t^{\psi_1}(\delta_A^{is})\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})$$
$$= \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})\lambda_A^{ist}\delta_A^{is}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})$$

and therefore using 4.9 (i) and (iii):

$$\mathfrak{a}(\tau_t^A(\delta_A^{is})) = (1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})(1 \underset{N}{{}_b\otimes_\alpha} \lambda^{ist})(\delta_A^{is} \underset{N}{{}_b\otimes_\alpha} \delta^{is})(1 \underset{N}{{}_b\otimes_\alpha} \hat{J}\hat{\delta}^{-it}\hat{J})$$
$$= \delta_A^{is} \underset{N}{{}_b\otimes_\alpha} \lambda^{ist}\tau_t\sigma_{-t}^{\Phi}(\delta^{is}) = \delta_A^{is} \underset{N}{{}_b\otimes_\alpha} \lambda^{ist}\lambda^{-ist}\delta^{is} = \delta_A^{is} \underset{N}{{}_b\otimes_\alpha} \delta^{is} = \mathfrak{a}(\delta_A^{is})$$

from which we get (iv) and finish the proof. ∎

4.10. Corollary. *Let $(b, \mathfrak{a})$ be a Galois action of the measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 be a normal semi-finite faithful weight on $A^{\mathfrak{a}}$, having the Galois density property defined in 4.1; let $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$. Then the following are equivalent:*

(i) There exists a positive non-singular operator δ_A affiliated to A such that, for all $t \in \mathbb{R}$, we have $K^{it} = J_{\psi_1}\delta_A^{it}J_{\psi_1} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \delta_A^{it}$.

(ii) There exists two strongly commuting positive non-singular operator δ_A and λ_A, affiliated to A, such that the normal semi-finite faithful weight ϕ on A defined by $(D\phi : D\psi_1)_t = \lambda_A^{it^2/2}\delta_A^{it}$ (by [V1], 5.1) is invariant under $\mathfrak{a}$.

(iii) There exists a normal semi-finite faithful weight ϕ on A, such that $(A, b, \mathfrak{a}, \phi, \psi_0)$ is a Galois system.

Moreover, δ_A is the modulus of the action $(b, \mathfrak{a})$, and $\lambda_A = b(q)$, where $q\eta Z(N)$ is such that $\lambda = \alpha(q) = \beta(q)$.

Proof. We have obtained in 4.5 that (i) implies (iii); in 4.9, we have obtained that (ii) implies (i) and (iii); and applying 4.9 to (iii), we obtain (ii). ∎

4.11. Proposition. *Let $(A, b, \mathfrak{a}, \phi, \psi_0)$ be a Galois system for the measured quantum groupoid $\mathfrak{G}$; let $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, $\mathfrak{T}$ be the normal semi-finite faithful weight from A onto $b(N)$ such that $\phi = \nu^o \circ b^{-1} \circ \mathfrak{T}$, and r the canonical injection of $A^{\mathfrak{a}}$ into A. Then:*

(i) The left ideal $\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_{\mathfrak{a}}} \cap \mathfrak{N}_\phi \cap \mathfrak{N}_{\mathfrak{T}}$ is dense in A.

(ii) The subspace $\Lambda_\phi(\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_{\mathfrak{a}}} \cap \mathfrak{N}_\phi \cap \mathfrak{N}_{\mathfrak{T}})$ is dense in H_ϕ.

(iii) The subspace $D((H_{\psi_1})_b, \nu^o) \cap D({}_rH_{\psi_1}, \psi_0)$ is dense in H_{ψ_1}, i.e. ψ_0 satisfies the Galois density property defined in 4.1.

Proof. Using [V1], we know that, if x in A is such that $x\delta_A^{1/2}$ is bounded and its closure $\overline{x\delta_A^{1/2}}$ belongs to $\mathfrak{N}_\phi$, then x belongs to $\mathfrak{N}_{\psi_1}$; we can then (and we shall) identify $\Lambda_{\psi_1}(x)$ with $\Lambda_\phi(\overline{x\delta_A^{1/2}})$ and J_{ψ_1} with $\lambda_A^{i/4}J_\phi$. In particular, using the selfadjoint elements of A given by the formula

$$e_n = \frac{2n^2}{\Gamma(1/2)\Gamma(1/4)}\int_{\mathbb{R}^2} e^{-n^2x^2-n^4y^4}\lambda_A^{ix}\delta_A^{iy}dxdy$$

which are analytic with respect to σ^ϕ and such that, for any $z \in \mathbb{C}$, the sequence $\sigma_z^\phi(e_n)$ is bounded and strongly converges to 1, we get that for any $x \in \mathfrak{N}_\phi$, $x(e_n\delta_A^{1/2})$ belongs to $\mathfrak{N}_{\psi_1}$.

Let $\mathfrak{T}$ be the normal faithful semi-finite operator-valued weight from A onto $b(N)$ such that $\phi = \nu^o \circ b^{-1} \circ \mathfrak{T}$. Let us suppose that x is positive in the Tomita algebra $\mathcal{T}_{\phi,\mathfrak{T}}$ ([E5], 2.2.1) associated to ϕ and $\mathfrak{T}$ (i.e. x belongs to $\mathfrak{N}_\phi \cap \mathfrak{N}_\mathfrak{T}$, is analytical with respect to σ^ϕ, and for all $z \in \mathbb{C}$, $\sigma_z^\phi(x)$ belongs to $\mathfrak{N}_\phi \cap \mathfrak{N}_\phi^* \cap \mathfrak{N}_\mathfrak{T} \cap \mathfrak{N}_\mathfrak{T}^*$. As in ([L], 5,17), let us define

$$x_{p,q} = f_p\sqrt{\frac{d}{\pi}}\int_{-\infty}^{+\infty} e^{-qt^2}\sigma_t^{\psi_1}(x)dt$$

with $f_p = \int_{1/p}^p de_t$, where $\lambda_A = \int_0^\infty t de_t$ and we get that $x_{p,q}$ belongs to $T_{\phi,\mathfrak{T}}$, is analytical with respect to σ^{ψ_1}, and that, for all $z \in \mathbb{C}$, $\sigma_z^{\psi_1}(x_{p,q})$ belongs to $T_{\phi,\mathfrak{T}}$. As $\sigma_t^{\psi_1}\sigma_{-t}^\phi = Ad\delta_A^{it}$, we get that, for all z in $\mathbb{C}$, $\delta_A^{iz}x_{p,q}\delta_A^{-iz}$ belongs to $T_{\phi,\mathfrak{T}}$; in particular, $\delta_A^{-1/2}x_{p,q}\delta_A^{1/2}$ belongs to $T_{\phi,\mathfrak{T}}$ and $e_n x_{p,q}\delta_A^{1/2} = (\delta_A^{1/2}e_n)\delta_A^{-1/2}x_{p,q}\delta_A^{1/2}$ belongs to $\mathfrak{N}_\phi \cap \mathfrak{N}_\mathfrak{T}$. We prove this way that the set $T_{\phi,\mathfrak{T}}^{\psi_1}$ of elements x in $\mathfrak{N}\phi \cap \mathfrak{N}_\phi^* \cap \mathfrak{N}_\mathfrak{T} \cap \mathfrak{N}_\mathfrak{T}^*$ which are analytic with respect, both, of σ^ϕ and σ^{ψ_1}, and such that $x\delta_A^{1/2}$ is bounded and belongs to $\mathfrak{N}_\phi \cap \mathfrak{N}_\mathfrak{T}$ is weakly dense in A, and its image under Λ_ϕ is a dense subspace of H_ϕ.

Let us take $x \in T_{\phi,\mathfrak{T}}^{\psi_1}$; using the fact that ψ_1 is a δ-invariant weight with respect to $\mathfrak{a}$ (2.5), we get (where a is the representation of N on H_{ψ_1} given by $a(n) = J_{\psi_1}b(n^*)J_{\psi_1}$) that

$$\begin{aligned}(T_\mathfrak{a} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x^*x) &= \delta^{1/2}\beta(< \Lambda_{\psi_1}(x), \Lambda_{\psi_1}(x) >_{a,\nu})\delta^{1/2}\\ &= \delta^{1/2}\beta(< J_{\psi_1}\Lambda_{\psi_1}(x), J_{\psi_1}\Lambda_{\psi_1}(x) >_{b,\nu^o})\delta^{1/2}\\ &= \delta^{1/2}\beta(< \lambda_A^{i/4}J_\phi\Lambda_\phi(x\delta_A^{1/2}), \lambda_A^{i/4}J_\phi\Lambda_\phi(x\delta_A^{1/2}) >_{b,\nu^o})\delta^{1/2}\end{aligned}$$

and therefore if η belongs to $D(_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$, we get that $(id \underset{N}{{}_b*_\alpha} \omega_\eta)\mathfrak{a}(x^*x)$ belongs to $\mathfrak{M}_{T_\mathfrak{a}}^+$ (and to $\mathfrak{M}_{\psi_1}^+$ by similar arguments).

Using now the fact that ϕ is invariant with respect to $\mathfrak{a}$, we get (where a means here the representation of N on H_ϕ given by $a(n) = J_\phi b(n^*)J_\phi$) that

$$(\mathfrak{T} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x^*x) = \beta(< \Lambda_\phi(x), \Lambda_\phi(x) >_{a,\nu}) = \beta(< J_\phi\Lambda_\phi(x), J_\phi\Lambda_\phi(x) >_{b,\nu^o})$$

and we get that $(id \underset{N}{{}_b*_\alpha} \omega_\eta)\mathfrak{a}(x^*x)$ belongs also to $\mathfrak{M}_\mathfrak{T}^+ \cap \mathfrak{M}_\phi^+$. So, we get that, for any $x \in T_{\phi,\mathfrak{T}}^{\psi_1}$, $\xi \in D(_\alpha H, \nu)$, $\eta \in D(_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$, the operator $(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\mathfrak{a}(x)$ belongs to $\mathfrak{N}_{T_\mathfrak{a}} \cap \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_\mathfrak{T} \cap \mathfrak{N}_\phi$.

So, we get that the weak closure of $\mathfrak{N}_{T_\mathfrak{a}} \cap \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_\mathfrak{T} \cap \mathfrak{N}_\phi$ contains all elements of the form $(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi}\mathfrak{a}(x)$ for any ξ, η in $D(_\alpha H, \nu)$ and $x \in A$; using now [E5], 11.5(ii), we get (i).

Let us suppose now that $\zeta \in H_\phi$ is orthogonal to $\Lambda_\phi(\mathfrak{N}_{T_\mathfrak{a}} \cap \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_\mathfrak{T} \cap \mathfrak{N}_\phi)$. Using 2.4 and ([E6], 7.7), we get that

$$(V'_\phi(\Lambda_\phi(x) \underset{N^o}{{}_a\otimes_\beta} \eta)|\zeta \underset{N}{{}_b\otimes_\alpha} \xi) = 0$$

for all $x \in T^{\psi_1}_{\phi,\mathfrak{T}}$, $\eta \in D_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$, $\xi \in D(_\alpha, \nu)$. As $\Lambda_\phi(T^{\psi_1}_{\phi,\mathfrak{T}})$ is dense in H_ϕ and V'_ϕ is a unitary, we get that $\zeta \,{}_b\underset{N}{\otimes}{}_\alpha\, \xi) = 0$ for all $\xi \in D(_\alpha H, \nu)$, and therefore, that $\zeta = 0$; which is (ii).

We know that $J_\phi \Lambda_\phi(\mathfrak{N}_\phi \cap \mathfrak{N}_{\mathfrak{T}}) \subset D((H_\phi)_b, \nu^o)$ and that $J_{\psi_1}\Lambda_{\psi_1}(\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_\mathfrak{a}}) \subset D(_r H_{\psi_1}, \psi^o)$. As the canonical isomorphism between H_ϕ and H_{ψ_1} exchanges the representations of A (and, therefore, of $b(N)$), and sends J_ϕ on $J_{\psi_1}\lambda_A^{i/4}$, we get that

$$J_{\psi_1}\Lambda_{\psi_1}(\mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{T_\mathfrak{a}} \cap \mathfrak{N}_\phi \cap \mathfrak{N}_{\mathfrak{T}}) \subset D((H_{\psi_1})_b, \nu^o) \cap D(_r H_{\psi_1}, \psi_0)$$

from which we get (iii). ∎

5. Through the looking-glass. In this chapter, we use the reflection technic introduced by De Commer in [DC1]; if we start from a Galois action $(b, \mathfrak{a})$ of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A, we obtain a co-involutive Hopf bimodule which has $A^\mathfrak{a}$ as basis (5.4). If we start from a Galois system $(A, b, \mathfrak{a}, \phi, \psi_0)$, we then construct a left-invariant operator-valued weight on this co-involutive Hopf bimodule, and obtain this way, "through the Galois system", another measured quantum groupoid. More precisely, we get in fact two measured quantum groupoids, one with basis $A^\mathfrak{a}$, called the reflected measured quantum groupoid of $\mathfrak{G}$, through the Galois system, whose underlying von Neumann algebra acts on H_{ψ_1} (5.11), and another one which will be a von Neumann algebra acting on $H_{\psi_1} \oplus H$, with the basis $A^\mathfrak{a} \oplus N$, and will be called the linking measured quantum groupoid, between the preceding two (5.12).

5.1. Notations. Let $(b, \mathfrak{a})$ be a Galois action of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra A; let ψ_0 be a normal semi-finite faithful weight on $A^\mathfrak{a}$ satisfying the density condition. Let us now consider the von Neumann algebra $\widetilde{N} = A^\mathfrak{a} \oplus N$, equipped with a normal faithful semi-finite weight $\psi_0 \oplus \nu$, its representation $\widetilde{\alpha} = r \oplus \alpha$, and its antirepresentation $\widetilde{\hat{\beta}} = s \oplus \hat{\beta}$ on the Hilbert space $H_{\psi_1} \oplus H$. For any $m' \in \widehat{M}'$, let us write $\mu(m') = \pi_\mathfrak{a}(1 \,{}_b\underset{N}{\otimes}{}_\alpha\, m')$, and consider the operator $\varpi(m') = \mu(m') \oplus m'$ on $H_{\psi_1} \oplus H$; we define this way a normal faithful representation ϖ of $\widehat{M}'$ on $H_{\psi_1} \oplus H$, and a faithful normal antirepresentation ϖ^o of $\widehat{M}$ given, for any $m \in \widehat{M}$ by

$$\varpi^o(m) = \mu(\hat{J}m^*\hat{J}) \oplus \hat{J}m^*\hat{J}.$$

We shall denote by $\widehat{Q}$ the commutant $\varpi^o(\widehat{M})'$. We shall use matrix notation for elements in $\widehat{Q}$, or, more generally, in $\mathcal{L}(H_{\psi_1} \oplus H)$. In particular, we shall write

$$\widehat{Q} = \begin{pmatrix} \widehat{P} & \widehat{I} \\ \widehat{I}^* & \widehat{M} \end{pmatrix}$$

where $\widehat{P} = \pi_\mathfrak{a}(1 \,{}_b\underset{N}{\otimes}{}_\alpha\, \widehat{M}')'$, and $\widehat{I}$ is the following closed linear set of intertwiners:

$$\widehat{I} = \{X \in \mathcal{L}(H, H_{\psi_1}), Xm = \pi_\mathfrak{a}(1 \,{}_b\underset{N}{\otimes}{}_\alpha\, m)X, \forall m \in \widehat{M}'\}$$

We see that $r(A^\mathfrak{a}) \subset \widehat{P}$ and $s(A^\mathfrak{a}) \subset \widehat{P}$, and therefore, $\widetilde{\alpha}(\tilde{N}) \subset \widehat{Q}$ and $\widetilde{\hat{\beta}}(\tilde{N}) \subset \widehat{Q}$.

Let us remark that, for any $\xi \in D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o)$, the operator $R^{\mu^o, \widehat{\Phi}^o}(\xi)$ belongs to $\widehat{I}$ (which implies that $\widehat{I}$ is not reduced to $\{0\}$). Using 3.7, we get that, if $X \in \widehat{I}$, we have, for any $m \in \widehat{M}'$:

$$J_{\psi_1} X J m = J_{\psi_1} X \hat{R}^c(m^*) J = J_{\psi_1} \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \hat{R}^c(m^*)) X J = \pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} m) J_{\psi_1} X J$$

from which we get that $J_{\psi_1} X J$ belongs to $\widehat{I}$.

In particular, for any $n \in N$, we get that $X \in \widehat{I}$ satisfies $b(n)X = X\beta(n)$, and we can define $1_H \underset{N^o}{{}_\alpha\otimes_\beta} X$ from $H \underset{N}{{}_\alpha\otimes_\beta} H$ to $H \underset{N}{{}_\alpha\otimes_b} H_{\psi_1}$. Applying this result to $J_{\psi_1} X J$, we get that

$$X\hat{\alpha}(n) = XJ\beta(n^*)J = J_{\psi_1} b(n^*) J_{\psi_1} X = a(n)X$$

and we can define $X \underset{N}{{}_{\hat{\alpha}}\otimes_\beta} 1_H$ from $H \underset{\nu^o}{{}_{\hat{\alpha}}\otimes_\beta} H$ to $H_{\psi_1} \underset{\nu^o}{{}_a\otimes_\beta} H$, and $X \underset{N}{{}_\beta\otimes_\alpha} 1_H$ from $H \underset{\nu}{{}_\beta\otimes_\alpha} H$ to $H_{\psi_1} \underset{\nu}{{}_b\otimes_\alpha} H$.

Using then 3.6, we get that

$$V_{\psi_1}^*(X \underset{N}{{}_\beta\otimes_\alpha} 1_H) = (X \underset{N}{{}_{\hat{\alpha}}\otimes_\beta} 1_H)(\sigma(\hat{J} \otimes \hat{J}) W (\hat{J} \otimes \hat{J})\sigma),$$

$$V_{\psi_1}(X \underset{N}{{}_{\hat{\alpha}}\otimes_\beta} 1_H) = (X \underset{N}{{}_\beta\otimes_\alpha} 1_H)(\sigma(\hat{J} \otimes \hat{J}) W^* (\hat{J} \otimes \hat{J})\sigma).$$

Let us denote $e_1 = 1_{A^{\mathfrak{a}}} \in \widetilde{N}$, and $e_2 = 1_N \in \widetilde{N}$; we get that $\widetilde{\alpha}(e_1) = \widetilde{\hat{\beta}}(e_1) = P_{H_{\psi_1}} \in \widehat{P}$, and that $\widetilde{\alpha}(e_2) = \widetilde{\hat{\beta}}(e_2) = P_H \in \widehat{M}$, and $\widehat{P} = \widehat{Q}_{\widetilde{\alpha}(e_1)}$, $\widehat{M} = \widehat{Q}_{\widetilde{\alpha}(e_2)}$. We can verify that $s(A^{\mathfrak{a}}) \subset \widehat{P}$ and $r(A^{\mathfrak{a}}) \subset \widehat{P}$.

Let us describe now the fiber product $\widehat{Q} \underset{\widetilde{N}}{{}_{\widetilde{\hat{\beta}}}*_{\widetilde{\alpha}}} \widehat{Q}$. This von Neumann algebra is defined on the Hilbert space

$$(H_{\psi_1} \oplus H) \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} (H_{\psi_1} \oplus H) = (H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}) \oplus (H \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} H)$$

where this direct sum decomposition can be seen with the projections

$$P_{H_{\psi_1} \underset{\psi_0}{{}_s\otimes_r} H_{\psi_1}} = \widetilde{\alpha}(e_1) \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} 1 = 1 \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} \widetilde{\alpha}(e_1) = \widetilde{\alpha}(e_1) \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} \widetilde{\alpha}(e_1),$$

$$P_{H \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} H} = \widetilde{\alpha}(e_2) \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} 1 = 1 \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} \widetilde{\alpha}(e_2) = \widetilde{\alpha}(e_2) \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} \widetilde{\alpha}(e_2).$$

So, we can also use matrix notations for elements in $\widehat{Q} \underset{\widetilde{N}}{{}_{\widetilde{\hat{\beta}}}*_{\widetilde{\alpha}}} \widehat{Q}$, or, more generally, in $\mathcal{L}((H_{\psi_1} \oplus H) \underset{\psi_0 \oplus \nu}{{}_{\widetilde{\hat{\beta}}}\otimes_{\widetilde{\alpha}}} (H_{\psi_1} \oplus H))$. In particular, we shall get

$$\widehat{Q} \underset{\widetilde{N}}{{}_{\widetilde{\hat{\beta}}}*_{\widetilde{\alpha}}} \widehat{Q} = \begin{pmatrix} \widehat{P} \underset{A^{\mathfrak{a}}}{{}_s*_r} \widehat{P} & \widehat{I} \underset{N}{{}_{\hat{\beta}}*_\alpha} \widehat{I} \\ (\widehat{I} \underset{N}{{}_{\hat{\beta}}*_\alpha} \widehat{I})^* & \widehat{M} \underset{N}{{}_{\hat{\beta}}*_\alpha} \widehat{M} \end{pmatrix}$$

where $\widehat{I} \underset{N}{{}_{\hat{\beta}}*_{\alpha}} \widehat{I}$ is the closed set of intertwiners:

$$\widehat{I} \underset{N}{{}_{\hat{\beta}}*_{\alpha}} \widehat{I} = \{Y \in \mathcal{L}(H \underset{\nu}{{}_{\hat{\beta}}\otimes_{\alpha}} H, H_{\psi_1} \underset{\psi_0}{{}_{s}\otimes_{r}} H_{\psi_1}), Y(m_1 \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} m_2) = (\mu(m_1) \underset{A^{\mathfrak{a}}}{{}_{s}\otimes_{r}} \mu(m_2))Y, \forall m_1, m_2 \in \widehat{M}'\}.$$

5.2. Lemma. *Let's use the notations of 5.1. Then we have, for all $X \in \widehat{I}$ and $m \in \widehat{M}'$:*

$$\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}(m \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} 1) = [\pi_{\mathfrak{a}}(1 \underset{N}{{}_{b}\otimes_{\alpha}} m] \underset{A^{\mathfrak{a}}}{{}_{s}\otimes_{r}} 1)\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W},$$

$$\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}(1 \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} m) = [1 \underset{A^{\mathfrak{a}}}{{}_{s}\otimes_{r}} \pi_{\mathfrak{a}}(1 \underset{N}{{}_{b}\otimes_{\alpha}} m)]\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W},$$

and therefore $\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}$ belongs to $\widehat{I} \underset{N}{{}_{\hat{\beta}}*_{\alpha}} \widehat{I}$. We obtain also that, for any $v \in D((H_{\psi_1})_s, \psi_0^o)$ and $\xi \in D(H_{\hat{\beta}}, \nu^o)$, the operator $(\omega_{v,\xi} * id)[\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}]$ belongs to $\widehat{I}$.

Proof. Using ([E5], 3.6(ii)) applied to $\widehat{\mathfrak{G}}$, and 3.8(iv), we get the first formula.

Using 3.10, we get that

$$[1 \underset{A^{\mathfrak{a}}}{{}_{s}\otimes_{r}} \pi_{\mathfrak{a}}(1 \underset{N}{{}_{b}\otimes_{\alpha}} m)]G^* = G^*V_{\psi_1}[1 \underset{N^o}{{}_{a}\otimes_{\beta}} \hat{J}\hat{R}^c(m^*)\hat{J}]V_{\psi_1}^*$$

and therefore using 3.10, 5.1 and 2.2.3:

$$\begin{aligned}
[1 \underset{A^{\mathfrak{a}}}{{}_{s}\otimes_{r}} \pi_{\mathfrak{a}}(1 \underset{N}{{}_{b}\otimes_{\alpha}} m)]\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X) &= G^*V_{\psi_1}[1 \underset{N^o}{{}_{a}\otimes_{\beta}} \hat{J}\hat{R}^c(m^*)\hat{J}]V_{\psi_1}^*(X \underset{N}{{}_{b}\otimes_{\alpha}} 1)\sigma_{\nu^o} \\
&= G^*V_{\psi_1}[1 \underset{N^o}{{}_{a}\otimes_{\beta}} \hat{J}\hat{R}^c(m^*)\hat{J}](X \underset{N^o}{{}_{\hat{\alpha}}\otimes_{\beta}} 1_H)(\sigma W^o\sigma)\sigma \\
&= G^*V_{\psi_1}(X \underset{N^o}{{}_{\hat{\alpha}}\otimes_{\beta}} 1_H)[1 \underset{N^o}{{}_{\hat{\alpha}}\otimes_{\beta}} \hat{J}\hat{R}^c(m^*)\hat{J}](\sigma W^o\sigma)\sigma \\
&= G^*((X \underset{N}{{}_{\beta}\otimes_{\alpha}} 1_H)(\sigma W^o\sigma)^*[1 \underset{N^o}{{}_{a}\otimes_{\beta}} \hat{J}\hat{R}^c(m^*)\hat{J}](\sigma W^o\sigma)\sigma \\
&= \tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)W^{o*}(\hat{J}\hat{R}^c(m^*)\hat{J} \underset{N}{{}_{\beta}\otimes_{\hat{\alpha}}} 1)W^o \\
&= \tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}(1 \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} m)\widehat{W}^*
\end{aligned}$$

from which we get the second formula and finish the proof. ∎

5.3. Proposition. *With the notations of 5.1, for any $X \in \widehat{I}$, we have*

$$\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} J_{\psi_1}XJ)\widehat{W} = \varsigma_N[(J_{\psi_1} \underset{A^{\mathfrak{a}}}{{}_{s}\otimes_{r}} J_{\psi_1})\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}(J \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} J)].$$

Proof. Using 3.13 (i), we get that $(J_{\psi_1} \underset{A^{\mathfrak{a}o}}{{}_{r}\otimes_{s}} J_{\psi_1})\sigma_{\psi_0}\tilde{G}^*(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}\sigma_{\nu^o}(J \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} J)$ is equal to

$$\begin{aligned}
&\tilde{G}^*(\hat{J} \underset{N}{{}_{\beta}\otimes_{a}} J_{\psi_1})\sigma_{\nu^o}V_{\psi_1}^*\sigma_{\nu^o}(1 \underset{N^o}{{}_{\alpha}\otimes_{\beta}} X)\widehat{W}\sigma_{\nu^o}(J \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} J) \\
&\qquad = \tilde{G}^*(\hat{J} \underset{N}{{}_{\beta}\otimes_{a}} J_{\psi_1})\sigma_{\nu^o}V_{\psi_1}^*(X \underset{N}{{}_{\beta}\otimes_{\alpha}} 1)\sigma_{\nu^o}\widehat{W}\sigma_{\nu^o}(J \underset{N}{{}_{\hat{\beta}}\otimes_{\alpha}} J).
\end{aligned}$$

Using now 5.1, we get it is equal to

$$\tilde{G}^*(\hat{J} \underset{N}{{}_\beta\otimes_a} J_{\psi_1})\sigma_{\nu^o}(X \underset{N^o}{{}_{\hat{\alpha}}\otimes_\beta} 1)(\sigma W^o\sigma)\sigma_{\nu^o}\widehat{W}\sigma_{\nu^o}(J \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} J)$$

which, by 3.13(ii), is equal to $\tilde{G}^*(1 \underset{N^o}{{}_\alpha\otimes_\beta} J_{\psi_1}XJ)\widehat{W}$. ∎

5.4. Proposition. *(i) With the notations of 5.1, for any element* $\begin{pmatrix} A & X \\ Y^* & m \end{pmatrix}$ *in* $\widehat{Q}$*, let us write*

$$\Gamma_{\widehat{Q}}(\begin{pmatrix} A & X \\ Y^* & m \end{pmatrix}) = \begin{pmatrix} \tilde{G}^*(1 \underset{N^o}{{}_\alpha\otimes_b} A)\tilde{G} & \tilde{G}^*(1 \underset{N^o}{{}_\alpha\otimes_\beta} X)\widehat{W} \\ \widehat{W}^*(1 \underset{N^o}{{}_\alpha\otimes_b} Y^*)\tilde{G} & \widehat{\Gamma}(m) \end{pmatrix}.$$

Then, we define a mapping $\Gamma_{\widehat{Q}}$ *from* $\widehat{Q}$ *into* $\widehat{Q} \underset{\tilde{N}}{{}_{\tilde{\tilde{\beta}}}*_{\tilde{\alpha}}} \widehat{Q}$ *which is a coproduct. So,* $(\tilde{N}, \widehat{Q}, \tilde{\alpha}, \tilde{\tilde{\beta}}, \Gamma_{\widetilde{Q}})$ *is a Hopf bimodule.*

(ii) Let us write

$$R_{\widehat{Q}}(\begin{pmatrix} A & X \\ Y^* & m \end{pmatrix}) = \begin{pmatrix} J_{\psi_1}A^*J_{\psi_1} & J_{\psi_1}YJ \\ JX^*J_{\psi_1} & Jm^*J \end{pmatrix}.$$

Then, we define an involutive $*$*-anti-isomorphism* $R_{\widehat{Q}}$ *of* $\widehat{Q}$*, which a co-involution for the coproduct* $\Gamma_{\widehat{Q}}$*.*

(iii) For any $A \in \widehat{P}$*, let us write* $\Gamma_{\widehat{P}}(A) = \tilde{G}^*(1 \underset{N^o}{{}_\alpha\otimes_b} A)\tilde{G}$*, and* $R_{\widehat{P}}(A) = J_{\psi_1}A^*J_{\psi_1}$*; then* $(A^{\mathfrak{a}}, \widehat{P}, r, s, \Gamma_{\widehat{P}})$ *is a Hopf bimodule, and* $R_{\widehat{P}}$ *is a co-involution for* $\Gamma_{\widehat{P}}$*.*

Proof. We have got in 5.2 that $\tilde{G}^*(1 \underset{N}{{}_\alpha\otimes_\beta} X)\widehat{W}$ belongs to $\widehat{I} \underset{N}{{}_{\hat{\beta}}*_\alpha} \widehat{I}$; so, for any ξ, η in $D({}_\mu H_{\psi_1}, \widehat{\Phi}')$, we get that $\tilde{G}^*(1 \underset{N}{{}_\alpha\otimes_b} \theta^{\mu,\widehat{\Phi}'}(\xi,\eta))\tilde{G}$ commutes with $\mu(\widehat{M}') \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \mu(\widehat{M}')$, and therefore, belongs to $\widehat{P} \underset{A^{\mathfrak{a}}}{{}_s*_r} \widehat{P}$; by continuity and density, this remains true for any A in $\widehat{P}$. So, we have got that $\Gamma_{\widehat{Q}}$ is an injective $*$-homomorphism from $\widehat{Q}$ into $\widehat{Q} \underset{\tilde{N}}{{}_{\tilde{\tilde{\beta}}}*_{\tilde{\alpha}}} \widehat{Q}$. The fact that it is a coassociative coproduct is given by 4.2(iv), which gives (i).

We have seen in 5.1 that $J_{\psi_1}YJ$ belongs to $\widehat{I}$; therefore, for any ξ, η in $D({}_\mu H_{\psi_1}, \widehat{\Phi}')$, we get that $J_{\psi_1}\theta^{\mu,\widehat{\Phi}'}(\xi,\eta)J_{\psi_1}$ belongs to $\widehat{P}$, and by density, that remains true for any A in $\widehat{P}$. The fact that we obtain a co-involution is given by 5.3, which gives (ii). As $\widehat{P} = \widehat{Q}_{\tilde{\alpha}(e_1)}$, we easily get (iii). ∎

5.5. Proposition. *Let* $(A, b, \mathfrak{a}, \phi, \psi_0)$ *be a Galois system for* $\mathfrak{G}$*; let* $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$*; let* δ_A *be the modulus introduced in 3.11, and* P_A *be the generator of the one-parameter group of unitaries introduced in 4.8. Then:*

(i) There exists a one-parameter group of unitaries $\widehat{\Delta}_A^{it} = P_A^{it}J_{\psi_1}\delta_A^{it}J_{\psi_1}$*.*

(ii) We have, for all $t \in \mathbb{R}$ *and* $m \in \widehat{M}'$*:*

$$\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \sigma_t^{\widehat{\Phi}^c}(m)) = \widehat{\Delta}_A^{-it}\pi_A(1 \underset{N}{{}_b\otimes_\alpha} m)\widehat{\Delta}_A^{it}$$

and, in particular, for any $n \in N$:

$$\widehat{\Delta}_A^{-it} b(n) \widehat{\Delta}_A^{it} = b(\sigma_{-t}^{\nu}(n)).$$

(iii) We have

$$(\Delta_\Phi^{it} \underset{N}{{}_\alpha\otimes_b} \widehat{\Delta}_A^{it})\tilde{G} = \tilde{G}(\Delta_{\psi_1}^{it} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \widehat{\Delta}_A^{it}), \qquad (\Delta_{\widehat{\Phi}}^{it} \underset{N}{{}_\alpha\otimes_b} P_A^{it})\tilde{G} = \tilde{G}(\widehat{\Delta}_A^{it} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} P_A^{it}\delta_A^{it}).$$

Proof. Using 4.8(vi), we get that P_A^{it} commutes with J_{ψ_1}; using 4.8(v), we get that P_A^{it} commutes with δ_A^{is}; so, we get (i).

Let now $x \in \mathfrak{N}_{\psi_1}$, and let ξ be in $D({}_\alpha H, \nu)$ and η in $D({}_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$, such that $\delta^{1/2}\eta$ belongs to $D(H_\beta, \nu^o)$. We have then, using 4.8(v), [E5], 8.4(iii), 4.9(i) and (iii), and again 4.8(v) and [E5], 8.4(iii):

$$\begin{aligned}(id * \omega_{\eta,\xi})(V_{\psi_1})\widehat{\Delta}_A^{-it}\Lambda_{\psi_1}(x) &= (id * \omega_{\eta,\xi})(V_{\psi_1})\Lambda_{\psi_1}(\lambda_A^{-t/2}\tau_t^A(x)\delta_A^{-it}) \\ &= \Lambda_{\psi_1}[(id \underset{N}{{}_b*_\alpha} \omega_{\delta^{1/2}\eta,\xi})\mathfrak{a}(\lambda_A^{-t/2}\tau_t^A(x)\delta_A^{-it})] \\ &= \Lambda_{\psi_1}[\lambda_A^{-t/2}\tau_t^A(id \underset{N}{{}_b*_\alpha} \omega_{P^{-it}\delta^{-it}\delta^{1/2}\eta, P^{-it}\xi})\mathfrak{a}(x)\delta_A^{-it}] \\ &= \widehat{\Delta_A}^{-it}\Lambda_{\psi_1}((id \underset{N}{{}_b*_\alpha} \omega_{P^{-it}\delta^{-it}\delta^{1/2}\eta, P^{-it}\xi})\mathfrak{a}(x)) \\ &= \widehat{\Delta_A}^{-it}(id * \omega_{P^{-it}\delta^{-it}\eta, P^{-it}\xi})(V_{\psi_1})\Lambda_{\psi_1}(x).\end{aligned}$$

Therefore, using 3.2 and 2.2.2 applied to $\mathfrak{G}^o$, we get that

$$\widehat{\Delta}_A^{-it}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} (\omega_{\xi,\eta} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})])\widehat{\Delta}_A^{it}$$

is equal to $\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} \sigma_t^{\widehat{\Phi}^c}(\omega_{\xi,\eta} * id)[(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{J})W^*(\hat{J} \underset{N}{{}_\beta\otimes_{\hat{\alpha}}} \hat{J})])$ and, by density and continuity, we get (ii).

Using 3.8 and 4.8(iii) and (ii), we get, where x is in $\mathfrak{N}_{\psi_1}$, ζ in $D((H_{\psi_1})_b, \nu^o)$ and $(e_i)_{i\in I}$ is an orthogonal (b, ν^o)-basis of H_{ψ_1}:

$$\begin{aligned}\tilde{G}(\Delta_{\psi_1}^{it}\Lambda_{\psi_1}(x) \underset{N}{{}_\alpha\otimes_b} \widehat{\Delta}_A^{it}\zeta) &= \sum_i \Lambda_\Phi[(\omega_{\widehat{\Delta}_A^{it}\zeta, e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(\sigma_t^{\psi_1}(x))] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\ &= \sum_i \Lambda_\Phi[(\omega_{\widehat{\Delta}_A^{it}\zeta, e_i} \underset{N}{{}_b*_\alpha} id)(\tau_t^A \underset{N}{{}_b*_\alpha} \sigma_t^\Phi)\mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\ &= \sum_i \Delta_\Phi^{it}\Lambda_\Phi[(\omega_{P_A^{it}\widehat{\Delta}_A^{it}\zeta, P_A^{-it}e_i} \underset{N}{{}_b*_\alpha} \mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\ &= \sum_i \Delta_\Phi^{it}\Lambda_\Phi[(\omega_{J_{\psi_1}\delta_A^{it}J_{\psi_1}\zeta, P_A^{-it}e_i} \underset{N}{{}_b*_\alpha} \mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\ &= \sum_i \Delta_\Phi^{it}\Lambda_\Phi[(\omega_{\zeta, P_A^{-it}J_{\psi_1}\delta_A^{it}J_{\psi_1}e_i} \underset{N}{{}_b*_\alpha} \mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} e_i\end{aligned}$$

and, using the fact that $(P_A^{-it}J_{\psi_1}\delta_A^{it}J_{\psi_1}e_i)_{i\in I}$ is another orthogonal (b, ν^o)-basis of H_{ψ_1}, and that the sum does not depend on the choice of the basis, we get it is equal to

$$\sum_i \Delta_\Phi^{it}\Lambda_\Phi[(\omega_{\zeta,e_i} \underset{N}{{}_b*_\alpha} \mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} P_A^{-it}J_{\psi_1}\delta_A^{it}J_{\psi_1}e_i = (\Delta_{\psi_1}^{it} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} \widehat{\Delta}_A^{it})\tilde{G}(\Lambda_{\psi_1}(x) \underset{N}{{}_\alpha\otimes_b} \zeta)$$

which gives the first formula of (iii).

Finally, we have, using similar arguments:

$$\begin{aligned}
\tilde{G}(\widehat{\Delta}_A^{it} \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} P_A^{it}\delta_A^{it})(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta) &= \tilde{G}(P_A^{it} J_{\psi_1}\delta_A^{it} J_{\psi_1}\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} P_A^{it}\delta_A^{it}\zeta) \\
&= \tilde{G}(\lambda_A^{-t/2} P_A^{it}\Lambda_{\psi_1}(x\delta_A^{-it}) \underset{\psi_0}{{}_s\otimes_r} P_A^{it}\delta_A^{it}\zeta) \\
&= \tilde{G}(\Lambda_{\psi_1}(\tau_t^A(x\delta_A^{-it})) \underset{\psi_0}{{}_s\otimes_r} P_A^{it}\delta_A^{it}\zeta) \\
&= \sum_i \Lambda_\Phi[(\omega_{P_A^{it}\delta_A^{it}\zeta, e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(\tau_t^A(x\delta_A^{-it}))] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\
&= \sum_i \Lambda_\Phi[(\omega_{P_A^{it}\delta_A^{it}\zeta, e_i} \underset{N}{{}_b*_\alpha} id)(\tau_t^A \underset{N}{{}_b*_\alpha} \tau_t)\mathfrak{a}(x\delta_A^{-it})] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\
&= \sum_i \lambda_A^{-t/2} P^{it}\Lambda_\Phi[(\omega_{\zeta, P_A^{-it}e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)\delta^{-it}] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\
&= \sum_i P^{it} J_\Phi \delta^{it} J_\Phi \Lambda_\Phi[(\omega_{\zeta, P_A^{-it}e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} e_i \\
&= \sum_i P^{it} J_\Phi \delta^{it} J_\Phi \Lambda_\Phi[(\omega_{\zeta, e_i} \underset{N}{{}_b*_\alpha} id)\mathfrak{a}(x)] \underset{\nu^o}{{}_\alpha\otimes_b} P_A^{it} e_i \\
&= (\Delta_{\widehat{\Phi}}^{it} \underset{N^o}{{}_\alpha\otimes_b} P_A^{it})\tilde{G}(\Lambda_{\psi_1}(x) \underset{\psi_0}{{}_s\otimes_r} \zeta)
\end{aligned}$$

which finishes the proof. ■

5.6. Proposition. *Let $(A, b, \mathfrak{a}, \phi, \psi_0)$ be a Galois system for $\mathfrak{G}$; let $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, and let $\widehat{\Delta}_A$ be the operator introduced in 5.5. Then:*

(i) There exists a normal semi-finite faithful weight $\Phi_{\hat{P}}$ on $\hat{P}$ such that $\frac{d\Phi_{\hat{P}}}{d\widehat{\Phi}^c} = \widehat{\Delta}_A$;

(ii) There exists a normal faithful semi-finite operator valued weight $T_L^{\hat{P}}$ from $\hat{P}$ on $r(A^{\mathfrak{a}})$, such that $\Phi_{\hat{P}} = \psi_0 \circ r^{-1} \circ T_L^{\hat{P}}$.

Proof. Using 5.5(ii) and the definition of the spatial derivative ([T], IX.3.11), one gets (i). Moreover, we then get that, for all $t \in \mathbb{R}$ and $x \in A^{\mathfrak{a}}$, we have, using 4.8:

$$\sigma_t^{\Phi_{\hat{P}}}(r(x)) = P_A^{it} J_{\psi_1}\delta_A^{it} J_{\psi_1} r(x) J_{\psi_1}\delta_A^{-it} J_{\psi_1} P_A^{-it} = P_A^{it} r(x) P_A^{-it} = r(\sigma_t^{\psi_0}(x))$$

which gives (ii). ■

5.7. Notations. $(A, b, \mathfrak{a}, \phi, \psi_0)$ be a Galois system for $\mathfrak{G}$; let $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, and let $\widehat{\Delta}_A$ be the operator introduced in 5.5; let $\Phi_{\hat{P}}$ the normal semi-finite faithful weight on $\hat{P}$ introduced in 5.6(i), and let $T_L^{\hat{P}}$ be the normal faithful semi-finite operator valued weight from $\hat{P}$ on $r(A^{\mathfrak{a}})$, introduced in 5.6(ii), such that $\Phi_{\hat{P}} = \psi_0 \circ r^{-1} \circ T_L^{\hat{P}}$.

Let us denote by $\Phi_{\hat{Q}}$ the diagonal faithful normal semi-finite weight $\Phi_{\hat{P}} \oplus \widehat{\Phi}$ on the von Neumann algebra introduced in 5.1. Let us first remark that we can also define a diagonal normal faithful semi-finite operator-valued weight $T_L^{\hat{Q}}$ from $\hat{Q}$ to $\tilde{\alpha}(\tilde{N})$, defined, for any positive element $\begin{pmatrix} A & X \\ Y^* & m \end{pmatrix}$ in $\widehat{Q}^+$ (which implies that $A \in \hat{P}^+$, $m \in \widehat{M}^+$ and $Y = X$), by

$$T_L^{\hat{Q}}\left(\begin{pmatrix} A & X \\ X^* & m \end{pmatrix}\right) = T_L^{\hat{P}}(A) \oplus \hat{T}_L(m)$$

and we get that $\Phi_{\hat{P}} \oplus \widehat{\Phi} = (\psi_0 \oplus \nu) \circ T_L^{\hat{Q}}$.

It is straightforward to get that $\begin{pmatrix} A & X \\ Y^* & m \end{pmatrix}$ in $\widehat{Q}$ belongs to $\mathfrak{N}_{\hat{Q}}$ if and only if A belongs to $\mathfrak{N}_{\hat{P}}$, m belongs to $\mathfrak{N}_{\widehat{\Phi}}$, X is such that $\widehat{\Phi}(X^*X) < \infty$, and Y is such that $\Phi_{\hat{P}}(YY^*) \leq \infty$.

Let us consider the polar decomposition $X = u|X|$; then u belongs to $\widehat{I}$, and $|X|$ belongs to $\mathfrak{N}_{\widehat{\Phi}}$. Writing $\xi = u\Lambda_{\widehat{\Phi}}(|X|)$, we get that, for all $m \in \mathfrak{N}_{\widehat{\Phi}}$, we have

$$\begin{aligned} J_{\psi_1}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} m^*)J_{\psi_1}\xi = J_{\psi_1}\pi_{\mathfrak{a}}(1 \underset{N}{{}_b\otimes_\alpha} m^*)J_{\psi_1}u\Lambda_{\widehat{\Phi}}(|X|) \\ = u\hat{J}m^*\hat{J}\Lambda_{\widehat{\Phi}}(|X|) = u|X|\hat{J}\Lambda_{\widehat{\Phi}}(m) \end{aligned}$$

which means that $\xi \in D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o)$ and $X = R^{\mu^o,\widehat{\Phi}^o}(\xi)$.

If now we suppose that $\begin{pmatrix} A & X \\ Y^* & m \end{pmatrix}$ belongs to $\mathfrak{N}_{\hat{Q}} \cap \mathfrak{N}_{\hat{Q}}^*$, we get that there exists η in $D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o)$, such that $Y = R^{\mu^o,\widehat{\Phi}^o}(\eta)$, and $YY^* = \theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta)$; by definition of the spatial derivative, the fact that $\Phi_{\hat{P}}(YY^*) < \infty$ implies that $\eta \in \mathcal{D}(\widehat{\Delta}_A^{1/2})$, and $\Phi_{\hat{P}}(\theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta)) = \|\widehat{\Delta}_A^{1/2}\eta\|^2$; more precisely, there exists an antilinear involutive isometry $\widetilde{J}$ on H_{ψ_1} such that $\tilde{J}\widehat{\Delta}_A^{1/2} = \widehat{\Delta}_A^{-1/2}\tilde{J}$, and we can write

$$\Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} A & R^{\mu,\widehat{\Phi}'}(\xi) \\ R^{\mu,\widehat{\Phi}'}(\eta)^* & m \end{pmatrix}) = \Lambda_{\Phi_{\hat{P}}}(A) \oplus \xi \oplus \widetilde{J}\widehat{\Delta}_A^{1/2}\eta \oplus \Lambda_{\widehat{\Phi}}(m)$$

and we identify this way $H_{\Phi_{\hat{Q}}}$ with $H_{\Phi_{\hat{P}}} \oplus H_{\psi_1} \oplus H_{\psi_1} \oplus H$; for simplification, we shall identify $\Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix})$ with $\Lambda_{\Phi_{\hat{P}}}(A)$, $\Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix})$ with $\Lambda_{\widehat{\Phi}}(m)$. We shall write $p^{1,2}_{H_{\psi_1}}$ for the projection on the first subspace H_{ψ_1} of $H_{\Phi_{\hat{Q}}}$, and $p^{2,1}_{H_{\psi_1}}$, for the projection on the second subspace H_{ψ_1}.

If $X \in \widehat{I}$ is such that $\widehat{\Phi}(X^*X) < \infty$, let us write $\Lambda^{1,2}(X) = \Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix})$ (and, therefore, $\Lambda^{1,2}(R^{\mu,\widehat{\Phi}'}(\xi)) = \xi$ for all $\xi \in D({}_\mu H_{\psi_1}, \widehat{\Phi}'))$.

If $Y \in \widehat{I}$ is such that $\Phi_{\widehat{P}}(YY^*) < \infty$, let us write $\Lambda^{2,1}(Y^*) = \Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} 0 & 0 \\ Y^* & 0 \end{pmatrix})$, and therefore, if $\eta \in D({}_\mu(H_{\psi_1}), \widehat{\Phi}') \cap \mathcal{D}(\widehat{\Delta}_A^{1/2})$, we have $\Lambda^{2,1}(R^{\mu,\widehat{\Phi}'}(\eta)^*) = \widetilde{J}\widehat{\Delta}_A^{1/2}\eta$.

The identification of $H_{\Phi_{\hat{Q}}}$ with $H_{\Phi_{\hat{P}}} \oplus H_{\psi_1} \oplus H_{\psi_1} \oplus H$ leads also to write

$$\Delta_{\Phi_{\hat{Q}}} = \Delta_{\Phi_{\hat{P}}} \oplus \widehat{\Delta}_A^{1/2} \oplus \widehat{\Delta}_A^{1/2} \oplus \Delta_{\widehat{\Phi}}$$

and $J_{\Phi_{\hat{Q}}} = J_{\Phi_{\hat{P}}} \oplus (\tilde{J} \oplus \tilde{J}) \circ \tau \oplus \hat{J}$, where $\tau(\xi \oplus \eta) = \eta \oplus \xi$, for any ξ, η in H_{ψ_1}.

For any $n \in N$, $x \in A^{\mathfrak{a}}$, we get that

$$\pi_{\Phi_{\hat{Q}}}(\widetilde{\alpha}(x \oplus n))\Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} A & R^{\mu,\widehat{\Phi}'}(\xi) \\ R^{\mu,\widehat{\Phi}'}(\eta)^* & m \end{pmatrix}) = \Lambda_{\Phi_{\hat{Q}}}(\begin{pmatrix} r(x)A & r(x)R^{\mu,\widehat{\Phi}'}(\xi) \\ \alpha(n)R^{\mu,\widehat{\Phi}'}(\eta)^* & \alpha(n)m \end{pmatrix})$$

Using 2.1, we get, for any $n \in N$, analytical with respect to ν, that

$$R^{\mu,\widehat{\Phi}'}(\eta)\alpha(n) = R^{\mu,\widehat{\Phi}'}(\mu(\sigma^{\widehat{\Phi}'}_{i/2}(\beta(n))\eta) = R^{\mu,\widehat{\Phi}'}(\mu(\beta(\sigma^\nu_{i/2}(n))\eta) = R^{\mu,\widehat{\Phi}'}(b(\sigma^\nu_{i/2}(n))\eta)$$

and therefore that

$$\Lambda^{2,1}(\alpha(n)R^{\mu,\widehat{\Phi}'}(\eta)^*) = \tilde{J}\widehat{\Delta}_A^{1/2} b(\sigma^\nu_{i/2}(n^*))\eta$$

and, using 5.5(i), we get that

$$\Lambda^{2,1}(\alpha(n)R^{\mu,\widehat{\Phi}'}(\eta)^*) = \tilde{J}b(n^*)\widehat{\Delta}_A^{1/2}\eta$$

which, by continuity, remains true for all $n \in N$; from which we obtain

$$\pi_{\Phi_{\widehat{Q}}}(\widetilde{\alpha}(x \oplus n)) = \pi_{\Phi_{\widehat{P}}}(r(x)) \oplus r(x) \oplus \tilde{a}(n) \oplus \alpha(n)$$

where we define $\tilde{a}(n) = \tilde{J}b(n^*)\tilde{J}$.

With similar arguments, we obtain

$$\pi_{\Phi_{\widehat{Q}}}(\widetilde{\hat{\beta}}(x \oplus n)) = \pi_{\Phi_{\widehat{P}}}(s(x)) \oplus s(x) \oplus \tilde{b}(n) \oplus \hat{\beta}(n)$$

where we define $\tilde{b}(n) = \tilde{J}a(n^*)\tilde{J}$. Therefore, we get that $\pi_{\Phi_{\widehat{Q}}}(e_1) = p_{H_{\Phi_{\widehat{P}}}} + p^{1,2}_{H_{\psi_1}}$ and $\pi_{\Phi_{\widehat{Q}}}(e_2) = p^{2,1}_{H_{\psi_1}} + p_H$.

5.8. Proposition. *Let's use the notations of 5.1 and 5.7. Then:*

(i) For any $\eta \in D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o)$, $v \in D({}_\alpha H, \nu) \cap D(H_{\hat{\beta}}, \nu^o)$, $\xi \in D((H_{\psi_1})_s, \psi_0^o)$, the element $X = (\omega_{v,\xi} * id)[\tilde{G}^*(1 {}_\alpha\underset{N}{\otimes}{}_\beta R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}]$, which belongs to $\widehat{I}$ by 5.2, is such that $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ belongs to $\mathfrak{N}_{\Phi_{\widehat{Q}}}$.

(ii) Let $(\xi_i)_{i\in I}$ be an orthogonal (s, ψ_0^o)-basis of H_{ψ_1}; there exists $\eta_i \in D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o)$ such that

$$(\omega_{v,\xi_i} * id)[\tilde{G}^*(1 {}_\alpha\underset{N}{\otimes}{}_\beta R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}] = R^{\mu^o,\widehat{\Phi}^o}(\eta_i).$$

Moreover,

$$\|v {}_\alpha\underset{\nu^o}{\otimes}{}_b \eta\|^2 = \sum_i \|\eta_i\|^2 = \|\sum_i \xi_i {}_s\underset{\psi_0}{\otimes}{}_r \eta_i\|^2.$$

(iii) We have

$$\tilde{G}^*(v {}_\alpha\underset{\nu^o}{\otimes}{}_b \eta) = \sum_i \xi_i {}_s\underset{\psi_0}{\otimes}{}_r \eta_i.$$

(iv) We have

$$\Lambda^{1,2}(X) = (\omega_{v,\xi} * id)(\tilde{G}^*)\eta.$$

Proof. We have

$$\begin{aligned} X^*X &= ((\omega_{v,\xi} * id)[\tilde{G}^*(1 {}_\alpha\underset{N^o}{\otimes}{}_\beta R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}])^*(\omega_{v,\xi} * id)[\tilde{G}^*(1 {}_\alpha\underset{N^o}{\otimes}{}_\beta R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}] \\ &= (\omega_v {}_{\hat{\beta}}\underset{N}{*}{}_\alpha id)[\widehat{W}^*(1 {}_\alpha\underset{N^o}{\otimes}{}_b R^{\mu^o,\widehat{\Phi}^o}(\eta))^*\tilde{G}(\theta^{s,\psi_0^o}(\xi,\xi) {}_s\underset{A^{\mathfrak{a}}}{\otimes}{}_r 1)\tilde{G}^*(1 {}_\alpha\underset{N}{\otimes}{}_\beta R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}] \\ &\le \|R^{s,\psi_0}(\xi)\|^2(\omega_v {}_{\hat{\beta}}\underset{N}{*}{}_\alpha id)[\widehat{W}^*(1_\alpha\underset{N^o}{\otimes}{}_\beta < \eta,\eta >_{\mu^o,\widehat{\Phi}^o})\widehat{W}] \\ &= \|R^{s,\psi_0}(\xi)\|^2(\omega_v {}_{\hat{\beta}}\underset{N}{*}{}_\alpha id)(\widehat{\Gamma}(< \eta,\eta >_{\mu^o,\widehat{\Phi}^o})) \end{aligned}$$

and therefore using the left-invariance of $\widehat{T_L}$, then, using 5.7:

$$\widehat{\Phi}(X^*X) \leq \|R^{s,\psi_0}(\xi)\|^2(\widehat{T_L}(<\eta,\eta>_{\mu^o,\widehat{\Phi}^o})v|v)$$
$$= \|R^{s,\psi_0}(\xi)\|^2\|v \underset{N^o}{_\alpha\otimes_\beta} \Lambda_{\widehat{\Phi}}(R^{\mu^o,\widehat{\Phi}^o}(\eta))\|^2 = \|R^{s,\psi_0}(\xi)\|^2\|v \underset{N^o}{_\alpha\otimes_b} \eta\|^2$$

from which we get that X^*X belongs to $\mathfrak{M}^+_{\widehat{\Phi}}$, and using 5.7, we finish the proof of (i).

The same calculation with $X_i = (\omega_{v,\xi_i} * id)[\tilde{G}^*(1 \underset{N^o}{_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}]$ shows that

$$\sum_i X_i^* X_i = (\omega_v \underset{N}{_{\hat\beta}*_\alpha} id)\widehat{\Gamma}(<\eta,\eta>_{\mu^o,\widehat{\Phi}^o})$$

and then we get $\widehat{\Phi}(\sum_i X_i^* X_i) = \|v \underset{N^o}{_\alpha\otimes_b} \eta\|^2$.

Using again 5.7, we get that there exists $\eta_i \in D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o)$ such that $X_i = R^{\mu^o,\widehat{\Phi}^o}(\eta_i)$; from which we get that

$$\sum_i \|\eta_i\|^2 = \sum_i \|\Lambda_{\widehat{\Phi}}(R^{\mu^o,\widehat{\Phi}^o}(\eta_i))\|^2 = \sum_i \widehat{\Phi}(X_i^*X_i) = \|v \underset{N^o}{_\alpha\otimes_b} \eta\|^2$$

which is (ii). Let now $m \in \mathfrak{N}_{\widehat{\Phi}}$; we have

$$(\hat{J}m^*\hat{J} \underset{N^o}{_\alpha\otimes_b} 1)\tilde{G}(\xi_i \underset{\psi_0^o}{_s\otimes_r} \eta_i) = \tilde{G}(\xi_i \underset{\psi_0^o}{_s\otimes_r} \mu(\hat{J}m^*\hat{J})\eta_i) = \tilde{G}(\xi_i \underset{\psi_0^o}{_s\otimes_r} R^{\mu^o,\widehat{\Phi}^o}(\eta_i)\hat{J}\Lambda_{\widehat{\Phi}}(m))$$
$$= \tilde{G}(\xi_i \underset{\psi_0^o}{_s\otimes_r} X_i\hat{J}\Lambda_{\widehat{\Phi}}(m))$$

and therefore

$$(\hat{J}m^*\hat{J} \underset{N^o}{_\alpha\otimes_b} 1)\tilde{G}\sum_i(\xi_i \underset{\psi_0^o}{_s\otimes_r} \eta_i)$$
$$= \tilde{G}\sum_i(\xi_i \underset{\psi_0^o}{_s\otimes_r} (\omega_{v,\xi_i} * id)[\tilde{G}^*(1 \underset{N}{_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}]\hat{J}\Lambda_{\widehat{\Phi}}(m)$$
$$= (1 \underset{N}{_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}(v \underset{N}{_{\hat\beta}\otimes_\alpha} \hat{J}\Lambda_{\widehat{\Phi}}(m))$$

Therefore, taking now $\zeta_1 \in D(_\alpha H, \nu)$ and $\zeta_2 \in D((H_{\psi_1})_b, \nu^o)$, we get that

$$((\hat{J}m^*\hat{J} \underset{N^o}{_\alpha\otimes_b} 1)\tilde{G}\sum_i(\xi_i \underset{\psi_0^o}{_s\otimes_r} \eta_i)|\zeta_1 \underset{\nu^o}{_\alpha\otimes_b} \zeta_2)$$
$$= ((1 \underset{N}{_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}(v \underset{N}{_{\hat\beta}\otimes_\alpha} \hat{J}\Lambda_{\widehat{\Phi}}(m))|\zeta_1 \underset{\nu^o}{_\alpha\otimes_b} \zeta_2)$$
$$= (R^{\mu^o,\widehat{\Phi}^o}(\eta)(\omega_{v,\zeta_1} * id)(\widehat{W})\hat{J}\Lambda_{\widehat{\Phi}}(m)|\zeta_2)$$

and, using now ([E5], 3.10(ii) applied to $\widehat{\mathfrak{G}}$, and 3.11(iii)), we get it is equal to

$$(R^{\mu^o,\widehat{\Phi}^o}(\eta)\hat{J}\Lambda_{\widehat{\Phi}}(\omega_{J\zeta_1,Jv} * id)\widehat{\Gamma}(m))|\zeta_2) = (\mu(\hat{J}(\omega_{J\zeta_1,Jv} * id)\widehat{\Gamma}(m)^*\hat{J})\eta|\zeta_2).$$

Taking the limit when m goes to 1, we get

$$(\tilde{G}(\sum_i \xi_i \underset{\psi_0^o}{_s\otimes_r} \eta_i)|\zeta_1 \underset{\nu^o}{_\alpha\otimes_b} \zeta_2) = (\beta(<Jv, J\zeta_1>_{\hat\beta,\nu^o})\eta|\zeta_2)$$
$$= (\beta(<\zeta_1, v>_{\alpha,\nu})\eta|\zeta_2) = (v \underset{\nu^o}{_\alpha\otimes_b} \eta|\zeta_1 \underset{\nu^o}{_\alpha\otimes_b} \zeta_2)$$

from which we get that $\tilde{G}(\sum_i \xi_i \,{}_s\underset{\psi_0^o}{\otimes}{}_r\, \eta_i) = v \,{}_\alpha\underset{\nu^o}{\otimes}{}_b\, \eta$, which is (iii); this can be written

$$(\omega_{v,\xi_i} * id)(\tilde{G}^*)\eta = \eta_i = \Lambda^{1,2}(X_i)$$

which, by linearity and continuity, gives (iv). ∎

5.9. Proposition. *Let us use the notations of 5.1, 5.7, 5.8 and take* $\eta \in D((H_{\psi_1})_{\mu^o}, \widehat{\Phi}^o) \cap \mathcal{D}(\widehat{\Delta}_A^{1/2})$; *let us define the antirepresentation* $\tilde{s}$ *of* $A^{\mathfrak{a}}$ *on* H_{ψ_1} *by* $\tilde{s}(x) = \tilde{J}r(x^*)\tilde{J}$, *for all* $x \in A^{\mathfrak{a}}$; *let us define the representation* $\tilde{a}$ *of* N *on* H_{ψ_1} *by* $\tilde{a}(n) = \tilde{J}b(n^*)\tilde{J}$, *for all* $n \in N$; *then, for any* $v \in D(H_{\hat{\beta}}, \nu^o)$ *and* $\xi \in D((H_{\psi_1})_s, \psi_0^o) \cap D({}_rH_{\psi_1}, \psi_0)$ *the element*

$$X = (\omega_{v,\xi} * id)[\tilde{G}^*(1 \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}]$$

is such that $\begin{pmatrix} 0 & 0 \\ X^* & 0 \end{pmatrix}$ *belongs to* $\mathfrak{N}_{\Phi_{\widehat{Q}}}$, *and we have*

$$\Lambda^{2,1}(X^*) = (\omega_{\xi,v} * id)[(J \,{}_\alpha\underset{N^o}{\otimes}{}_b\, \tilde{J})\tilde{G}(J_{\psi_1} \,{}_r\underset{A^{\mathfrak{a}o}}{\otimes}{}_{\tilde{s}}\, \tilde{J})]\Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta)).$$

Proof. Let us first take η such that $\Lambda_{\Phi_{\widehat{Q}}}(\begin{pmatrix} 0 & R^{\mu^o,\widehat{\Phi}^o}(\eta) \\ 0 & 0 \end{pmatrix})$ belongs to the Tomita algebra $\mathcal{T}_{\Phi_{\widehat{Q}}}$, x in the Tomita algebra $\mathcal{T}_{\psi_1,T_{\mathfrak{a}}}$, and y, z in $\mathcal{T}_{\Phi,T_L}$. Then, $\Lambda_{\psi_1}(x)$ belongs to $D((H_{\psi_1})_s, \psi_0^o)$, and $J\Lambda_\Phi(y^*z)$ belongs to $D({}_\alpha H, \nu) \cap D(H_{\hat{\beta}}, \nu^o)$. Therefore, we can apply 5.8(i) to the element

$$X = (\omega_{J\Lambda_\Phi(y^*z),\Lambda_{\psi_1}(x)} * id)[\tilde{G}^*(1 \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}].$$

Using 5.7 and ([E5], 3.11 applied to $\mathfrak{G}$), we get that $\sigma_t^{\Phi_{\widehat{Q}}}(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix})$ is of the form $\begin{pmatrix} 0 & X_t \\ 0 & 0 \end{pmatrix}$, with $X_t = \widehat{\Delta}_A^{it} X \Delta_{\widehat{\Phi}}^{-it}$. Using now 5.5(iii), we get that

$$X_t = (\omega_{\Delta_\Phi^{it} J\Lambda_\Phi(y^*z), \Delta_{\psi_1}^{-it}\Lambda_{\psi_1}(x)} * id)[\tilde{G}^*(1 \,{}_\alpha\underset{N^o}{\otimes}{}_\beta\, \widehat{\Delta}_A^{it} R^{\mu^o,\widehat{\Phi}^o}(\eta)\widehat{\Delta}^{-it})\widehat{W}]$$

and the hypothesis on η, x, y, z give that the function $t \mapsto X_t$ extends to an analytic function; in particular, we get that $\Lambda^{1,2}(X)$ belongs to $\mathcal{D}(\widehat{\Delta}_A^{1/2})$, and using 5.8(iv) and 4.2(iii), we get

$$\begin{aligned}\widehat{\Delta}_A^{1/2}\Lambda^{1,2}(X) = \Lambda^{1,2}(X_{-i/2}) &= (\omega_{\Delta_\Phi^{1/2} J\Lambda_\Phi(y^*z), \Delta_{\psi_1}^{-1/2}\Lambda_{\psi_1}(x)} * id)(\tilde{G}^*)\tilde{J}\widehat{\Delta}_A^{1/2}\eta \\ &= (\omega_{\Delta_{\psi_1}^{-1/2}\Lambda_{\psi_1}(x), \Delta_\Phi^{1/2} J\Lambda_\Phi(y^*z)} * id)(\tilde{G})^*\tilde{J}\widehat{\Delta}_A^{1/2}\eta \\ &= (\omega_{J_{\psi_1}\Lambda_{\psi_1}(x), \Lambda_\Phi(y^*z)} * id)(\tilde{G})\tilde{J}\widehat{\Delta}_A^{1/2}\eta\end{aligned}$$

and therefore

$$\begin{aligned}\Lambda^{2,1}(X^*) = \tilde{J}\widehat{\Delta}_A^{1/2}\Lambda^{1,2}(X) &= \tilde{J}(\omega_{J_{\psi_1}\Lambda_{\psi_1}(x), \Lambda_\Phi(y^*z)} * id)(\tilde{G})\tilde{J}\widehat{\Delta}_A^{1/2}\eta \\ &= (\omega_{\Lambda_{\psi_1}(x), J\Lambda_\Phi(y^*z)} * id)[(J \,{}_\alpha\underset{N^o}{\otimes}{}_b\, \tilde{J})\tilde{G}(J_{\psi_1} \,{}_r\underset{A^{\mathfrak{a}o}}{\otimes}{}_{\tilde{s}}\, \tilde{J})]\Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta)^*)\end{aligned}$$

As $\Lambda_{\widehat{Q}}$ (and therefore $\Lambda^{2,1}$) is closed, we get, for any $v \in D(H_{\widehat{\beta}}, \nu^o)$ and $\xi \in D((H_{\psi_1})_s, \psi_0^o) \cap D({}_rH_{\psi_1}, \psi_0)$, that $X = (\omega_{v,\xi} * id)[\tilde{G}^*(1 \underset{N^o}{{}_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}]$ is such that X^* belongs to $\mathcal{D}(\Lambda^{2,1})$ and that

$$\Lambda^{1,2}(X^*) = (\omega_{\xi,v} * id)[(J \underset{N^o}{{}_\alpha\otimes_b} \tilde{J})\tilde{G}(J_{\psi_1} \underset{A^{\mathfrak{a}o}}{{}_r\otimes_{\tilde{s}}} \tilde{J})]\Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta)^*).$$

Using again the closedness of $\Lambda^{2,1}$, we get that this result remains true for any η such that $\begin{pmatrix} 0 & R^{\mu^o,\widehat{\Phi}^o}(\eta) \\ 0 & 0 \end{pmatrix}$ belongs to $\mathfrak{N}_{\Phi_{\widehat{Q}}} \cap \mathfrak{N}^*_{\Phi_{\widehat{Q}}}$ (i.e., using 5.7, if η belongs to $\mathcal{D}(\widehat{\Delta}_A^{1/2})$). ■

5.10. Theorem. *The operator-valued weight $T_L^{\widehat{P}}$ is left-invariant.*

Proof. Let η in $D((H_{\psi_1})\mu^o, \widehat{\Phi}^o) \cap {}^{1/2}_{A})$; let $(v_i)_{i\in I}$ a $(\widehat{\beta}, \nu^o)$ orthogonal basis of H, and ξ in $D((H_{\psi_1})_s, \psi_0^o) \cap D({}_rH_{\psi_1}, \psi_0)$; let us write

$$X_i = (\omega_{v_i,\xi} * id)[\tilde{G}^*(1 \underset{N^o}{{}_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}].$$

We then get

$$\omega_\xi(id * \Phi_{\widehat{P}})(\Gamma_{\widehat{P}}(\theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta))) = \Phi_{\widehat{P}}(\omega_\xi * id)[\tilde{G}(1 \underset{N^o}{{}_\alpha\otimes_b} \theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta))\tilde{G}^*]$$

is equal to

$$\sum_i \Phi_{\widehat{P}}((\omega_\xi * id)(\tilde{G}(1 \underset{N}{{}_\alpha\otimes_\beta} R^{\mu^o,\widehat{\Phi}^o}(\eta))\widehat{W}(\theta^{\widehat{\beta}}(v_i,v_i) \underset{N}{{}_{\widehat{\beta}}\otimes_\alpha} 1)\widehat{W}^*(1 \underset{N^o}{{}_\alpha\otimes_b} R^{\mu^o,\widehat{\Phi}^o}(\eta)^*\tilde{G}^*]$$

which can be written, using 5.9:

$$\begin{aligned}
\sum_i \Phi_{\widehat{P}}(X_iX_i^*) &= \sum_i \|\Lambda^{2,1}(X_i^*)\|^2 \\
&= \sum_i \|(\omega_{\xi,v_i} * id)[(J \underset{N^o}{{}_\alpha\otimes_b} \tilde{J})\tilde{G}(J_{\psi_1} \underset{A^{\mathfrak{a}o}}{{}_r\otimes_{\tilde{s}}} \tilde{J})]\Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta))\|^2 \\
&= \sum_i \|v_i \underset{N}{{}_{\widehat{\beta}}\otimes_\alpha} (\omega_{\xi,v_i} * id)[(J \underset{N^o}{{}_\alpha\otimes_b} \tilde{J})\tilde{G}(J_{\psi_1} \underset{A^{\mathfrak{a}o}}{{}_r\otimes_{\tilde{s}}} \tilde{J})]\Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta))\|^2 \\
&= \|[(J \underset{N^o}{{}_\alpha\otimes_b} \tilde{J})\tilde{G}(J_{\psi_1} \underset{A^{\mathfrak{a}o}}{{}_r\otimes_{\tilde{s}}} \tilde{J})](\xi \underset{A^{\mathfrak{a}o}}{{}_r\otimes_{\tilde{s}}} \Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta))\|^2 \\
&= \|\xi \underset{A^{\mathfrak{a}o}}{{}_r\otimes_{\tilde{s}}} \Lambda^{2,1}(R^{\mu^o,\widehat{\Phi}^o}(\eta))\|^2 = (T_L^{\widehat{P}}(\theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta))\xi|\xi)
\end{aligned}$$

from which we get that $(id * \Phi_{\widehat{P}})(\Gamma_{\widehat{P}}(\theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta))) = T_L^{\widehat{P}}(\theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta))$. As any element in $\mathfrak{M}^+_{\Phi_{\widehat{P}}}$ can be approximated from below by finite sums of operators of the form $\theta^{\mu^o,\widehat{\Phi}^o}(\eta,\eta)$, we get the result. ■

5.11. Theorem. *With the notations of 5.1 and 5.7, we have:*

(i) $(A^{\mathfrak{a}}, \widehat{P}, r, s, \Gamma_{\widehat{P}}, T_L^{\widehat{P}}, R_{\widehat{P}} \circ T_L^{\widehat{P}} \circ R_{\widehat{P}}, \psi_0)$ is a measured quantum groupoid. We shall denote this measured quantum groupoid by $\mathfrak{G}_1(A, b, \mathfrak{a}, \phi, \psi_0)$, or simply by $\mathfrak{G}_1(\mathfrak{a})$. Following [DC1], its dual $\widehat{\mathfrak{G}_1(\mathfrak{a})}$ will be called the reflected measured quantum groupoid of $\mathfrak{G}$ through the Galois system $(A, b, \mathfrak{a}, \phi, \psi_0)$, or simply, through $\mathfrak{a}$.

(ii) $(\tilde{N}, \widehat{Q}, \widetilde{\alpha}, \widetilde{\widehat{\beta}}, \Gamma_{\widehat{Q}}, T_L^{\widehat{Q}}, R_{\widehat{Q}} \circ T_L^{\widehat{Q}} \circ R_{\widehat{Q}}, \psi_0 \oplus \nu)$ *is a measured quantum groupoid. We shall denote this measured quantum groupoid by* $\mathfrak{G}_2(A, b, \mathfrak{a}, \phi, \psi_0)$, *or simply by* $\mathfrak{G}_2(\mathfrak{a})$.

Proof. By 5.4(iii), we know that $(A^{\mathfrak{a}}, \widehat{P}, r, s, \Gamma_{\widehat{P}})$ is a Hopf bimodule, and by 5.10, that $T_L^{\widehat{P}}$ is left-invariant. Using again 5.4(iii), we get that $R_{\widehat{P}} \circ T_L^{\widehat{P}} \circ R_{\widehat{P}}$ is right-invariant. The only result needed is that the modular automorphism groups $\sigma^{\Phi_{\widehat{P}}}$ and $\sigma^{\Phi_{\widehat{P}} \circ R_{\widehat{P}}}$ commute. By definition, we have, for all $A \in \widehat{P}$, we have, using 5.5(i) $\sigma_t^{\Phi_{\widehat{P}}}(A) = \widehat{\Delta}_A^{it} A \widehat{\Delta}_A^{-it} = P_A^{it} J_{\psi_1} \delta_A^{it} J_{\psi_1} A J_{\psi_1} \delta_A^{-it} J_{\psi_1} P_A^{-it}$, and, using 5.5(i) and 4.8(v) and (vi):

$$\begin{aligned}\sigma_s^{\Phi_{\widehat{P}} \circ R_{\widehat{P}}}(A) &= R_{\widehat{P}} \circ \sigma_{-s}^{\Phi_{\widehat{P}}} \circ R_{\widehat{P}}(A) = J_{\psi_1} \widehat{\Delta}_A^{-is} J_{\psi_1} A J_{\psi_1} \widehat{\Delta}_A^{is} J_{\psi_1} \\ &= J_{\psi_1} P_A^{-is} J_{\psi_1} \delta_A^{-is} A \delta_A^{is} J_{\psi_1} P_A^{is} J_{\psi_1} = P_A^{-is} \delta_A^{-is} A \delta_A^{is} P_A^{is}\end{aligned}$$

and, as $P_A^{it} J_{\psi_1} \delta_A^{it} J_{\psi_1}$ commutes with $P_A^{-is} \delta_A^{-is}$, we obtain the result, and we finish the proof of (i).

We have obtained in 5.4(i), that $(\tilde{N}, \widehat{Q}, \widetilde{\alpha}, \widetilde{\widehat{\beta}}, \Gamma_{\widehat{Q}})$ is a Hopf bimodule; using 5.10 and the definition of $T_L^{\widehat{Q}}$ (5.7), we get that $T_L^{\widehat{Q}}$ is left-invariant; using 5.4(ii), we get that $R_{\widehat{Q}} \circ T_L^{\widehat{Q}} \circ R_{\widehat{Q}}$ is right-invariant. The calculation made in (i) proves as well that the automorphism groups $\sigma^{\Phi_{\widehat{Q}}}$ and $\sigma^{\Phi_{\widehat{Q}} \circ R_{\widehat{Q}}}$ commute, which finishes the proof. ∎

5.12. Theorem. *Let* $\mathfrak{G}$ *a measured quantum groupoid, and* $(A, b, \mathfrak{a}, \phi, \psi_0)$ *a Galois system for* $\mathfrak{G}$*; let us denote by* $(\tilde{N}, Q, \widetilde{\alpha}, \widetilde{\beta}, \Gamma_Q, T_L^Q, R_Q T_L^Q R_Q, \psi_0 \oplus \nu)$ *the dual measured quantum groupoid* $\widehat{\mathfrak{G}_2(\mathfrak{a})}$. *This measured quantum groupoid will be called the linking measured quantum groupoid between* $\mathfrak{G}$ *and the reflected measured quantum groupoid* $\widehat{\mathfrak{G}_1(\mathfrak{a})}$. *We shall consider that the von Neumann algebra* Q *acts on* $H_{\Phi_{\widehat{Q}}} = H_{\Phi_{\widehat{P}}} \oplus H_{\psi_1} \oplus H_{\psi_1} \oplus H$. *Then:*

(i) $\widetilde{\alpha}(e_1)$, $\widetilde{\alpha}(e_2)$, $\widetilde{\beta}(e_1)$, $\widetilde{\beta}(e_2)$ *belong to* $Z(Q)$.

(ii) We have $p_{H_{\Phi_{\widehat{P}}}} = \tilde{\alpha}(e_1)\tilde{\beta}(e_1)$; $p^{1,2}_{H_{\psi_1}} = \tilde{\alpha}(e_1)\tilde{\beta}(e_2)$; $p^{2,1}_{H_{\psi_1}} = \tilde{\alpha}(e_2)\tilde{\beta}(e_1)$, *and* $p_H = \tilde{\alpha}(e_2)\tilde{\beta}(e_2)$; *all these projections belong to* $Z(Q)$.

(iii) We have $Q_{p_{H_{\Phi_{\widehat{P}}}}} = P$, $Q_{p^{1,2}_{H_{\psi_1}}} = A$, $Q_{p^{2,1}_{H_{\psi_1}}} = \tilde{J} A \tilde{J}$, *and* $Q_{p_H} = M$. *Therefore, we have* $Q = P \oplus A \oplus A^o \oplus M$.

(iv) If $x \in P$, $y \in M$, $z \in A$, *we have*

$$\Gamma_P(x) = \Gamma_Q(x)_{\tilde{\alpha}(e_1)\tilde{\beta}(e_1)_{\tilde{\beta}} \otimes_{\tilde{\alpha}} \tilde{\alpha}(e_1)\tilde{\beta}(e_1)},\\ \tilde{N}$$

$$\Gamma(y) = \Gamma_Q(y)_{\tilde{\alpha}(e_2)\tilde{\beta}(e_2)_{\tilde{\beta}} \underset{\tilde{N}}{\otimes}_{\tilde{\alpha}} \tilde{\alpha}(e_2)\tilde{\beta}(e_2)},$$

$$\mathfrak{a}(z) = \Gamma_Q(z)_{\tilde{\alpha}(e_1)\tilde{\beta}(e_2)_{\tilde{\beta}} \underset{\tilde{N}}{\otimes}_{\tilde{\alpha}} \tilde{\alpha}(e_2)\tilde{\beta}(e_2)}.$$

(v) Let R *(resp.* R_P*, resp.* R_Q*) be the co-inverse of* $\mathfrak{G}$ *(resp. of the reflected measured quantum groupoid, resp. of the linking measured quantum groupoid); let* τ_t, τ_t^P, τ_t^Q *be the scaling groups of these measured quantum groupoids,* γ_t, γ_t^P, γ_t^Q *be the automorphism groups on the basis of these measured quantum groupoids, as defined in 2.2 or [E5], 3.8(i), (ii) and (v); we have, for any* $x \in P$, $y \in M$, z_1, z_2 *in* A, $n \in N$, $u \in A^{\mathfrak{a}}$*:*

$$R_Q(x \oplus z_1 \oplus z_2^o \oplus y) = R_P(x) \oplus z_2 \oplus z_1^o \oplus R(y),$$

$$\tau_t^Q(x \oplus z_1 \oplus z_2^o \oplus y) = \tau_t^P(x) \oplus \widehat{\Delta}_A^{it} z_1 \widehat{\Delta}_A^{-it} \oplus (\widehat{\Delta}_A^{it} z_2 \widehat{\Delta}_A^{-it})^o \oplus \tau_t(y),$$
$$\gamma_t^Q(u \oplus n) = \gamma_t^P(u) \oplus \gamma_t(n).$$

Proof. As $\widetilde{\alpha}(e_1) = \widetilde{\widehat{\beta}}(e_1)$ (5.1), we get that $\widetilde{\alpha}(e_1)$ belongs to $Z(Q)$; so $\widetilde{\alpha}(e_2) = 1 - \widetilde{\alpha}(e_1)$ belongs also to $Z(Q)$, and as $\widetilde{\beta}(e_1) = R_Q(\widetilde{\alpha}(e_1))$ and $\widetilde{\beta}(e_2) = R_Q(\widetilde{\alpha}(e_2))$, we get (i).

We have seen in 5.7 that $\pi_{\Phi_{\widehat{Q}}}(\widetilde{\alpha}(e_1) = p_{H_{\Phi_{\widehat{P}}}} + p^{1,2}_{H_{\psi_1}}$ (as we shall consider that Q is acting on $H_{\Phi_{\widehat{Q}}}$, we shall now skip the representation $\pi_{\Phi_{\widehat{Q}}}$). Using now the formula obtained for $J_{\widehat{Q}}$, we obtain

$$\tilde{\alpha}(e_1) = p_{H_{\Phi_{\widehat{Q}}}} + p^{1,2}_{H_{\psi_1}},$$
$$\tilde{\alpha}(e_2) = p^{2,1}_{H_{\psi_1}} + p_H,$$
$$\tilde{\beta}(e_1) = p_{H_{\Phi_{\widehat{Q}}}} + p^{2,1}_{H_{\psi_1}},$$
$$\tilde{\beta}(e_2) = p^{1,2}_{H_{\psi_1}} + p_H,$$

from which we get (ii).

Let $W_{\widehat{Q}}$ be the pseudo-multiplicative unitary associated to $\mathfrak{G}_2(A, b, \mathfrak{a}, \Phi, \psi_0)$; then, Q is the weak closure of the linear set generated by all operators of the form $(\omega_{w,v} * id)(W^*_{\widehat{Q}})$, for all $v \in D({}_{\tilde{\alpha}}H_{\Phi_{\widehat{Q}}}, \nu \oplus \psi_0) \cap D((H_{\Phi_{\widehat{Q}}})_{\tilde{\hat{\beta}}}, \nu \oplus \psi_0)$, and $w \in D((H_{\Phi_{\widehat{Q}}})_{\tilde{\hat{\beta}}}, \nu \oplus \psi_0)$. Using now [E5], 3.10 (ii), we get that, for A in $\mathfrak{N}_{\Phi_{\widehat{P}}}$, $\xi \in D({}_{\mu}H_{\psi_1}, \widehat{\Phi}')$, $\eta \in D({}_{\mu}H_{\psi_1}, \widehat{\Phi}') \cap \mathcal{D}(\widehat{\Delta}_A^{1/2})$ and $m \in \mathfrak{N}_{\widehat{\Phi}'}$:

$$p_{H_{\Phi_{\widehat{P}}}}(\omega_{w,v} * id)(W^*_{\widehat{Q}}) p_{H_{\Phi_{\widehat{P}}}} \Lambda_{\Phi_{\widehat{Q}}}(\begin{pmatrix} A & R^{\mu,\widehat{\Phi}'}(\xi) \\ R^{\mu,\widehat{\Phi}'}(\eta)^* & m \end{pmatrix})$$

is, using 5.4(i) and (iii), equal to

$$p_{H_{\Phi_{\widehat{P}}}} \Lambda_{\Phi_{\widehat{Q}}}[(\omega_{w,v} \underset{\tilde{N}}{{}_{\tilde{\hat{\beta}}}*_{\tilde{\alpha}}} id)\Gamma_{\widehat{Q}}(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix})] = \Lambda_{\Phi_{\widehat{P}}}[(\omega_{p_{H_{\Phi_{\widehat{P}}}} w, p_{H_{\Phi_{\widehat{P}}}} v} \underset{A^{\mathfrak{a}}}{{}_{s}*_{r}} id)\Gamma_{\widehat{P}}(A)]$$
$$= (\omega_{p_{H_{\Phi_{\widehat{P}}}} w, p_{H_{\Phi_{\widehat{P}}}} v} * id)(W^*_{\widehat{P}})\Lambda_{\Phi_{\widehat{P}}}(A)$$

from which we get that $Q_{p_{H_{\Phi_{\widehat{P}}}}} = P$. The proof for Q_{p_H} is similar.

The same way, we get that

$$p^{1,2}_{H_{\psi_1}}(\omega_{w,v} * id)(W^*_{\widehat{Q}}) p^{1,2}_{H_{\psi_1}} \Lambda_{\Phi_{\widehat{Q}}}(\begin{pmatrix} A & R^{\mu,\widehat{\Phi}'}(\xi) \\ R^{\mu,\widehat{\Phi}'}(\eta)^* & m \end{pmatrix})$$

is equal, using 5.8(iv) to $(\omega_{p_H w, p^{1,2}_{H_{\psi_1}} v} * id)(\tilde{G}^*)\xi$, and therefore, using 4.2(iv), we get that A is the the weak closure of the linear set generated by all elements of the form $p^{1,2}_{H_{\psi_1}}(\omega_{w,v} * id)(W^*_{\widehat{Q}}) p^{1,2}_{H_{\psi_1}}$. For $Q_{p^{2,1}_{H_{\psi_1}}}$, the proof is the same, using 5.9, which finishes the proof of (iii).

The restriction of $(p_{H_{\Phi_{\widehat{P}}}} \underset{\tilde{N}}{{}_{\tilde{\beta}}\otimes_{\tilde{\alpha}}} p_{H_{\Phi_{\widehat{P}}}}) W^*_{\widehat{Q}}((p_{H_{\Phi_{\widehat{P}}}} \underset{\tilde{N}^o}{{}_{\tilde{\alpha}}\otimes_{\tilde{\hat{\beta}}}} p_{H_{\Phi_{\widehat{P}}}})$ to $H_{\Phi_{\widehat{P}}} \underset{\psi_0}{{}_{r}\otimes_{s}} H_{\Phi_{\widehat{P}}}$ is equal to $W^*_{\widehat{P}}$, that the restriction of $(p_H \underset{\tilde{N}}{{}_{\tilde{\beta}}\otimes_{\tilde{\alpha}}} p_H) W^*_{\widehat{Q}}(p_H \underset{\tilde{N}^o}{{}_{\tilde{\alpha}}\otimes_{\tilde{\hat{\beta}}}} p_H)$ to $H \underset{\nu^o}{{}_{\alpha}\otimes_{\hat{\beta}}} H$ is equal to $\widehat{W}$, and the restriction of $(p^{1,2}_{H_{\psi_1}} \underset{\tilde{N}}{{}_{\tilde\beta}\otimes_{\tilde\alpha}} p^{1,2}_{H_{\psi_1}})W^*_{\widehat{Q}}(p_H \underset{N^o}{{}_\alpha\otimes_b} p^{1,2}_{H_{\psi_1}})$ to $H \underset{\nu^o}{{}_\alpha\otimes_b} H_{\psi_1}$ is equal to $\tilde{G}^*$. Then the result (iv) comes from ([E5], 3.6(ii)) applied to $\widehat{\mathfrak{G}_2(\mathfrak{a})}$, $\widehat{\mathfrak{G}_1(\mathfrak{a})}$ and $\mathfrak{G}$, and 3.8(iv).

For any $X \in Q$, we have $R_Q(X) = J_{\Phi_{\widehat{Q}}} X^* J_{\Phi_{\widehat{Q}}}$ ([E5], 3.10(v)), and $\tau^Q_t(X) = \Delta^{it}_{\Phi_{\widehat{Q}}} X \Delta^{-it}_{\Phi_{\widehat{Q}}}$ ([E5], 3.10 (vii)). So, the result about R_Q (resp. τ^Q_t) is then given by the formula about $J_{\Phi_{\widehat{Q}}}$ (resp. $\Delta_{\Phi_{\widehat{Q}}}$) obtained in 5.7. Let's look at the automorphism group $\gamma^{\widehat{Q}}_t$; we have, using 5.1 and 5.7:

$$\begin{aligned}\widetilde{\hat{\beta}}(\gamma^{\widehat{Q}}_t(u\oplus n)) &= \sigma^{\Phi_{\widehat{Q}}}_t(\widetilde{\hat{\beta}}(u\oplus n)) = \sigma^{\Phi_{\widehat{Q}}}_t(s(u)\oplus\hat{\beta}(n))\\ &= s(\gamma^P_t(u))\oplus\hat{\beta}(\hat{\gamma}_t(n)) = \widetilde{\hat{\beta}}(\gamma^P_t(u)\oplus\hat{\gamma}_t(n))\end{aligned}$$

from which we get $\gamma^{\widehat{Q}}_t(u\oplus n) = \gamma^P_t(u))\oplus\hat{\gamma}_t(n)$, and using [E5]3.10 (vii), $\gamma^Q_t(u\oplus n) = \gamma^P_t(u)\oplus\gamma_t(n)$. ■

5.13. Proposition. *Let $\mathfrak{G}$ be a measured quantum groupoid, and $(A, b, \mathfrak{a}, \phi, \psi_0)$ be a Galois system for $\mathfrak{G}$; let $A \subset \tilde{A}$ be a unital inclusion of von Neumann algebras, and $(b, \tilde{\mathfrak{a}})$ be an action of $\mathfrak{G}$ on $\tilde{A}$; let us suppose that $\tilde{A}^{\tilde{\mathfrak{a}}} = A^{\mathfrak{a}}$, and that $\tilde{\mathfrak{a}}_{|A} = \mathfrak{a}$. Then $\tilde{A} = A$.*

Proof. As the restriction of $T_{\tilde{\mathfrak{a}}}$ to A is equal to $T_{\mathfrak{a}}$, we get clearly that $\tilde{\mathfrak{a}}$ is integrable. Let now ψ_0 be a normal faithful semi-finite weight on $A^{\mathfrak{a}}$, and $\psi_1 = \psi_0 \circ T_{\mathfrak{a}}$, $\tilde{\psi_1} = \psi_0 \circ T_{\tilde{\mathfrak{a}}}$; clearly, we get that theses two weights are normal faithful semi-finite, and that ψ_1 is equal to the restriction of $\tilde{\psi_1}$ to A; from which we get that there exists a normal faithful conditional expectation E from $\tilde{A}$ onto A, such that $\tilde{\psi_1} = \psi_1 \circ E$, and a projection p in $\mathcal{L}(H_{\tilde{\psi_1}})$ such that $p\Lambda_{\tilde{\psi_1}}(x) = \Lambda_{\tilde{\psi_1}}(Ex)$, for any $x \in \mathfrak{N}_{\tilde{\psi_1}}$; moreover, as $\tilde{\psi_1}$ is δ-relatively invariant and has the density property (3.2), we get, using the implementation $V_{\tilde{\psi_1}}$ of $\tilde{\mathfrak{a}}$ recalled in 2.4, that, for any $x \in \mathfrak{N}_{\tilde{\psi_1}}$, $\xi \in D({}_\alpha H, \nu)$ and $\eta \in D({}_\alpha H, \nu) \cap \mathcal{D}(\delta^{1/2})$ such that $\delta^{1/2}\eta$ belongs to $D(H_\beta, \nu^o)$, we get

$$\Lambda_{\tilde{\psi_1}}[(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(x)] = (id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})\Lambda_{\tilde{\psi_1}}(x)$$

from which we get that

$$\begin{aligned}(id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})p\Lambda_{\tilde{\psi_1}}(x) &= (id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})\Lambda_{\tilde{\psi_1}}(Ex) = \Lambda_{\tilde{\psi_1}}[(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(Ex)]\\ &= \Lambda_{\tilde{\psi_1}}[E(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(Ex)] = p\Lambda_{\tilde{\psi_1}}[(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(Ex)]\\ &= p(id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})p\Lambda_{\tilde{\psi_1}}(x)\end{aligned}$$

from which we get $(id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})p = p(id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})p$. Using now 3.6 and 2.2, we get that p belongs to $\pi_{\tilde{\mathfrak{a}}}(1 \underset{N}{{}_b\otimes_\alpha} \widehat{M}')'$. Returning to the same calculation, we then get that

$$\begin{aligned}\Lambda_{\tilde{\psi_1}}[(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(Ex)] &= (id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})p\Lambda_{\tilde{\psi_1}}(x)\\ &= p(id * \omega_{\delta^{1/2}\eta,\xi})(V_{\tilde{\psi_1}})\Lambda_{\tilde{\psi_1}}(x)\\ &= p\Lambda_{\tilde{\psi_1}}[(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(x)]\\ &= \Lambda_{\tilde{\psi_1}}[E(id \underset{N}{{}_b*_\alpha} \omega_{\eta,\xi})\tilde{\mathfrak{a}}(x)]\end{aligned}$$

from which we get that $\mathfrak{a} \circ E = (E \underset{N}{{}_b*_\alpha} id)\tilde{\mathfrak{a}}$. Hence, $E \underset{N}{{}_b*_\alpha} id$ can be extended to a faithful conditional expectation from $\tilde{A} \rtimes_{\tilde{\mathfrak{a}}} \mathfrak{G}$ onto $A \rtimes_{\mathfrak{a}} \mathfrak{G}$, and we easily get that, for any $X \in \tilde{A} \rtimes_{\tilde{\mathfrak{a}}} \mathfrak{G}$, $\pi_{\mathfrak{a}}(E \underset{N}{{}_b*_\alpha} id)(X) = p\pi_{\tilde{\mathfrak{a}}}(X)p$; as the action $\mathfrak{a}$ is Galois by hypothesis, $\pi_{\mathfrak{a}}$ is faithful, and we then get that $\pi_{\tilde{\mathfrak{a}}}$ is also faithful, and therefore that $\tilde{\mathfrak{a}}$ is also Galois. Let $\tilde{G}_{\tilde{\mathfrak{a}}}$ be its Galois unitary, as defined in 3.11.

Moreover, if $\mathfrak{T}$ is the normal faithful semi-finite operator-valued weight from A onto $b(N)$ such that $\phi = \nu \circ b^{-1} \circ \mathfrak{T}$, we get that $E \circ \mathfrak{T}$ is a normal faithful semi-finite operator-valued weight from $\tilde{A}$ onto $b(N)$, which satisfies $(E \circ \mathfrak{T} \underset{N}{{}_b*_\alpha} id)\tilde{\mathfrak{a}} = (E \underset{N}{{}_b*_\alpha} id)\mathfrak{a} \circ \mathfrak{T} = \mathfrak{a} \circ E \circ \mathfrak{T}$, which gives that $\phi \circ E$ is invariant by $\mathfrak{a}$. For all $t \in \mathbb{R}$, using the notations of 3.11, we get that $(D\phi \circ E : D\tilde{\psi_1})_t = (D\phi : D\psi_1)_t = \lambda_A^{it^2/2}\delta_A^{it}$, which proves that the modular automorphism groups of $\phi \circ E$ and $\tilde{\psi_1}$ commute, and therefore we have obtained that $(\tilde{A}, b, \tilde{\mathfrak{a}}, \phi \circ E, \psi_0)$ is a Galois system for $\mathfrak{G}$.

So, using 5.11, we get that $p \in \widehat{P}$, where $(A^{\mathfrak{a}}, \widehat{P}, r, s, \Gamma_{\widehat{P}}, T_L^{\widehat{P}}, R_{\widehat{P}} \circ T_L^{\widehat{P}} \circ R_{\widehat{P}}, \psi_0)$ is the measured quantum groupoid $\mathfrak{G}_1(\tilde{A}, b, \tilde{\mathfrak{a}}, \phi \circ E, \psi_0)$; more precisely, using the definition of p, we get that $p \in r(A^{\mathfrak{a}})' \cap b(N)'$, and that $J_{\tilde{\psi_1}} p J_{\tilde{\psi_1}} = p$, which gives (5.4(iii)) that $R_{\widehat{P}}(p) = p$, and therefore that $p \in s(A^{\mathfrak{a}})'$; using now the definition of $G_{\tilde{\alpha}}$ given in 3.8, we get that $G_{\tilde{\mathfrak{a}}}(p \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p) = (p \underset{N}{{}_b\otimes_\alpha} 1)G_{\tilde{\mathfrak{a}}}$, which gives then, using again 5.4(iii), that $\Gamma_{\widehat{P}}(p) = p \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p \leq 1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p$. Applying now the faithful operator valued weight $id \underset{A^{\mathfrak{a}}}{{}_s*_r} T_L^{\widehat{P}}$ to the positive operator $1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p - \Gamma_{\widehat{P}}(p)$, we obtain that $\Gamma_{\widehat{P}}(p) = 1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p$, and therefore that $p \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p = 1 \underset{A^{\mathfrak{a}}}{{}_s\otimes_r} p$, and $p = 1$; from which we infer the result. ∎

6. Morita equivalence for measured quantum groupoids. In that chapter, we begin (6.1 and 6.2) by the converse result of 5.11; starting from a measured quantum groupoid with a basis of the form $N_1 \oplus N_2$, we see under which conditions it is a linking measured quantum groupoid between a measured quantum groupoid $\mathfrak{G}_1$ (with basis N_1) and a measured quantum groupoid $\mathfrak{G}_2$ (with basis N_2). This leads to some technical additional results about the reflected groupoid of a measured quantum groupoid $\mathfrak{G}$ through some Galois system (6.3 and 6.4). Then, we can define Morita equivalence of measured quantum groupoids (6.5), prove it is indeed an equivalence relation (6.7), and give a complete link between Morita equivalence and Galois systems (6.10). We finish this chapter by giving some examples of Morita equivalences between locally compact quantum groups and measured quantum groupoids (6.12).

6.1. Proposition. *Let $\mathfrak{G}_{1,2}$ be a measured quantum groupoid with a basis which is a sum $N_1 \oplus N_2$, we shall denote $(N_1 \oplus N_2, M, \alpha, \beta, \Gamma, T, RTR, \nu_1 \oplus \nu_2)$, with a co-inverse R; we shall identify $H_{\nu_1 \oplus \nu_2}$ with $H_{\nu_1} \oplus H_{\nu_2}$; let us denote by e_1 the unit of N_1, considered as a projection in $N_1 \oplus N_2$, and $e_2 = 1 - e_1$. Let us suppose that $\alpha(e_1)$ belongs to $Z(M)$; let us denote by α_i (resp. β_i) the restriction of α (resp. β) to N_i ($i = 1, 2$). Let us write $\Phi = (\nu_1 \oplus \nu_2) \circ \alpha^{-1} \circ T$, and $H = H_\Phi$. Let us write $M_{i,j} = M_{\alpha(e_i)\beta(e_j)}$. Let $\widehat{M}$ be the underlying von Neumann algebra of the dual measured quantum groupoid $\widehat{\mathfrak{G}_{1,2}}$. Then:*

(i) The projections $\alpha(e_2)$, $\beta(e_1)$ *and* $\beta(e_2)$ *belong to* $Z(M)$. *Moreover, the projection* $\alpha(e_1)$ *belongs to* $Z(\widehat{M})$ *if and only if* $\alpha(e_1)\beta(e_2) = 0$.

(ii) If η *belongs to* $D({}_\alpha H, \nu_1 \oplus \nu_2)$, *then* $\alpha(e_i)\eta$ *belongs to* $D({}_{\alpha_i}H, \nu_i)$ $(i = 1, 2)$, *and*

$$R^{\alpha,\nu_1\oplus\nu_2}(\eta) = R^{\alpha_1,\nu_1}(\alpha(e_1)\eta) \oplus R^{\alpha_2,\nu_2}(\alpha(e_2)\eta)$$

and for η_1, η_2 *in* $D({}_\alpha H, \nu)$, $<\eta_1, \eta_2>^o_{\alpha,\nu_1\oplus\nu_2}$ *is equal to*

$$<\alpha(e_1)\eta_1, \alpha(e_1)\eta_2>^o_{\alpha_1,\nu_1} e_1 + <\alpha(e_2)\eta_1, \alpha(e_2)\eta_2>^o_{\alpha_2,\nu_2} e_2.$$

(iii) The map which sends (for $\eta \in D({}_\alpha H, \nu_1 \oplus \nu_2)$ *and* ξ *in* H*) the vector* $\xi \underset{\nu_1\oplus\nu_2}{{}_\beta\otimes_\alpha} \eta$ *on* $(\beta(e_1)\xi \underset{\nu_1}{{}_{\beta_1}\otimes_{\alpha_1}} \alpha(e_1)\eta) \oplus (\beta(e_2)\xi \underset{\nu_2}{{}_{\beta_2}\otimes_{\alpha_2}} \alpha(e_2)\eta)$ *extends to an isometry, which leads to the identification of* $H \underset{\nu_1\oplus\nu_2}{{}_\beta\otimes_\alpha} H$ *with*

$$(\beta(e_1)H \underset{\nu_1}{{}_{\beta_1}\otimes_{\alpha_1}} \alpha(e_1)H) \oplus (\beta(e_2)H \underset{\nu_2}{{}_{\beta_2}\otimes_{\alpha_2}} \alpha(e_2)H)$$

and for all $(i, j) = 1, 2$, *we have*

$$\Gamma(\alpha(e_i)\beta(e_j)) = [\alpha(e_i)\beta(e_1) \underset{N_1}{{}_{\beta_1}\otimes_{\alpha_1}} \alpha(e_1)\beta(e_j)] \oplus [\alpha(e_i)\beta(e_2) \underset{N_2}{{}_{\beta_2}\otimes_{\alpha_2}} \alpha(e_2)\beta(e_j)]$$

and $R(M\alpha(e_i)\beta(e_j)) = M\alpha(e_j)\beta(e_i)$.

(iv) For $(i, j) = 1, 2$, *let us write* $M_{i,j} = M_{\alpha(e_i)\beta(e_j)}$; *we can define* $*$*-anti-isomorphisms* $R_{i,j}$ *from* $M_{i,j}$ *onto* $M_{j,i}$ *by writing* $R_{i,j}(x_{\alpha(e_i)\beta(e_j)}) = R(x)_{\alpha(e_j)\beta(e_i)}$. *So, we get that* $M_{j,i}$ *is isomorphic to* $M^o_{i,j}$; *using (i), we get that* $M_{1,2} \neq \{0\}$ *if and only if* $\alpha(e_1)$ *does not belong to* $Z(\widehat{\mathfrak{G}_{1,2}})$. *Moreover, we can define, for all* $x \in M$:

$$\Gamma^1_{i,j}(x_{\alpha(e_i)\beta(e_j)}) = \Gamma(x)_{\alpha(e_i)\beta(e_1) \underset{N_1}{{}_{\beta_1}\otimes_{\alpha_1}} \alpha(e_1)\beta(e_j)},$$

$$\Gamma^2_{i,j}(x_{\alpha(e_i)\beta(e_j)}) = \Gamma(x)_{\alpha(e_i)\beta(e_2) \underset{N_2}{{}_{\beta_2}\otimes_{\alpha_2}} \alpha(e_2)\beta(e_j)},$$

which satisfies, for $k = 1, 2$, *for any* $n_i \in N_i$, *and* $n_j \in N_j$:

$$\Gamma^k_{i,j}(\alpha_i(n_i)) = \alpha_i(n_i) \underset{N_k}{{}_{\beta_k}\otimes_{\alpha_k}} 1, \quad \Gamma^k_{i,j}(\beta_j(n_j)) = 1 \underset{N_k}{{}_{\beta_k}\otimes_{\alpha_k}} \beta_j(n_j),$$

and $\Gamma^k_{i,j}$ *are normal injective* $*$*-homomorphisms from* $M_{i,j}$ *into* $M_{i,k} \underset{N_k}{{}_{\beta_k}*_{\alpha_k}} M_{k,j}$. *These homomorphisms satisfy:*

$$(\Gamma^i_{i,i} \underset{N_i}{{}_{\alpha_i}*_{\beta_i}} id)\Gamma^i_{i,i} = (id \underset{N_i}{{}_{\alpha_i}*_{\beta_i}} \Gamma^i_{i,i})\Gamma^{,}_{i,i}i,$$

$$(\Gamma^j_{i,j} \underset{N_j}{{}_{\beta_j}*_{\alpha_j}} id)\Gamma^j_{i,j} = (id \underset{N_j}{{}_{\beta_j}*_{\alpha_j}} \Gamma^j_{j,j})\Gamma^j_{i,j},$$

$$(\Gamma^i_{i,i} \underset{N_i}{{}_{\beta_i}*_{\alpha_i}} id)\Gamma^i_{i,j} = (id \underset{N_i}{{}_{\beta_i}*_{\alpha_i}} \Gamma^i_{i,j})\Gamma^i_{i,j},$$

$$(\Gamma^i_{i,j} \underset{N_j}{{}_{\beta_j}*_{\alpha_j}} id)\Gamma^j_{i,j} = (id \underset{N_i}{{}_{\beta_i}*_{\alpha_i}} \Gamma^j_{i,j})\Gamma^i_{i,j},$$

and therefore $(N_i, M_{i,i}, \alpha_i, \beta_i, \Gamma^i_{i,i})$ *is a Hopf bimodule, with* $R_{i,i}$ *as a co-inverse, and if* $\alpha(e_1)$ *does not belong to* $Z(\widehat{\mathfrak{G}_{1,2}})$, $(\beta_j, \Gamma^j_{i,j})$ *is an action of* $(N_j, M_{j,j}, \alpha_j, \beta_j, \Gamma^j_{j,j})$ *on* $M_{i,j}$, *and* $(\alpha_i, \Gamma^i_{i,j})$ *is a left action of* $(N_i, M_{i,i}, \alpha_i, \beta_i, \Gamma^i_{i,i})$ *on* $M_{i,j}$. *Moreover, these two actions commute.*

Proof. As $\alpha(e_2) = 1 - \alpha(e_1)$, $\beta(e_1) = R(\alpha(e_1))$, $\beta(e_2) = R(\alpha(e_2))$, the beginning of (i) is clear. If $\alpha(e_1)$ belongs to $Z(\widehat{\mathfrak{G}_{1,2}})$, we have $\alpha(e_1) = \beta(e_1)$, and therefore $\alpha(e_1)\beta(e_2) = 0$. Conversely, if $\alpha(e_1)\beta(e_2) = 0$, we have $\alpha(e_1) \leq \beta(e_1)$, and $\alpha(e_1) = \alpha(e_1)\beta(e_1)$. Applying R, we get $\beta(e_1) = \alpha(e_1)\beta(e_1)$, and therefore $\alpha(e_1) = \beta(e_1)$, from which we get that $\alpha(e_1)$ belongs to $Z(\widehat{\mathfrak{G}_{1,2}})$, which finishes the proof of (i). Result (ii) and (iii) are straightforward. ∎

6.2. Proposition. *Let's use the notations of 6.1. Then:*

(i) Let us remark that, for any $(i,j) = 1,2$, *we have* $T(M\alpha(e_i)\beta(e_j)) = \alpha(N_i)$, *and* $RTR(M\alpha(e_i)\beta(e_j)) = \beta(N_j)$; *this leads to define normal semi-finite faithful operator valued weights* $T_{i,j}$ *from* $M_{i,j}$ *onto* $\alpha_i(N_i)$, *and* $T'_{i,j} = R_{j,i}T_{j,i}R_{i,j}$ *from* $M_{i,j}$ *onto* $\beta_j(N_j)$. *Moreover, the left-invariance of* T *(resp. the right-invariance of* RTR*) gives then the following formulae, for any* $x_{i,j} \in M^+_{i,j}$:

$$(id \underset{N_1}{{}_{\beta_1}*_{\alpha_1}} T_{1,1})\Gamma^1_{1,1}(x_{1,1}) = T_{1,1}(x_{1,1}), \qquad (id \underset{N_2}{{}_{\beta_2}*_{\alpha_2}} T_{2,2})\Gamma^2_{1,2}(x_{1,2}) = T_{1,2}(x_{1,2}),$$

$$(T'_{1,1} \underset{N_1}{{}_{\beta_1}*_{\alpha_1}} id))\Gamma^1_{1,1}(x_{1,1}) = T'_{1,1}(x_{1,1}), \qquad (T'_{1,2} \underset{N_2}{{}_{\beta_2}*_{\alpha_2}} id)\Gamma^2_{1,2}(x_{1,2}) = T'_{1,2}(x_{1,2}),$$

$$(id \underset{N_2}{{}_{\beta_2}*_{\alpha_2}} T_{2,2})\Gamma^2_{2,2}(x_{2,2}) = T_{2,2}(x_{2,2}), \qquad (id \underset{N_1}{{}_{\beta_1}*_{\alpha_1}} T_{1,1})\Gamma^1_{2,1}(x_{2,1}) = T_{2,1}(x_{2,1}),$$

$$(T'_{2,2} \underset{N_2}{{}_{\beta_2}*_{\alpha_2}} id)\Gamma^2_{2,2}(x_{2,2}) = T'_{2,2}(x_{2,2}), \qquad (T'_{2,1} \underset{N_1}{{}_{\beta_1}*_{\alpha_1}} id)\Gamma^1_{2,1}(x_{2,1}) = T'_{2,1}(x_{2,1}),$$

from which we get that $T_{1,1}$ *(resp.* $T'_{1,1}$*) is left-invariant (resp. right-invariant) with respect to* $\Gamma^1_{1,1}$, *that* $T_{2,2}$ *(resp.* $T'_{2,2}$*) is left-invariant (resp. right-invariant) with respect to* $\Gamma^2_{2,2}$, *and that (if* $\alpha(e_1)$ *does not belong to* $Z(\widehat{\mathfrak{G}_{1,2}})$*) both actions* $\Gamma^2_{1,2}$ *and* $\Gamma^1_{2,1}$ *are integrable and have invariant weights.*

(ii) Let us define

$$\Phi_1 = \nu_1 \circ \alpha_1^{-1} \circ T_{1,1}, \qquad \psi_{1,2} = \nu_1 \circ \alpha_1^{-1} \circ T_{1,2},$$
$$\Psi_1 = \nu_1 \circ \beta_1^{-1} \circ T'_{1,1}, \qquad \phi_{1,2} = \nu_2 \circ \beta_2^{-1} \circ T'_{1,2},$$
$$\Phi_2 = \nu_2 \circ \alpha_2^{-1} \circ T_{2,2}, \qquad \psi_{2,1} = \nu_2 \circ \alpha_2^{-1} \circ T_{2,1},$$
$$\Psi_2 = \nu_2 \circ \beta_2^{-1} \circ T'_{2,2}, \qquad \phi_{2,1} = \nu_1 \circ \beta_1^{-1} \circ T'_{2,1}.$$

The fact that $\nu_1 \oplus \nu_2$ *is relatively invariant leads to the commutation of* σ^{Φ_1} *and* σ^{Ψ_1}, *of* σ^{Φ_2} *and* σ^{Ψ_2}, *of* $\sigma^{\psi_{1,2}}$ *and* $\sigma^{\phi_{1,2}}$, *and of* $\sigma^{\psi_{2,1}}$ *and* $\sigma^{\phi_{2,1}}$.

(iii) $\mathfrak{G}_i = (N_i, M_{i,i}, \alpha_i, \beta_i, \Gamma^1_{i,i}, T_{i,i}, T'_{i,i}, \nu_i)$ $(i = 1,2)$ *are two measured quantum groupoids. Moreover,* $R_{i,i}$ *is the co-inverse of* $\mathfrak{G}_i$.

(iv) If $\alpha(e_1) \in Z(\widehat{\mathfrak{G}_{1,2}})$, *then* $\mathfrak{G}_{1,2} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$; *if* $\alpha(e_1)$ *does not belong to* $Z(\widehat{\mathfrak{G}_{1,2}})$, *then* $(M_{1,2}, \beta_2, \Gamma^2_{1,2}, \phi_{1,2}, \nu_1)$ *is a Galois system for* $\mathfrak{G}_2$ *(and* $\mathfrak{G}_1$ *is the measured quantum groupoid reflected from* $\mathfrak{G}_2$ *through this Galois system), and* $(M_{2,1}, \beta_1, \Gamma^1_{2,1}, \phi_{2,1}, \nu_2)$ *is a Galois system for* $\mathfrak{G}_1$ *(and* $\mathfrak{G}_2$ *is the measured quantum groupoid reflected from* $\mathfrak{G}_1$ *through this Galois system). Moreover, the left action* $(\alpha_1, \Gamma^1_{1,2})$ *of* $\mathfrak{G}_1$ *on* $M_{1,2}$ *leads (2.4) to an*

action $(\alpha_1, \varsigma_{N_1}\Gamma^1_{1,2})$ *of* $\mathfrak{G}_1^o$ *on* $M^o_{1,2}$*, which, by the identification of* $M_{2,1}$ *with* $M^o_{1,2}$ *made in 6.1(iii), is equal to* $(\beta_1, \Gamma^1_{2,1})^o$.

Proof. Results (i), (ii), (iii) are straightforward. If $\alpha(e_1) \in Z(\widehat{\mathfrak{G}_{1,2}})$, then $M_{1,2} = M_{2,1} = \{0\}$, and we get that $\mathfrak{G}_{1,2} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ (in the sense of 2.3(v)). Otherwise, we have got in (i) that $(\beta_2, \Gamma^2_{1,2})$ is an integrable action of $\mathfrak{G}_2$ on $M_{1,2}$, with an invariant normal faithful semi-finite weight $\phi_{1,2}$; moreover, the invariant algebra $M_{1,2}^{\Gamma^2_{1,2}}$ is $\alpha_1(N_1)$, and the modular automorphism group of the lifted weight $\psi_{1,2}$ commutes with the modular automorphism group $\phi_{1,2}$, which gives that ν_1 is $\Gamma^2_{1,2}$-relatively invariant, in the sense of 3.11. Therefore, to get that $(M_{1,2}, \beta_2, \Gamma^2_{1,2}, \phi_{1,2}, \nu_1)$ is a Galois system for $\mathfrak{G}_2$, we have only to prove that the Galois homomorphism $\pi_{\Gamma^2_{1,2}}$ is faithful, or, equivalently (3.9(iv)), that the isometry G constructed in 3.8 from $\Gamma^2_{1,2}$ is surjective. As $\Gamma^2_{1,2}$ is "part of " Γ, we get, using 3.12(ii) that ςG is the restriction and co-restriction of

$$(\alpha(e_1)\beta(e_2) \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \alpha(e_2)\beta(e_2))W^*(\alpha(e_1)\beta(e_2) \underset{\nu^o}{{}_\alpha\otimes_\beta} \alpha(e_2)\beta(e_2))$$

which is a unitary. The proof for $(\beta_1, \Gamma^1_{2,1})$ is identical. ∎

6.3. Theorem. *Let* $\mathfrak{G}$ *be a measured quantum groupoid,* $(A, b, \mathfrak{a}, \phi, \psi_0)$ *a Galois system for* $\mathfrak{G}$*, and* $\mathfrak{G}_1$ *be the measured quantum groupoid reflected from* $\mathfrak{G}$ *through* $(A, b, \mathfrak{a}, \phi, \psi_0)$*; let's use the notations of 5.12; then, for* $z \in A$*, let us write*

$$\mathfrak{b}(z) = \Gamma_Q(z)_{\tilde\alpha(e_1)\tilde\beta(e_1) \underset{\tilde N}{{}_{\tilde\beta}\otimes_{\tilde\alpha}} \tilde\alpha(e_1)\tilde\beta(e_2)}.$$

Then, $\mathfrak{b}(z)$ *belongs to* $P \underset{A^{\mathfrak{a}}}{{}_{\hat s}*_r} A$*, with* $\hat s(x) = J_{\Phi_{\widehat P}} r(x)^* J_{\Phi_{\widehat P}}$ *for all* $x \in A^{\mathfrak{a}}$*, and* $(r, \mathfrak{b})$ *is a left action of* $\mathfrak{G}_1$ *on* A*, with* $A^{\mathfrak{b}} = b(N)$*; the left action* $(r, \mathfrak{b})$ *commutes with* $\mathfrak{a}$*, and leads to a Galois system for* $\mathfrak{G}_1$.

Proof. Let us denote by $(\tilde N, Q, \tilde\alpha, \tilde\beta, \Gamma_Q, T^Q_L, R_Q T^Q_L R_Q, \psi_0 \oplus \nu)$ the linking measured quantum groupoid between $\mathfrak{G}$ and $\mathfrak{G}_1$, as in 5.12. Then, the result comes from 6.2(iv). ∎

6.4. Theorem. *Let* $\mathfrak{G}$ *be a measured quantum groupoid,* $(A, b, \mathfrak{a}, \phi, \psi_0)$ *a Galois system for* $\mathfrak{G}$*, and* $\mathfrak{G}_1$ *be the measured quantum groupoid reflected from* $\mathfrak{G}$ *through* $(A, b, \mathfrak{a}, \phi, \psi_0)$*, and* $\mathfrak{G}_2$ *the linking measured groupoid between* $\mathfrak{G}$ *and* $\mathfrak{G}_1$*; we have, for* $x \in P^+$*,* $y \in M^+$*,* z_1*,* z_2 *in* A^+*:*

(i) $T^Q_L(x \oplus z_1 \oplus z_2^o \oplus y) = (T_L(x) + r \circ T_{\mathfrak{a}}(z_1)) \oplus (T_L(y) + \alpha \circ b^{-1}\mathfrak{T}(z_2))$*, where* $\mathfrak{T}$ *is the normal semi-finite faithful operator-valued weight from* A *onto* $b(N)$ *defined by* $\phi = \nu \circ b^{-1} \circ \mathfrak{T}$.

(ii) $\delta_Q = \delta_P \oplus \delta_A \oplus (\delta_A^{-1})^o \oplus \delta$.

(iii) $\lambda_P = \lambda_A = r(b \circ \beta^{-1}(\lambda))$.

Proof. Applying 6.2(i) to the measured quantum groupoid $\mathfrak{G}_2$, we see that the map $x \mapsto T^Q_L(x)$ is a left-invariant weight on (P, Γ_P); moreover, using 5.12(v), we get that, for all $t \in \mathbb{R}$, we have ${\tau^Q_t}_{|P} = \tau^P_t$ and $(\gamma^Q_t)_{|A^{\mathfrak{a}}} = \gamma^P_t$; therefore, we can use Lesieur's theorem ([L], 5.21), and we get that there exists a non-singular positive operator h affiliated to $Z(A^{\mathfrak{a}})$ such that, for all $x \in P^+$, we have $T^Q_L(x) = T^P_L(r(h)x)$. Therefore, we have then $\Phi_Q(x) = \Phi_P(r(h)x)$; but using now the link between $W^*_{\widehat Q}$ and $W^*_{\widehat P}$ found in 5.12, we get, using [E5], 3.10(v), that, for an operator of the form $x = (\omega * id)(W^*_{\widehat{P}})$, with $\omega \in I_{\Phi_{\widehat{P}}}$ (with the notations of [E5], 3.10 (v)), we have $\Phi_Q(x^*x) = \Phi_P(x^*x)$, from which we infer that $h = 1$, and $T_L^Q(x) = T_L^P(x)$, for all $x \in P^+$.

The fact that $T_L^Q(y) = T_L(y)$, for all $y \in M^+$, is proved by similar arguments.

Using now 5.12(iv) and 6.2(i), we get that $T_L^Q(z_1) = T_{\mathfrak{a}}(z_1)$; we have obtained that $\Phi_Q(x) = \Phi_P(x)$, $\Phi_Q(y) = \Phi(y)$, $\Phi_Q(z_1) = \psi_1(z_1)$, and using 5.12(v), that $\Phi_Q \circ R_Q(x) = \Phi_P \circ R_P(x)$, $\Phi_Q \circ R_Q(y) = \Phi \circ R(y)$, $\Phi_Q \circ R_Q(z_2) = \psi_1(z_2)$.

Let's look now at the operator P_Q^{it} which is the canonical implementation of τ_t^Q; using the results obtained for Φ_Q and for τ_t^Q (6.2(v)), we easily get that $(P_Q)_{H_{\Phi_{\widehat{P}}}} = P_P$, $(P_Q)_{H_\Phi} = P$, and using 4.8(v), that $(P_Q)_{H^{1,2}_{\psi_1}} = P_A$. With same arguments, we get that $(\lambda_Q)_{H_{\Phi_{\widehat{P}}}} = \lambda_P$, $(\lambda_Q)_{H_\Phi} = \lambda$ and $(\lambda_Q)_{H^{1,2}_{\psi_1}} = \lambda_A$. But using now [E5], 3.10 (vii), and the result about $\Delta_{\widehat{\Phi}}$ obtained in 5.7, we get that $(\delta_Q)_{H_{\Phi_{\widehat{P}}}} = \delta_P$, $(\delta_Q)_{H_\Phi} = \delta$ and using 5.5(i), that $(\delta_Q)_{H^{1,2}_{\psi_1}} = \delta_A$.

So, we get, for all $t \in \mathbb{R}$, using 3.11:

$$(D(\Phi_Q \circ R_Q)_{|A} : D(\Phi_Q)_{|A})_t = \lambda_A^{it^2/2}\delta_A^{it} = (D\phi : D\psi_1)_t$$

from we we infer that $\Phi_Q \circ R_Q(z_1) = \phi(z_1)$, for all positive z_1 in A; so, we have $\Phi_Q(z_2^o) = \phi(z_2)$ for all positive z_2 in A, from which we finish the proof of (i).

Now we have

$$(D(\Phi_Q \circ R_Q)_{|A^o} : D(\Phi_Q)_{|A^o})_t = (D\Phi^o : D\psi_1^o)_t = [(\lambda_A)^o]^{it^2/2}[(\delta_A)^o]^{-it}$$

from which we get (ii). Finally, there is $p \in Z(N)$ such that $\lambda = \alpha(p) = \beta(p)$, and $u \in Z(A^{\mathfrak{a}})$ such that $\lambda_P = r(u) = \hat{s}(u)$; on the other hand, there are $q \in Z(N)$ and $v \in Z(A^{\mathfrak{a}})$ such that $\lambda_Q = r(v) \oplus \alpha(q) = \hat{s}(v) \oplus \beta(q)$. From all our calculations above, we infer that $q = p$, $v = u$, $\lambda_A = r(v)$ and $\lambda_A^o = \alpha(p)$; from which we get (iii). ∎

6.5. Definition. For $i = 1, 2$, let $\mathfrak{G}_i = (N_i, M_i, \alpha_i, \beta_i, T_i, T'_i, \nu_i)$ be a measured quantum groupoid. We shall say that $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_2$ if there exists a von Neumann algebra A, a Galois action $(b, \mathfrak{a})$ of $\mathfrak{G}_1$ on A, a Galois left action $(a, \mathfrak{b})$ of $\mathfrak{G}_2$ on A, such that

(i) $A^{\mathfrak{a}} = a(N_2)$, $A^{\mathfrak{b}} = b(N_1)$, and the actions $(b, \mathfrak{a})$ and $(a, \mathfrak{b})$ commute;

(ii) the modular automorphism groups of the normal semi-finite faithful weights $\nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}$ and $\nu_2 \circ a^{-1} \circ T_{\mathfrak{a}}$ commute.

Then A (or, more precisely, $(A, b, \mathfrak{a}, a, \mathfrak{b})$) will be called the imprimitivity bi-comodule for $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

6.6. Remark. Then, using 3.3, we get that the system $(A, b, \mathfrak{a}, \nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}, \nu_2 \circ a^{-1})$ is Galois for $\mathfrak{G}_1$ and that the system $(A, a, \mathfrak{b}, \nu_2 \circ a^{-1} \circ T_{\mathfrak{a}}, \nu_1 \circ b^{-1})$ is left-Galois for $\mathfrak{G}_2$. Therefore, we can construct, following 5.12, the reflected measured quantum groupoid $\tilde{\mathfrak{G}_2}$ of $\mathfrak{G}_1$ through the Galois system $(A, b, \mathfrak{a}, \nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}, \nu_2 \circ a^{-1})$, and the reflected measured quantum groupoid $\tilde{\mathfrak{G}_1}$ of $\mathfrak{G}_2$ through the left-Galois system $(A, a, \mathfrak{b}, \nu_2 \circ a^{-1} \circ T_{\mathfrak{a}}, \nu_1 \circ b^{-1})$, and using 6.3, an action $\tilde{\mathfrak{a}_1}$ of $\tilde{\mathfrak{G}_1}$ on A, and a left action of $\tilde{\mathfrak{G}_2}$ on A; let us first remark that the basis of $\tilde{\mathfrak{G}_2}$ is $A^{\mathfrak{a}} = a(N_2)$ and is therefore isomorphic to N_2 which is the basis of $\mathfrak{G}_2$. Similarly, the $\mathfrak{G}_1$ and $\tilde{\mathfrak{G}_1}$ has the same basis.

As the action $\tilde{\mathfrak{a}_1}$ is Galois, the homomorphism $\pi_{\tilde{\mathfrak{a}_1}}$ is an isomorphism from the crossed product $A \rtimes_{\tilde{\mathfrak{a}}} \tilde{\mathfrak{G}_1}$ onto the algebra $\tilde{A_2}$ constructed by basic construction made from the inclusion $A^{\tilde{\mathfrak{a}}} \subset A$; as $A^{\tilde{\mathfrak{a}}} = a(N_2) = A^{\mathfrak{a}}$, we get that $\tilde{A_2}$ is equal to the algebra $s(A^{\mathfrak{a}})'$ constructed by basic construction made from the inclusion $A^{\mathfrak{a}} \subset A$, which is isomorphic, via $\pi_{\mathfrak{a}}^{-1}$, to $A \rtimes_{\mathfrak{a}} \mathfrak{G}_1$; therefore, there exists an isomorphism $\mathcal{I}_1$ from $A \rtimes_{\mathfrak{a}} \mathfrak{G}_1$ onto $A \rtimes_{\tilde{\mathfrak{a}}} \tilde{\mathfrak{G}_1}$ such that $\mathcal{I}_1 \circ \mathfrak{a} = \tilde{\mathfrak{a}}$; similarly, there exists an isomorphism $\mathcal{I}_2$ from $A \ltimes_{\mathfrak{b}} \mathfrak{G}_2$ onto $A \ltimes_{\tilde{\mathfrak{b}}} \tilde{\mathfrak{G}_2}$ such that $\mathcal{I}_2 \circ \mathfrak{b} = \tilde{\mathfrak{b}}$; using all these remarks, we easily get that $(A, b, \tilde{\mathfrak{a}}, a, \tilde{\mathfrak{b}})$ is an imprimitivity bi-comodule between $\tilde{\mathfrak{G}_1}$ and $\tilde{\mathfrak{G}_2}$; we can prove also that if A is an imprimitivity bi-comodule for $\tilde{\mathfrak{G}_1}$ and $\tilde{\mathfrak{G}_2}$, it is also an imprimitivity bi-comodule for $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

6.7. Theorem. *Morita equivalence is indeed an equivalence relation.*

Proof. Using the Galois system $(M, \beta, \Gamma, \Phi \circ R, \nu)$ (3.12(ii)), we get the left-Galois system $(M, \alpha, \Gamma, \Phi, \nu)$, and that $\mathfrak{G}$ is Morita equivalent to $\mathfrak{G}$, with M as imprimitivity bi-comodule; so, Morita equivalence is indeed reflexive.

If $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_2$, with A as imprimitivity co-bimodule, we get, using 3.11, that $(b^o, \sigma_N \mathfrak{a}^o)$ and $(a^o, (\sigma_N \mathfrak{b})^o)$ make $\mathfrak{G}_2$ be Morita equivalent to $\mathfrak{G}_1$, with A^o as imprimitivity co-bimodule; so, Morita equivalence is indeed symmetric.

Let us suppose now that $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3$ are three measured quantum groupoids, and that $(A_1, b_1, \mathfrak{a}_1, a_1, \mathfrak{b}_1)$ is an imprimitivity bi-comodule for $\mathfrak{G}_1$ and $\mathfrak{G}_2$, and $(A_2, b_2, \mathfrak{a}_2, a_2, \mathfrak{b}_2)$ is an imprimitivity bi-comodule for $\mathfrak{G}_2$ and $\mathfrak{G}_3$. Using 6.6, we know there exists an action $(b_1, \tilde{\mathfrak{a}_1})$ of the reflected measured quantum groupoid $\tilde{\mathfrak{G}_1}$ of $\mathfrak{G}_2$ through the Galois system $(A_1, a_1, \mathfrak{b}_1, \nu_2 \circ a_1^{-1} \circ T_{\mathfrak{a}}, \nu_1 \circ b_1^{-1})$ such that $(A_1, b_1, \tilde{\mathfrak{a}_1}, a_1, \mathfrak{b}_1)$ is an imprimitivity bi-comodule between $\tilde{\mathfrak{G}_1}$ and $\mathfrak{G}_2$; similarly, we shall consider $(A_2, b_2, \mathfrak{a}_2, a_2, \tilde{\mathfrak{b}_2})$ which is an imprimitivity bi-comodule between $\mathfrak{G}_2$ and the reflected measured quantum groupoid $\tilde{\mathfrak{G}_3}$ of $\mathfrak{G}_2$ through the left Galois system $(A_2, a_2, \mathfrak{b}_2, \nu_3 \circ a_2^{-1} \circ T_{\mathfrak{a}}, \nu_2 \circ b_2^{-1})$.

Let $A_3 = \{X \in A_2 \underset{N_2}{{}_{b_2}*_{a_2}} A_1; (id \underset{N_2}{{}_{b_2}*_{a_2}} \mathfrak{b}_1)(X) = (\mathfrak{a}_2 \underset{N_2}{{}_{b_2}*_{a_2}} id)(X)\}$. It is straightforward to check that $a_2(N_3) \underset{N_2}{{}_{b_2}*_{a_2}} 1 \subset A_3$ and $1 \underset{N_2}{{}_{b_2}*_{a_2}} b_1(N_1) \subset A_3$, and that:

(i) $(1 \underset{N_2}{{}_{b_2}*_{a_2}} b_1, (id \underset{N_2}{{}_{b_2}*_{a_2}} \tilde{\mathfrak{a}_1})_{|A_3}$ is an action of $\tilde{\mathfrak{G}_1}$ on A_3, we shall denote it by $(b_3, \mathfrak{a}_3)$ for simplification.

(ii) $(a_2 \underset{N_2}{{}_{b_2}*_{a_2}} 1, (\tilde{\mathfrak{b}_2} \underset{N_2}{{}_{b_2}*_{a_2}} id)_{|A_3})$ is a left action of $\tilde{\mathfrak{G}_3}$ on A_3, we shall denote it by $(a_3, \mathfrak{b}_3)$ for simplification.

(iii) We have $A_3^{\mathfrak{a}_3} = a_3(N_3)$, and $A_3^{\mathfrak{b}_3} = b_3(N_1)$, and the actions $\mathfrak{a}_3$ and $\mathfrak{b}_3$ commute.

To prove that we get an imprimitivity system, we shall make a detour.

So, let us consider a Galois system for $\mathfrak{G}_2$, with $\tilde{\mathfrak{G}_1}$ as reflected measured quantum groupoid, and another Galois system for $\mathfrak{G}_2$, with $\tilde{\mathfrak{G}_3}$ as reflected measured quantum groupoid. Let us consider now, as in 5.1, the representation μ_1 of $\widehat{M_2}'$ on H_1 and the representation μ_3 of $\widehat{M_2}'$ on H_3, and the representation ϖ of $\widehat{M_2}'$ on $H_3 \oplus H_2 \oplus H_1$ given by

$\mu_3 \oplus id \oplus \mu_1$, and $\widehat{Q} = \varpi(\widehat{M'})'$; using again matrix notations for elements in $\widehat{Q}$, we get that

$$\widehat{Q} = \begin{pmatrix} \widehat{\tilde{M}_3} & \widehat{Q}_{2,3} & \widehat{Q}_{1,3} \\ \widehat{Q}^*_{2,3} & \widehat{M_2} & \widehat{Q}_{1,2} \\ \widehat{Q}^*_{1,3} & \widehat{Q}^*_{1,3} & \widehat{\tilde{M}_1} \end{pmatrix}$$

where, for instance:

$$\widehat{Q}_{1,3} = \{X \in \mathcal{L}(H_1, H_3), X\mu_1(m) = \mu_3(m)X, \forall m \in \widehat{M_2}'\}.$$

We have clearly $\widehat{Q}_{2,3}\widehat{Q}_{1,2} \subset \widehat{Q}_{1,3}$; using again an orthogonal basis as in the proof of 5.4(i), we get that the linear set generated by the products in $\widehat{Q}_{2,3}\widehat{Q}_{1,2}$ is weakly dense in $\widehat{Q}_{1,3}$. But, as in 5.2, we can construct a coproduct from $\widehat{Q}_{1,2}$ into $\widehat{Q}_{1,2} \underset{N_1}{{}_{\hat{\beta_1}}*_{\alpha_1}} \widehat{Q}_{1,2}$ and a coproduct from $\widehat{Q}_{2,3} \underset{N_2}{{}_{\hat{\beta_2}}*_{\alpha_2}} \widehat{Q}_{2,3}$, and, by product, we obtain therefore a coproduct from $\widehat{Q}_{1,3}$ into $\widehat{Q}_{1,3} \underset{N_1}{{}_{\hat{\beta_1}}*_{\alpha_1}} \widehat{Q}_{1,3}$, then, as in 5.4, a coproduct for $\widehat{Q}$. The proof that $\widehat{Q}$ has a structure of measured quantum groupoid is completely similar to 5.8, 5.9 and 5.10. So, as in 5.12, we can look at the dual measured quantum groupoid, which will be on the basis $N_1 \oplus N_2 \oplus N_3$; let us denote by α_Q and β_Q the canonical homomorphism and antihomomorphism from $N_1 \oplus N_2 \oplus N_3$ into Q; as in 5.12, we can prove that $\alpha_Q(e_i) \in Z(Q)$ and $\beta_Q(e_i) \in Z(Q)$, where, for $(i = 1, 2, 3)$, e_i is the unit of N_i, considered as a projection in $N_1 \oplus N_2 \oplus N_3$. Then, it is easy to get that the reduced algebra on $H_1 \oplus H_3$ has a structure of measured quantum groupoid, over the basis $N_1 \oplus N_3$. As $\widehat{Q}_{1,3} \neq \{0\}$, we can use 6.2(iv), and we get the existence of a Galois system for $\tilde{\mathfrak{G}}_3$, with $\tilde{\mathfrak{G}}_1$ as reflected measured quantum groupoid, which means that $\tilde{\mathfrak{G}}_3$ is Morita equivalent to $\tilde{\mathfrak{G}}_1$ (and, by the reflexivity, that $\tilde{\mathfrak{G}}_1$ is Morita equivalent to $\tilde{\mathfrak{G}}_3$); using then arguments analogous to 6.6, we get that $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_3$, which proves the transitivity. To get the imprimitivity bi-comodule, we must look at the dual $Q = \oplus^3_{i,j=1} Q_{i,j}$, which has a coproduct Γ_Q, which can be split into maps $(\Gamma_Q)^k_{i,j} : Q_{i,j} \mapsto Q_{i,k} * Q_{k,j}$.

We know that $Q_{1,1} = \tilde{M}_1$, $Q_{2,2} = M_2$, $Q_{3,3} = M_3$, $Q_{2,1} = A_1$, $Q_{3,2} = A_2$, and we are looking for $Q_{3,1}$. We know also that $(\Gamma_Q)^1_{1,1} = \tilde{\Gamma_1}$, $(\Gamma_Q)^2_{2,2} = \Gamma_2$, $(\Gamma_Q)^3_{3,3} = \tilde{\Gamma_3}$, $(\Gamma_Q)^1_{2,1} = \tilde{\mathfrak{a}_1}$, $(\Gamma_Q)^2_{2,1} = \mathfrak{b}_1$, $(\Gamma_Q)^2_{3,2} = \mathfrak{a}_2$, $(\Gamma_Q)^3_{3,2} = \tilde{\mathfrak{b}_2}$.

So, $(\Gamma_Q)^2_{3,1}$ sends $Q_{3,1}$ into $A_2 * A_1$, and it is easy, with the co-associativity condition of Γ_Q, to get that $(\Gamma_Q)^2_{3,1}$ sends $Q_{3,1}$ into A_3, and that $(\Gamma_Q)^2_{3,1}$ sends the action $(\Gamma_Q)^1_{3,1}$ on $id \underset{N_2}{{}_{b_2}*_{a_2}} \tilde{\mathfrak{a}_1}$ and the left action $(\Gamma_Q)^3_{3,1}$ on $\tilde{\mathfrak{b}_2} \underset{N_2}{{}_{b_2}*_{a_2}} id$; using then 5.13, we get that A_3 is the image of $(\Gamma_Q)^2_{3,1}$, which allow us to identify $Q_{3,1}$ with A_3, $(\Gamma_Q)^1_{3,1}$ with $\mathfrak{a}_3$, and $(\Gamma_Q)^3_{3,1}$ with $\mathfrak{b}_3$. By these identifications, we prove that $(A_3, \mathfrak{a}_3, \mathfrak{b}_3)$ is an imprimitivity bi-comodule between $\tilde{\mathfrak{G}}_1$ and $\tilde{\mathfrak{G}}_3$. By similar arguments to 6.6, we get an imprimitivity bi-comodule between $\mathfrak{G}_1$ and $\mathfrak{G}_3$. ∎

6.8. Notations. Let $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ be three measured quantum groupoids; let us suppose that $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_2$, with $(A_1, \mathfrak{a}_1, \mathfrak{b}_1)$ (or A_1 for simplification) as imprimitivity bi-comodule and that $\mathfrak{G}_2$ is Morita equivalent to $\mathfrak{G}_3$ with $(A_2, \mathfrak{a}_2, \mathfrak{b}_2)$ (or simply A_2) as imprimitivity co-bimodule; we have proved in 6.7 that $\mathfrak{G}_1$ is Morita equivalent to

$\mathfrak{G}_3$, with $(A_3, \mathfrak{a}_3, \mathfrak{b}_3)$ as imprimitivity co-bimodule, with

$$A_3 = \{X \in A_2 \underset{N_2}{{}_{b_2}*_{a_2}} A_1; (id \underset{N_2}{{}_{b_2}*_{a_2}} \mathfrak{b}_1)(X) = (\mathfrak{a}_2 \underset{N_2}{{}_{b_2}*_{a_2}} id)(X)\}$$

and $\mathfrak{a}_3 = (id \underset{N_2}{{}_{b_2}*_{a_2}} \mathfrak{a}_1)_{|A_3}$, $\mathfrak{b}_3 = (\mathfrak{b}_2 \underset{N_2}{{}_{b_2}*_{a_2}} 1)_{|A_3}$.

We shall write $(A_3, \mathfrak{a}_3, \mathfrak{b}_3) = (A_2, \mathfrak{a}_2, \mathfrak{b}_2) \circ (A_1, \mathfrak{a}_1, \mathfrak{b}_1)$, or, simply $A_3 = A_2 \circ A_1$; we can check that this product is associative, and that, if we write M_1 for the imprimitivity bi-comodule $(M_1, \Gamma_1, \Gamma_1)$ between $\mathfrak{G}_1$ and itself, we easily get that $A_1 \circ M_1 = A_1$ and $M_2 \circ A_1 = A_1$.

6.9. Proposition. *Let $\mathfrak{G}$, $\mathfrak{G}_1$, $\mathfrak{G}_2$ be measured quantum groupoids; let us use the notations of 6.8.*

(i) Suppose that $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_2$ with an imprimitivity co-bimodule A. Then $A^o \circ A = M_1$.

(ii) Suppose that $\mathfrak{G}$ is Morita equivalent to $\mathfrak{G}$ with an imprimitivity co-bimodule A; then $A = M$.

(iii) Suppose that $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_2$ with an imprimitivity co-bimodule A_1, and with another imprimitivity co-bimodule A_2; then $A_1 = A_2$.

(iv) Suppose that $\mathfrak{G}_1$ is Morita equivalent to $\mathfrak{G}_2$ with an imprimitivity bi-comodule $(A, \mathfrak{a}, \mathfrak{b})$; then $\mathfrak{G}_2$ is the reflected measured quantum groupoid of $\mathfrak{G}_1$ through the Galois system $(A, b, \mathfrak{a}, \nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}, \nu_2 \circ a^{-1})$.

Proof. Let us use the Galois system $(A, b, \mathfrak{a}, \nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}, \nu_2 \circ a^{-1})$, and apply the constructions and results of 5.11 applied to this Galois system; for any $y \in M_1$, the operator $\Gamma_Q(y)_{\tilde{\alpha}(e_2)\tilde{\beta}(e_1) \underset{\tilde{N}}{{}_{\tilde{\beta}}\otimes_{\tilde{\alpha}}} \tilde{\alpha}(e_1)\tilde{\beta}(e_2)}$ belongs to $A^o \underset{N_2}{{}_{a_2^o}*_{a_2}} A$, and more precisely, using the coassociativity of the coproduct Γ_Q, we can check it belongs to the subagebra $A^o \circ A$; we define this way an injective morphism from M_1 into $A^o \circ A$, which sends Γ_1 to the action (and on the left action) canonically defined on $A^o \circ A$; therefore, using 5.13, we get (i).

Let us now use the Galois system $(A, b, \mathfrak{a}, \nu \circ b^{-1} \circ T_{\mathfrak{b}}, \nu \circ a^{-1})$, and apply the constructions and results of 5.11 to this Galois system. Then, for $x \in A$, the operator $\Gamma_Q(x)_{\tilde{\alpha}(e_2)\tilde{\beta}(e_1) \underset{\tilde{N}}{{}_{\tilde{\beta}}\otimes_{\tilde{\alpha}}} \tilde{\alpha}(e_1)\tilde{\beta}(e_2)}$ belongs to $A^o \circ A$, and therefore, using (i), to M; we define this way an injective morphism from A into M, which sends $\mathfrak{a}$ to Γ; using again 5.13, we get (ii).

As $A_2^o \circ A_1$ is an imprimitivity bi-comodule for a Morita equivalence between $\mathfrak{G}_1$ and $\mathfrak{G}_1$, we get, using (ii), that $A_2^o \circ A_1 = M_1$; therefore, we have, using (i):

$$A_1 = M_2 \circ A_1 = A_2 \circ A_2^o \circ A_1 = A_2 \circ M_1 = A_2$$

which is (iii).

Let $\tilde{\mathfrak{G}_2}$ be the reflected measured quantum groupoid of $\mathfrak{G}_1$ through the Galois system $(A, b, \mathfrak{a}, \nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}, \nu_2 \circ a^{-1})$; there exists a left action $\tilde{\mathfrak{b}}$ of $\tilde{\mathfrak{G}_2}$ on A, and $\tilde{A} = (A, \mathfrak{a}, \tilde{\mathfrak{b}})$ is an imprimitivity bi-comodule which makes $\mathfrak{G}_1$ and $\tilde{\mathfrak{G}_2}$; therefore, using 6.8, we get that $\tilde{A} \circ A^o$ (whose underlying von Neumann algebra is M_2 by (i), and that we shall denote by P) is an imprimitivity bi-comodule between $\tilde{\mathfrak{G}_2}$ and $\mathfrak{G}_2$; we then get, using again (i), that $P^o \circ P = \tilde{M_2}$ and $P \circ P^o = M_2$, which leads, using 5.11, to define injective morphisms $M_2 \mapsto \tilde{M}_2$ and $\tilde{M}_2 \mapsto M_2$ as parts of the coproduct of the same measured quantum groupoid. Using then the co-associativity of this coproduct, we get that these mappings are each other's inverse, which leads to the isomorphism of M_2 and $\tilde{M}_2$, which is (iv). ■

6.10. Theorem. *Let $\mathfrak{G}_i = (N_i, M_i, \alpha_i, \beta_i, \Gamma_i, T_i, T'_i, \nu_i)$ $(i = 1, 2)$ be two measured quantum groupoids. Then the following are equivalent:*

(i) $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are Morita equivalent, with a imprimitivity bi-comodule $(A, \mathfrak{a}, \mathfrak{b})$.

(ii) There exists a Galois system $(A, b, \mathfrak{a}, \phi, \psi_0)$ for $\mathfrak{G}_1$, such that $\mathfrak{G}_2$ is the reflected measured groupoid of $\mathfrak{G}_1$ through this Galois system.

(iii) There exists a measured quantum groupoid

$$\mathfrak{G}_{1,2} = (N_1 \oplus N_2, M, \alpha, \beta, \Gamma, T, T', \nu_1 \oplus \nu_2)$$

such that $\alpha(e_1)$ belongs to $Z(M)$, and does not belong to $Z(\widehat{M})$, where e_1 is the unit of N_1, considered as a projection in $N_1 \oplus N_2$, and $(\mathfrak{G}_{1,2})_{\alpha(e_1)} = \mathfrak{G}_1$, $(\mathfrak{G}_{1,2})_{\alpha(1-e_1)} = \mathfrak{G}_2$.

Proof. The result (i) implies (ii) by 6.9(iv); the result (ii) implies (i) was obtained in 6.3; the result (ii) implies (iii) is given by 5.11(ii), and 6.1 gives that (iii) implies (ii). ■

6.11. Remark. A morphism between an action $(b_1, \mathfrak{a}_1)$ of $\mathfrak{G}$ on a von Neumann algebra A_1, and an action $(b_2, \mathfrak{a}_2)$ on a von Neumann algebra A_2 will be a $*$-homomorphism h from A_1 in A_2 such that $h \circ b_1 = b_2$, and $(h \underset{N}{{}_{b_1}*_{\alpha}} id)\mathfrak{a}_1 = \mathfrak{a}_2$; clearly this leads to a category $\mathcal{A}(\mathfrak{G})$; it is easy to get that , if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are two measured quantum groupoids which are Morita equivalent, then these categories $\mathcal{A}(\mathfrak{G}_1)$ and $\mathcal{A}(\mathfrak{G}_2)$ are equivalent too.

6.12. Examples of locally compact quantum groups Morita equivalent to measured quantum groupoids. Here we are looking to examples of locally compact quantum groups which are Morita equivalent to measured quantum groupoids. I am indebted to S. Vaes who called my attention to this question. We first give two constructions in which any locally compact quantum group is Morita equivalent to a measured quantum groupoid, whose basis is a given factor N (6.12.2, 6.12.3). More convincing is K. De Commer's example (6.12.4, [DC4]): he proves that the compact quantum $SU_q(2)$ is Morita equivalent to some measured quantum groupoid (whose basis is a finite sum of type I factors).

6.12.1. Ampliation of a locally compact quantum group. If **G**$= (M, \Gamma, \varphi, \psi)$ is a locally compact quantum group, and N is a von Neumann algebra, we shall call the measured quantum groupoid $\mathfrak{G}(N) \otimes \mathbf{G}$ the ampliation of **G** by N, where $\mathfrak{G}(N)$ is the N-measured quantum groupoid defined in 2.3(viii) and the tensor product of measured quantum groupoids has been defined in 2.3(ix). Morover, the measured quantum groupoid $\widehat{\mathfrak{G}(N)} \otimes \mathbf{G}$ is, using also 2.3(viii) and (ix), another measured quantum groupoid, we shall call the dual ampliation of $\mathfrak{G}$ by N.

6.12.2. Theorem. *Let* **G**$= (M, \Gamma, \varphi, \psi)$ *be a locally compact quantum group, N a factor, $\mathfrak{G}(N) \otimes \mathbf{G}$ the ampliation of $\mathfrak{G}$ by N, as defined in 6.12.1. Then, the locally compact quantum group* **G** *and the measured quantum groupoid $\mathfrak{G}(N) \otimes \mathbf{G}$ are Morita equivalent.*

Proof. Let us consider the von Neumann algebra $N \otimes M$; then, $(id \otimes \Gamma)$ is an action of $\mathbf{G}$ on this algebra; we get that the invariant subalgebra is $(N \otimes M)^{(id\otimes\Gamma)} = N \otimes \mathbb{C}$, and that the crossed product is $N \otimes \mathcal{L}(H_\varphi)$. Therefore, we get also that this action is Galois, and that $T_{id\otimes\Gamma} = id \otimes \varphi$. Let us choose a normal semi-finite faithful trace ν on N; we get $\nu \circ T_{id\otimes\Gamma} = \nu \otimes \varphi$.

Taking now on this algebra the restriction of the coproduct of $\mathfrak{G}(N) \otimes \mathbf{G}$, we obtain a left action $\mathfrak{b}$ of $\mathfrak{G}(N) \otimes \mathbf{G}$ on $N \otimes M$, and we get that $T_\mathfrak{b} = \nu \otimes \psi$. (Taking for τ the canonical finite trace on $\mathbb{C} = Z(N)$, we get that the operator-valued weight T_ν defined in 2.3(viii) is ν). So, we then get that $\mathfrak{b}$ is ergodic and Galois. Moreover, as the modular groups of φ and ψ commute, we get, by the definition (6.5) that the locally compact quantum group $\mathbf{G}$ and the measured quantum groupoid $\mathfrak{G}(N) \otimes \mathbf{G}$ are Morita equivalent, with $N \otimes M$ as imprimitivity bi-comodule. ∎

6.12.3. Proposition. *Let $\mathbf{G} = (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group, N a factor, $\widehat{\mathfrak{G}(N)} \otimes \mathbf{G}$ the dual ampliation of $\mathfrak{G}$ by N, as defined in 6.12.1. Then, the locally compact quantum group $\mathbf{G}$ and the measured quantum groupoid $\widehat{\mathfrak{G}(N)} \otimes \mathbf{G}$ are Morita equivalent.*

Proof. The proof is very similar to 6.12.2. ∎

6.12.4. Another example. In [DC2], [DC3], Kenny De Commer has studied Morita equivalences between the compact quantum group $SU_q(2)$ and various quantum groups, and, in [DC4], with a mesurable quantum groupoid. Indeed, he constructs an integrable Galois action of a $SU_q(2)$, which is not ergodic (the subalgebra of invariants is then a finite sum of type I factors), and therefore, this construction leads to measured quantum groupoid (whose basis is that finite sum of factors), which is Morita equivalent to the initial compact quantum group. This construction is a particular case of 4.7.

7. Application to deformation of a measured quantum groupoid by a 2-cocycle. In this section, we try to answer the problem of deformation of a measured quantum groupoid by a 2-cocycle. With this deformed coproduct constructed in 7.2, does this new Hopf bimodule still has a left-invariant (and a right-invariant) Haar operator-valued weight, and therefore remains a measured quantum groupoid? Following De Commer's strategy, we are able to answer this question positively for any 2-cocycle only in the case when the basis N is a finite sum of factors (7.7(xii)). In the general case, we can obtain (7.9) sufficient conditions, which leads to positive answers in particular cases (7.11, 7.12).

7.1. Definition. Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule, in the sense 2.2; a unitary Ω in $(M \cap \alpha(N)' \underset{N}{{}_\beta *_\alpha} (M \cap \beta(N)')$ is called a 2-cocycle for $(N, M, \alpha, \beta, \Gamma)$ if Ω satisfies the following relation:

$$(1 \underset{N}{{}_\beta\otimes_\alpha} \Omega)(id \underset{N}{{}_\beta *_\alpha} \Gamma)(\Omega) = (\Omega \underset{N}{{}_\beta\otimes_\alpha} 1)(\Gamma \underset{N}{{}_\beta *_\alpha} id)(\Omega).$$

If $\mathcal{G}$ is a measured qroupoid, equipped with a left Haar system and a quasi-invariant mesure on the set of units, and if Ω is a 2-cocycle for the measured quantum groupoid

$\mathfrak{G}(\mathcal{G})$(2.3(ii)), then Ω is just a measurable function from $\mathcal{G}^{(2)}$ to $\mathbb{T}$, such that, for all (g_1, g_2) and (g_2, g_3) in $\mathcal{G}^{(2)}$:

$$\Omega(g_2, g_3)\Omega(g_1, g_2g_3) = \Omega(g_1, g_2)\Omega(g_1g_2, g_3)$$

Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid, and let Ω be a 2-cocycle for $\mathfrak{G}$; let us define, for any $t \in \mathbb{R}$:

$$\Omega_t = (\tau_t \underset{N}{{}_\beta*_\alpha} \tau_t)(\Omega),$$

$$\Omega'_t = (\delta^{it} \underset{N}{{}_\beta\otimes_\alpha} \delta^{it})\Omega(\delta^{-it} \underset{N}{{}_\beta\otimes_\alpha} \delta^{-it}) = (\sigma_t^{\Phi\circ R}\sigma_{-t}^{\Phi} \underset{N}{{}_\beta*_\alpha} \sigma_t^{\Phi\circ R}\sigma_{-t}^{\Phi})(\Omega).$$

One can easily check that Ω_t and Ω'_t are also 2-cocycles for $\mathfrak{G}$.

7.2. Proposition. *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule, and let Ω be a 2-cocycle for $(N, M, \alpha, \beta, \Gamma)$; let us write, for all $x \in M$:*

$$\Gamma_\Omega(x) = \Omega\Gamma(x)\Omega^*.$$

Then, $(N, M, \alpha, \beta, \Gamma_\Omega)$ is a Hopf bimodule, that we shall call the deformation of the initial one by Ω.

Proof. We have, thanks to the definition of a 2-cocycle, for any $n \in N$:

$$\Gamma_\Omega(\alpha(n)) = \alpha(n) \underset{N}{{}_\beta\otimes_\alpha} 1, \quad \Gamma_\Omega(\beta(n)) = 1 \underset{N}{{}_\beta\otimes_\alpha} \beta(n)$$

which allows us to write

$$\begin{aligned}(\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)\Gamma_\Omega(x) &= (\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)(\Omega\Gamma(x)\Omega^*)\\ &= (\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)(\Omega)(\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)\Gamma(x)(\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)(\Omega)^*.\end{aligned}$$

But

$$(\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)(\Omega) = (\Omega \underset{N}{{}_\beta\otimes_\alpha} 1)(\Gamma \underset{N}{{}_\beta*_\alpha} id)(\Omega)(\Omega \underset{N}{{}_\beta\otimes_\alpha} 1)^*$$

and therefore

$$(\Gamma_\Omega \underset{N}{{}_\beta*_\alpha} id)\Gamma_\Omega(x) = (\Omega \underset{N}{{}_\beta\otimes_\alpha} 1)(\Gamma \underset{N}{{}_\beta*_\alpha} id)(\Omega)(\Gamma \underset{N}{{}_\beta*_\alpha} id)\Gamma(x)(\Gamma \underset{N}{{}_\beta*_\alpha} id)(\Omega)^*(\Omega \underset{N}{{}_\beta\otimes_\alpha} 1)^*$$

and, by a similar calculation, we get

$$(id \underset{N}{{}_\beta*_\alpha} \Gamma_\Omega)\Gamma_\Omega(x) = (1 \underset{N}{{}_\beta\otimes_\alpha} \Omega)(id \underset{N}{{}_\beta*_\alpha} \Gamma)(\Omega)(id \underset{N}{{}_\beta*_\alpha} \Gamma)\Gamma(x)(id \underset{N}{{}_\beta*_\alpha} \Gamma)(\Omega)^*(1 \underset{N}{{}_\beta\otimes_\alpha} \Omega)^*$$

which is equal, thanks to the definition of a 2-cocycle, and the coassociativity of Γ. ∎

7.3. Proposition. *Let $\mathfrak{G}$ be a measured quantum groupoid, and Ω be a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$; let us write $\widetilde{W} = W\Omega^*$, which is a unitary from $H \underset{\nu}{{}_\beta\otimes_\alpha} H$ onto $H \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} H$. Then:*

(i) The operator $\widetilde{W}$ satisfies

$$(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W})(\widetilde{W} \underset{N}{{}_\beta\otimes_\alpha} 1) = (W \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)\sigma^{2,3}_{\alpha,\beta}(\widetilde{W} \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o})(1 \underset{N}{{}_\beta\otimes_\alpha} \widetilde{W})$$

(with the notations of 2.2).

(ii) For all ξ, ξ' *in* $D(H_\beta,\nu^o)$, η, η' *in* $D(_\alpha H,\nu)$, *we have*

$$(\omega_{\xi,\xi'} \underset{N}{{}_\beta*_\alpha} id)[\widetilde{W}^*((id*\omega_{\eta,\eta'})(\sigma_{\nu^o}W) \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)\widetilde{W}] = \omega_{(\omega_{\xi,\eta'}*id)(\widetilde{W})^*\xi',\eta} * id)(\widetilde{W})^*.$$

(iii) The weakly closed linear space generated by the operators of the form $(\omega_{\xi,\eta} * id)(\widetilde{W})$, *for all* $\xi\in D(H_\beta,\nu^o)$ *and* $\eta\in D(_\alpha H,\nu)$ *is a non-degenerate involutive algebra, therefore a von Neumann algebra on* H, *we shall denote* A_Ω.

(iv) We have $\alpha(N)\subset A_\Omega$, $\hat\beta(N)\subset A_\Omega$, *and* $A_\Omega\subset\beta(N)'$, $A_\Omega\subset\hat\alpha(N)'$.

(v) A unitary v *on* H *belongs to* A'_Ω *if and only if* $v\in\alpha(N)'\cap\hat\beta(N)'$ *and*

$$\widetilde{W}(1 \underset{N}{{}_\beta\otimes_\alpha} v) = (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} v)\widetilde{W}.$$

(vi) For any $x\in M$, *we have*

$$\Gamma_\Omega(x) = \widetilde{W}^*(1 \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} x)\widetilde{W}$$

and the weakly closed linear space generated by the operators of the form $(id*\omega_{\zeta_1,\zeta_2})(\widetilde{W})$, *for* $\zeta_1\in D(_\alpha H,\nu)$ *and* $\zeta_2\in D(H_{\hat\beta},\nu^o)$ *is equal to* M.

Proof. We have, using the definition of a 2-cocycle,

$$\begin{aligned}(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W})(\widetilde{W} \underset{N}{{}_\beta\otimes_\alpha} 1) &= (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} W)(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \Omega^*)(W \underset{N}{{}_\beta\otimes_\alpha} 1)(\Omega^* \underset{N}{{}_\beta\otimes_\alpha} 1)\\
&= (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} W)(W \underset{N}{{}_\beta\otimes_\alpha} 1)(W^* \underset{N}{{}_\beta\otimes_\alpha} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \Omega^*)(W \underset{N}{{}_\beta\otimes_\alpha} 1)(\Omega^* \underset{N}{{}_\beta\otimes_\alpha} 1)\\
&= (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} W)(W \underset{N}{{}_\beta\otimes_\alpha} 1)(\Gamma \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)(\Omega^* \underset{N}{{}_\beta\otimes_\alpha} 1)\\
&= (1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} W)(W \underset{N}{{}_\beta\otimes_\alpha} 1)(id \underset{N}{{}_\beta*_\alpha} \Gamma)(\Omega^*)(1 \underset{N}{{}_\beta\otimes_\alpha} \Omega^*)\end{aligned}$$

which is equal to

$$(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} W)(W \underset{N}{{}_\beta\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} W^*)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_\nu)(\Omega^* \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o})(1 \underset{N}{{}_\beta\otimes_\alpha} W)(1 \underset{N}{{}_\beta\otimes_\alpha} \Omega^*)$$

and, using the pentagonal equation for W, is equal to

$$(W \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)\sigma^{2,3}_{\alpha,\beta}(W\Omega^* \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o})(1 \underset{N}{{}_\beta\otimes_\alpha} W\Omega^*)$$

which is (i).

For ζ, ζ' in H, we get that

$$((\omega_{\xi,\xi'} \underset{N}{{}_\beta*_\alpha} id)[\widetilde{W}^*[(id*\omega_{\eta,\eta'})(\sigma_{\nu^o}W) \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)]\widetilde{W}]\zeta|\zeta')$$

is equal to

$$(\widetilde{W}^*[(id*\omega_{\eta,\eta'})(\sigma_{\nu^o}W) \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)]\widetilde{W}(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|\xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta') =$$

$$[(\omega_{\eta,\eta'}*id)(W\sigma_\nu) \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)]\widetilde{W}(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|\widetilde{W}(\xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta')) =$$

$$((W \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)(\sigma_\nu \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W}(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|\eta' \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W}(\xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta')) =$$

$$((1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W}^*)(W \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)(\sigma_\nu \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W})(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|\eta' \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta')$$

which, using (i), is equal to

$$((\widetilde{W} \underset{N}{{}_\beta\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \widetilde{W}^*)(\xi \underset{\nu}{{}_\beta\otimes_\alpha} (\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta)|\eta' \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta')$$
$$= (\xi \underset{\nu}{{}_\beta\otimes_\alpha} \widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta)|\widetilde{W}^*(\eta' \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi') \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \zeta').$$

Let $(f_i)_{i\in I}$ be an orthogonal (β, ν^o)-basis of H; there exist δ_i such that

$$\widetilde{W}^*(\eta' \underset{\nu}{{}_\alpha\otimes_{\hat\beta}} \xi') = \sum_i f_i \underset{\nu}{{}_\beta\otimes_\alpha} \delta_i$$

and, as in ([E3], 3.11), we can prove that $\sum_i \alpha(< f_i, \xi >_{\beta,\nu^o})\delta_i$ is equal to $(\omega_{\xi,\eta'} * id)(\widetilde{W})^*\xi'$, and therefore

$$(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta)|\widetilde{W}^*(\eta' \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi') \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \zeta')$$
$$= (\xi \underset{\nu}{{}_\beta\otimes_\alpha} \widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta)|\sum_i f_i \underset{\nu}{{}_\beta\otimes_\alpha} \delta_i \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \zeta')$$
$$= (\widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta)|\sum_i f_i \underset{\nu}{{}_\beta\otimes_\alpha} \delta_i \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \zeta')$$
$$= (\widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta)|(\omega_{\xi,\eta'} * id)(\widetilde{W})^*\xi' \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \zeta')$$

from which we get (ii).

We have

$$((\omega_{\xi,\eta} * id)(\widetilde{W})(\omega_{\xi',\eta'} * id)(\widetilde{W})\zeta|\zeta') = (\widetilde{W}(\xi \underset{\nu}{{}_\beta\otimes_\alpha} (\omega_{\xi',\eta'} * id)(\widetilde{W})\zeta)|\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta')$$
$$= (\xi \underset{\nu}{{}_\beta\otimes_\alpha} (\omega_{\xi',\eta'} * id)(\widetilde{W})\zeta|\widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta'))$$

which is equal to

$$((1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o}\widetilde{W})(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|\widetilde{W}^*(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta') \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} \eta')$$
$$= ((\widetilde{W} \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o}\widetilde{W})(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|(\eta \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta') \underset{N}{{}_\beta\otimes_\alpha} \eta')$$
$$= (\sigma^{2,3}_{\alpha,\beta}(\widetilde{W} \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)(1 \underset{N}{{}_\beta\otimes_\alpha} \sigma_{\nu^o})(1 \underset{N}{{}_\beta\otimes_\alpha} \widetilde{W})(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|(\eta \underset{\nu}{{}_\beta\otimes_\alpha} \eta') \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta')$$

and, using (i), is equal to

$$((1 \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \widetilde{W})(\widetilde{W} \underset{N}{{}_\beta\otimes_\alpha} 1)(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \xi' \underset{\nu}{{}_\beta\otimes_\alpha} \zeta)|(W(\eta \underset{\nu}{{}_\beta\otimes_\alpha} \eta') \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \zeta').$$

Let now $(e_i)_{i\in I}$ be an orthogonal (α, ν)-basis. As in ([E3], 3.4), we can prove that there exist a family $(\eta_i)_{i\in I}$ in $D(_\alpha H,\nu)$ such that

$$W(\eta \underset{\nu}{_\beta\otimes_\alpha} \eta') = \sum_i e_i \underset{\nu^o}{_\alpha\otimes_{\hat{\beta}}} \eta_i$$

and a family $(\xi_i)_{i\in I}$ in $D(H_\beta,\nu^o)$ such that

$$\widetilde{W}(\xi \underset{\nu}{_\beta\otimes_\alpha} \xi') = \sum_i e_i \underset{\nu^o}{_\alpha\otimes_{\hat{\beta}}} \xi_i$$

and we get that

$$((\omega_{\xi,\eta} * id)(\widetilde{W})(\omega_{\xi',\eta'} * id)(\widetilde{W})\zeta|\zeta') = (\sum_i(\omega_{\xi_i,\eta_i} * id)(\widetilde{W})\zeta|\zeta')$$

from which we get that $(\omega_{\xi,\eta} * id)(\widetilde{W})(\omega_{\xi',\eta'} * id)(\widetilde{W})$ is equal to the weak limit of the finite sums $\sum_1^n(\omega_{\xi_i,\eta_i} * id)(\widetilde{W})$. From which we get that the weakly closed linear set generated by the operators of the form $(\omega_{\xi,\eta} * id)(\widetilde{W})$ is an algebra A_Ω.

Let us now use (ii); the weak regularity of the pseudo-multiplicative unitary W ([E5], 3.8) means that $\alpha(N)'$ is the closed linear space generated by the operators $(id * \omega_{\eta,\eta'})(\sigma_{\nu^o}W)$, for all η, η' in $D(_\alpha H,\nu)$ ([E3], 4.1); in particular, there exists a family in the linear space generated by these operators which weakly converges to 1. Using then (ii), we get that, for any ξ, ξ' in $D(H_\beta,\nu^o)$, there exists a family in the linear space generated by the operators of the form $(\omega_{(\omega_{\xi,\eta'}*id)(\widetilde{W})^*\xi',\eta} * id)(\widetilde{W})$, for all η, η' in $D(_\alpha H,\nu)$, which is weakly converging to $\alpha(<\xi,\xi'>_{\beta,\nu^o})$; therefore, we get by density that $\alpha(N)$ is included in A_Ω, and therefore that 1 belongs to A_Ω.

So, there exists a family of operators of the form (with finite sums) $\sum_i(\omega_{\xi_i,\eta'_i} * id)(\widetilde{W})$ which is weakly converging to 1. Using now the intertwining properties of W and the definition of Ω, that $(\omega_{\xi,\eta'} * id)(\widetilde{W})^*$ commutes with $\beta(N)$, and we get that $R^{\beta,\nu^o}(\sum_i(\omega_{\xi_i,\eta'_i} * id)(\widetilde{W})^*\xi') = \sum_i(\omega_{\xi_i,\eta'_i} * id)(\widetilde{W})^*R^{\beta,\nu^o}(\xi')$ is converging to $R^{\beta,\nu^o}(\xi')$; finally, we get that A_Ω is the weakly closed linear set generated by all operators of the type $(\omega_{(\omega_{\xi,\eta'}*id)(\widetilde{W})^*\xi',\eta} * id)(\widetilde{W})$, for all ξ, ξ' in $D(H_\beta,\nu^o)$ and η, η' in $D(_\alpha H,\nu)$; using again (ii) and the weak regularity of W, we get that A_Ω is closed under the involution, which finishes the proof of (iii).

For any $n\in N$, we have, using ([E5], 3.2):

$$\begin{aligned}((\omega_{\xi,\eta} * id)(\widetilde{W})\hat{\beta}(n)\zeta_1|\zeta_2) &= (\widetilde{W}(\xi \underset{\nu}{_\beta\otimes_\alpha} \hat{\beta}(n)\zeta_1)|\eta \underset{\nu^o}{_\alpha\otimes_{\hat{\beta}}} \zeta_2)\\ &= (\widetilde{W}(\xi \underset{\nu}{_\beta\otimes_\alpha} \zeta_1)|\beta(n^*)\eta \underset{\nu^o}{_\alpha\otimes_{\hat{\beta}}} \zeta_2)\\ &= ((\omega_{\xi,\beta(n^*)\eta} * id)(\widetilde{W})\zeta_1|\zeta_2)\end{aligned}$$

from which we get that $(\omega_{\xi,\eta} * id)(\widetilde{W})\hat{\beta}(n) = (\omega_{\xi,\beta(n^*)\eta} * id)(\widetilde{W})$, which gives that $\hat{\beta}(n)$ belongs to A_Ω.

We have seen that $\beta(n)$ commutes with $(\omega_{\xi,\eta} * id)(\Omega W^*)$; using then ([E5], 3.11 (iii)), we get also that $\hat{\alpha}(n)$ commutes with $(\omega_{\xi,\eta} * id)(\Omega W^*)$, which finishes the proof of (iv).

Then, using (iii), the proof of (v) is clear.

It is clear that $(id * \omega_{\zeta_1,\zeta_2})(\widetilde{W})$ belongs to M. Let us denote by M_Ω the closed linear set generated by these operators. Using (i), we get, for $\zeta'_1 \in D(_\alpha H,\nu)$ and $\zeta'_2 \in D(H_{\hat{\beta}},\nu^o)$, that

$$(id * \omega_{\zeta_1',\zeta_2'})(W)(id * \omega_{\zeta_1,\zeta_2})(\widetilde{W}) = [id * \omega_{\widetilde{W}^*(\zeta_1' {}_\alpha\otimes_{\hat\beta}{}_\nu \zeta_1), \widetilde{W}^*(\zeta_2' {}_\alpha\otimes_{\hat\beta}{}_\nu \zeta_2)}](\widetilde{W} {}_\beta\otimes_\alpha{}_N 1)$$

which belongs to M_Ω. By linearity and weak closure, we get that M_Ω is a left ideal of M.

Moreover, the formula $\Gamma_\Omega(x) = \widetilde{W}^*(1 {}_\alpha\otimes_{\hat\beta}{}_{\nu^o} x)\widetilde{W}$ is clear by the definition of Γ_Ω (7.2) and $\widetilde{W}$. Using that, we get, for any $(\hat\beta, \nu^o)$-orthogonal basis $(e_i)_{i\in I}$ of H, and any $\eta \in D({}_\alpha H, \nu)$, by taking $x = 1$:

$$\beta(<\eta,\eta>_{\alpha,\nu}) = (id {}_\beta *_\alpha{}_N \omega_\eta)\Gamma_\Omega(1) = \sum_i (id * \omega_{\eta,e_i})(\widetilde{W})^*(id * \omega_{\eta,e_i})(\widetilde{W})$$

from which we get that $\beta(<\eta,\eta>_{\alpha,\nu})$ belongs to M_Ω; by density, we get that $\beta(N)$ belongs to M_Ω, and therefore that $1 \in M_\Omega$, which finishes the proof. ∎

7.4. Theorem. *Let $\mathfrak{G}$ be a measured quantum groupoid, and Ω be a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$, A_Ω the von Neumann algebra on H defined in 7.3; let us write $\widetilde{W} = W\Omega^*$, and, for any $x \in A_\Omega$, let us write*

$$\mathfrak{a}(x) = W^c(x {}_{\hat\alpha}\otimes_{\hat\beta}{}_{N^o} 1)(W^c)^*.$$

Then:

(i) For any $\xi \in D(H_\beta, \nu^o)$ and $\eta \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$, we have

$$\mathfrak{a}[(\omega_{\xi,\eta} * id)(\widetilde{W})] = (\omega_{\xi,\eta} * id * id)[(1 {}_\alpha\otimes_{\hat\beta}{}_{N^o} \sigma_{\nu^o})(W {}_\alpha\otimes_{\hat\beta}{}_{N^o} 1)\sigma^{2,3}_{\beta,\alpha}(\widetilde{W} {}_{\hat\beta}\otimes_\alpha{}_{N^o} 1)].$$

(ii) $(\hat\beta, \mathfrak{a})$ is an action of $\widehat{\mathfrak{G}}$ on A_Ω.

(iii) This action is integrable and Galois.

Proof. Using ([E5], 5.6), we get that W^c is a corepresentation of $\widehat{\mathfrak{G}}$ on ${}_{\hat\alpha}H_{\hat\beta}$. Therefore, the formula, for $y \in \hat\alpha(N)'$:

$$\mathfrak{a}(y) = W^c(y {}_{\hat\alpha}\otimes_{\hat\beta}{}_{N^o} 1)(W^c)^*$$

leads to an action $(\hat\beta, \mathfrak{a})$ of $\widehat{\mathfrak{G}}$ on $\hat\alpha(N)'$. Using ([E5], 3.12), we get, for any $n \in N$ that $\mathfrak{a}(\alpha(n)) = \widehat{\Gamma}(\alpha(n)) = \alpha(n) {}_{\hat\beta}\otimes_\alpha{}_N 1$, and $\mathfrak{a}(\hat\beta(n)) = \widehat{\Gamma}(\hat\beta(n)) = 1 {}_{\hat\beta}\otimes_\alpha{}_N \hat\beta(n)$.

For any orthogonal (β, ν^o)-basis $(e_i)_{i\in I}$ of H, we get

$$\begin{aligned}\mathfrak{a}((\omega_{\xi,\eta} * id)(\widetilde{W})) &= W^c((\omega_{\xi,\eta} * id)(\widetilde{W}) {}_{\hat\alpha}\otimes_{\hat\beta}{}_{N^o} 1)W^{c*}\\ &= \sum_i W^c[(\omega_{e_i,\eta} {}_\beta *_\alpha{}_N id)(W)(\omega_{\xi,e_i} * id)(\Omega^*) {}_{\hat\alpha}\otimes_\beta{}_{N^o} 1]W^{c*}\\ &= \sum_i \widehat{\Gamma}((\omega_{e_i,\eta} {}_\beta *_\alpha{}_N id)(W))[(\omega_{\xi,e_i} * id)(\Omega^*) {}_{\hat\beta}\otimes_\alpha{}_N 1].\end{aligned}$$

Applying 2.2.1 to $\widehat{\mathfrak{G}}$, we get that $\widehat{\Gamma}[(id * \omega_{\eta,e_i})(\sigma W^*\sigma)]$ is equal to

$$(id {}_{\hat\beta}*_\alpha{}_N id * \omega_{\eta,e_i})[\sigma^{2,3}_{\alpha,\hat\beta}(\sigma W^*\sigma {}_\beta\otimes_\alpha{}_N 1)(1 {}_{\hat\beta}\otimes_\alpha{}_N \sigma_{\nu^o})(1 {}_{\hat\beta}\otimes_\alpha{}_N \sigma W^*\sigma)]$$

from which we get that

$$\widehat{\Gamma}[(\omega_{\eta,e_i} * id)(W^*)] = (\omega_{\eta,e_i} * id \underset{N}{_{\hat\beta}*_\alpha} id)[(W^* \underset{N^o}{_\alpha\otimes_{\hat\beta}} 1)\sigma^{2,3}_{\hat\beta,\alpha}(W^* \underset{N^o}{_\alpha\otimes_{\hat\beta}} 1)(1 \underset{N^o}{_\alpha\otimes_{\hat\beta}} \sigma_\nu)]$$

and, for any orthogonal (α,ν)-basis $(f_j)_{j\in J}$ of H, it is equal to

$$\sum_j ((\omega_{f_j,e_i} * id)(W^*) \underset{N}{_\beta\otimes_\alpha} 1)(\omega_{\eta,f_j} * id * id)[\sigma^{2,3}_{\hat\beta,\alpha}(W^* \underset{N}{_\alpha\otimes_{\hat\beta}} 1)(1 \underset{N^o}{_\alpha\otimes_{\hat\beta}} \sigma_\nu)]$$

and therefore $\mathfrak{a}((\omega_{\xi,\eta} * id)(\widetilde{W}))$ is equal to

$$\sum_{i,j}[(\omega_{f_j,\eta} * id * id)[(1 \underset{N^o}{_\alpha\otimes_{\hat\beta}} \sigma_{\nu^o})(W \underset{N^o}{_\alpha\otimes_{\hat\beta}} 1)\sigma^{2,3}_{\beta,\alpha}]\dots$$
$$\dots(\omega_{e_i,f_j} * id)(W) \underset{N}{_{\hat\beta}\otimes_\alpha} 1)(\omega_{\xi,e_i} \underset{N}{_\beta*_\alpha} id)(\Omega^*) \underset{N}{_{\hat\beta}\otimes_\alpha} 1]$$

which is equal to

$$\sum_j[(\omega_{f_j,\eta} * id * id)[(1 \underset{N^o}{_\alpha\otimes_{\hat\beta}} \sigma_{\nu^o})(W \underset{N^o}{_\alpha\otimes_{\hat\beta}} 1)\sigma^{2,3}_{\beta,\alpha}](\omega_{\xi,f_j} * id)(\widetilde{W}) \underset{N}{_{\hat\beta}\otimes_\alpha} 1]$$

from which we get (i).

For any δ_1, δ_2 in $D(H_{\hat\beta},\nu^o)$, we get that

$$\begin{aligned}(\omega_{\eta,f_j} * \omega_{\delta_1,\delta_2} * id)&[\sigma^{2,3}_{\hat\beta,\alpha}(W^* \underset{N}{_\alpha\otimes_{\hat\beta}} 1)(1 \underset{N^o}{_\alpha\otimes_{\hat\beta}} \sigma_\nu)] \\ &= (\omega_{\eta,f_j} * id)[(\alpha(< \delta_1,\delta_2 >_{\hat\beta,\nu^o} \underset{N}{_\beta\otimes_\alpha} 1)W^*] \\ &= (\omega_{\eta,f_j} * id)(W^*)\alpha(< \delta_1,\delta_2 >_{\hat\beta,\nu^o})\end{aligned}$$

as $\hat\beta(N) \subset A_\Omega$, any unitary $u \in A'_\Omega$ commutes with $\hat\beta(N)$, and we have

$$< u\delta_1, u\delta_2 >_{\hat\beta,\nu^o} = < \delta_1,\delta_2 >_{\hat\beta,\nu^o}$$

from which we get that $(\omega_{\eta,f_j} * id * id)[\sigma^{2,3}_{\hat\beta,\alpha}(W^* \underset{N}{_\alpha\otimes_{\hat\beta}} 1)(1 \underset{N^o}{_\alpha\otimes_{\hat\beta}} \sigma_\nu)]$ commutes with $A'_\Omega \underset{N}{_{\hat\beta}\otimes_\alpha} 1$, and therefore belongs to $A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \mathcal{L}(H)$, and more precisely to $A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \hat{M}$.

Then, using 7.3, we easily get that $\mathfrak{a}((\omega_{\xi,\eta} * id)(\widetilde{W}))$ belongs to $A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \hat{M}$; using again 7.3, we get $\mathfrak{a}(A_\Omega) \subset A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \hat{M}$, which gives (ii).

Using ([E5], 11.2), we know that the von Neumann algebra $A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \mathcal{L}(H)$ is isomorphic to the double crossed product $(A \rtimes_{\mathfrak{a}} \widehat{\mathfrak{G}}) \rtimes_{\tilde{\mathfrak{a}}} \mathfrak{G}^c$ and that this isomorphism sends the bidual action to the action $\underline{\mathfrak{a}}$ defined, for any $X \in A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \mathcal{L}(H)$ by

$$\underline{\mathfrak{a}}(X) = (1 \underset{N}{_{\hat\beta}\otimes_\alpha} W^*)(id \underset{N}{_{\hat\beta}*_\alpha} \varsigma_N)(\mathfrak{a} \underset{N}{_{\hat\beta}*_\alpha} id)(X)(1 \underset{N}{_{\hat\beta}\otimes_\alpha} W)$$

Let us define $\mathcal{I}(X) = \varsigma_N(\widetilde{W}^*)X\varsigma_N(\widetilde{W})$; then $\mathcal{I}$ is an isomorphism from $A_\Omega \underset{N}{_{\hat\beta}*_\alpha} \mathcal{L}(H)$ onto

$A_\Omega \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H)$, and the above calculations show that

$$(\mathcal{J} \underset{N}{{}_{\hat\beta} *_\alpha} id)\underline{\mathfrak{a}}(X) = (id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)(\mathfrak{I}(X))$$

from which we get that $(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)$ is an integrable and Galois action of $\widehat{\mathfrak{G}}$ on $A_\Omega \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H)$. As $(A_\Omega \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H))^{(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)} = A_\Omega^{\mathfrak{a}} \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H)$, we easily get that $T_{(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)} = T_{\mathfrak{a}} \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H)$; as $T_{(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)}$ is semi-finite, we get that $T_{\mathfrak{a}}$ is also semi-finite, and $\mathfrak{a}$ is integrable; moreover, as $(A_\Omega \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H)) \rtimes_{(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)} \widehat{\mathfrak{G}} = (id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)[(A \rtimes_{\mathfrak{a}} \widehat{\mathfrak{G}}) \underset{N^o}{{}_\alpha *_\beta} \mathcal{L}(H)]$, we get that $\pi_{(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)} = (\pi_{\mathfrak{a}} \underset{N^o}{{}_\alpha *_\beta} id)(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)$. As $\pi_{(id \underset{N}{{}_{\hat\beta} *_\alpha} \varsigma_N)(\mathfrak{a} \underset{N^o}{{}_\alpha *_\beta} id)}$ is injective, we easily get that $\pi_{\mathfrak{a}}$ is injective also, which is (iii). ■

7.5. Proposition. *Let $\mathfrak{G}$ be a measured quantum groupoid, Ω a 2-cocycle for $\mathfrak{G}$, let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$, A_Ω the von Neumann algebra on H defined in 7.3 and $(\hat\beta, \mathfrak{a})$ the action of $\widehat{\mathfrak{G}}$ on A_Ω defined in 2.4; let us write $\widetilde{W} = W\Omega^*$. Then, for $\xi \in D(H_\beta, \nu^o)$ and $\eta \in D({}_\alpha H, \nu)$ such that $\omega_{\xi,\eta}$ belongs to I_Φ, in the sense of ([E3], 4.1), which implies ([E3], 4.6) that $(\omega_{\xi,\eta} * id)(W)$ belongs to $\mathfrak{N}_{\hat\Phi}$, we have:*

(i) Let $\mathcal{P}_\eta$ be the element of the positive extension of M' defined in ([E3], 4.1); then, $R^{\beta,\nu^o}(\xi)^*\mathcal{P}_\eta R^{\beta,\nu^o}(\xi)$ belongs to the positive extension of N, and we have

$$T_{\mathfrak{a}}[((\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))] = \alpha(R^{\beta,\nu^o}(\xi)^*\mathcal{P}_\eta R^{\beta,\nu^o}(\xi)).$$

(ii) $A_\Omega^{\mathfrak{a}} = \alpha(N)$.

(iii) Let us write $\psi_1 = \nu \circ \alpha^{-1} \circ T_{\mathfrak{a}}$; then ψ_1 is a normal semi-finite faithful weight on A_Ω, $\hat\delta$-invariant with respect to $\mathfrak{a}$, satisfying the density condition, and we have

$$\psi_1[((\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))] = \hat\Phi[(\omega_{\xi,\eta} * id)(W)^*(\omega_{\xi,\eta} * id)(W)].$$

For all $n \in N$ and $t \in \mathbb{R}$, we have $\sigma_t^{\psi_1}(\alpha(n)) = \alpha(\sigma_t^\nu(n))$ and $\sigma_t^{\psi_1}(\hat\beta(n)) = \hat\beta(\gamma_{-t}(n))$.

(iv) There exists a unitary u from H onto H_{ψ_1} such that

$$u\Lambda_{\hat\Phi}((\omega_{\xi,\eta} * id)(W)) = \Lambda_{\psi_1}((\omega_{\xi,\eta} * id)(\widetilde{W}))$$

and we have, for all $n \in N$:

$$u\alpha(n) = \pi_{\psi_1}(\alpha(n))u,$$

$$u\hat\beta(n) = \pi_{\psi_1}(\hat\beta(n))u,$$

$$u\beta(n) = J_{\psi_1}\pi_{\psi_1}(\alpha(n^*))J_{\psi_1}u,$$

$$u\hat\alpha(n) = J_{\psi_1}\pi_{\psi_1}(\hat\beta(n^*))J_{\psi_1}u.$$

(v) The normal faithful semi-finite weight $\nu \circ \alpha^{-1}$ on $\alpha(N) = A_\Omega^{\mathfrak{a}}$ satisfies the Galois density condition defined in 4.1 for the Galois action $(\hat\beta, \mathfrak{a})$ of $\widehat{\mathfrak{G}}$ on A_Ω.

(vi) The operator $(u \underset{N}{{}_{\hat\beta} \otimes_\alpha} 1)W^c(u^* \underset{N^o}{{}_{\hat\alpha} \otimes_{\hat\beta}} 1)$ is the standard implementation V_{ψ_1} of the action $(\hat\beta, \mathfrak{a})$ associated to the weight ψ_1 on A_Ω.

(vii) For any $x \in A_\Omega$, we have $\pi_{\psi_1}(x) = uxu^*$.

(viii) If N is a finite sum of factors, then there exists a normal semi-finite faithful weight ϕ on A_Ω such that $(A_\Omega, \hat{\beta}, \mathfrak{a}, \phi, \nu)$ is a Galois system.

Proof. Using the calculations made in 2.4, with an orthogonal (α, ν)-basis $(f_j)_{j\in J}$ of H, we get that $\mathfrak{a}[((\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))]$ is equal to

$$\sum_{j,j'}((\omega_{\xi,f_{j'}} * id)(\widetilde{W})^* \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} 1)(\omega_{\eta,f_{j'}} * id * id)[\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})] \dots$$
$$\dots(\omega_{f_j,\eta} * id * id)([\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})]^*)((\omega_{\xi,f_j} * id)(\widetilde{W}) \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} 1)$$

and therefore $T_{\mathfrak{a}}[((\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))]$ is equal to

$$\sum_{j,j'}((\omega_{\xi,f_{j'}} * id)(\widetilde{W})^*) \dots$$
$$\dots(id \underset{N}{{}_{\hat{\beta}}*_\alpha} \hat{\Phi})[(\omega_{\eta,f_{j'}} * id * id)[\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})] \dots$$
$$\dots(\omega_{f_j,\eta} * id * id)([\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})]^*)] \dots$$
$$\dots((\omega_{\xi,f_j} * id)(\widetilde{W}) \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} 1).$$

Let δ be in $D(H_{\hat{\beta}}, \nu^o)$, and let $(\delta_i)_{i\in I}$ be an orthogonal $(\hat{\beta}, \nu^o)$-basis of H, we get that

$$(\omega_\delta \underset{N}{{}_{\hat{\beta}}*_\alpha} \hat{\Phi})[(\omega_{\eta,f_{j'}} * id * id)[\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})] \dots$$
$$\dots(\omega_{f_j,\eta} * id * id)([\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})]^*)]$$

is equal to

$$\sum_i \hat{\Phi}(\omega_{\eta,f_{j'}} * \omega_{\delta_i,\delta} * id)[\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})] \dots$$
$$\dots(\omega_{f_j,\eta} * \omega_{\delta,\delta_i} * id)([\sigma^{2,3}_{\hat{\beta},\alpha}(W^* \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})]^*)$$

which, using the calculations made in 2.4, is equal to

$$\sum_i \hat{\Phi}[(\omega_{\eta,f_{j'}} * id)(W^*)\alpha(< \delta_i, \delta >_{\hat{\beta},\nu^o})\alpha(< \delta, \delta_i >_{\hat{\beta},\nu^o})(\omega_{f_j,\eta} * id)(W)]$$
$$= \hat{\Phi}[(\omega_{\eta,f_{j'}} * id)(W^*)\alpha(< \delta, \delta >_{\hat{\beta},\nu^o})(\omega_{f_j,\eta} * id)(W)].$$

Therefore, if f_j and $f_{j'}$ are in $\mathcal{D}(\pi'(\eta))$, we finally get that it is equal to

$$(\alpha(< \delta, \delta >_{\hat{\beta},\nu^o})\Lambda_{\hat{\Phi}}[(\omega_{f_j,\eta} * id)(W)]|\Lambda_{\hat{\Phi}}[(\omega_{f_{j'},\eta} * id)(W)])$$
$$= (\alpha(< \delta, \delta >_{\hat{\beta},\nu^o})\pi'(\eta)^* f_j|\pi'(\eta)f^*_{j'})$$
$$= (\delta \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} \pi'(\eta)^* f_j|\delta \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} \pi'(\eta)^* f_{j'})$$
$$= (\pi'(\eta)^* f_j \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} \delta|\pi'(\eta)^* f_{j'} \underset{\nu^o}{{}_\alpha\otimes_{\hat{\beta}}} \delta).$$

Let now $\delta'_k \in D(H_{\hat\beta}, \nu^o)$ be an orthogonal basis of H; then

$$< T_{\mathfrak{a}}[((\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))]), \omega_\delta >$$

is equal to

$$\sum_{k,j,j'} (\tilde W^*(f'_j \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \delta'_k))|\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta)(\pi'(\eta)^* f_j \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \delta'_k|\pi'(\eta)^* f_{j'} \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \delta'_k) \dots$$
$$\dots (\tilde W(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta)|f_{j'} \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \delta'_k).$$

As the family $(f_j {}_\alpha \underset{N^o}{\otimes_{\hat\beta}} \delta'_k)_{(k,j)}$ is an orthogonal basis of $H \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} H$, we can use the Plancherel formula, which gives that

$$< T_{\mathfrak{a}}[((\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))]), \omega_\delta >= (\tilde W(\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta)|\tilde W(\mathcal{P}_\eta\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta))$$
$$= (\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta|(\mathcal{P}_\eta\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta) = \|\pi'(\eta)^*\xi \underset{\nu}{{}_\beta\otimes_\alpha} \delta\|^2$$

from which one gets (i).

We have seen in 2.4 that $\alpha(N) \subset A_\Omega^{\mathfrak{a}}$; on the other side, if $x \in \mathfrak{M}^+_{T_{\mathfrak{a}}}$, using 7.3, one gets that x is the upper limit of an increasing positive sum of elements of the form $(\omega_{\xi,\eta}*id)(\widetilde{W}))^*((\omega_{\xi,\eta}*id)(\widetilde{W}))$; therefore, $T_{\mathfrak{a}}(x)$ is, by (i), the upper limit of an increasing sequence of elements in $\alpha(N)$, and therefore we get that $T_{\mathfrak{a}}(x) \in \alpha(N)$; as $T_{\mathfrak{a}}(\mathfrak{M}_{T_{\mathfrak{a}}})$ is dense in A_Ω, we get (ii). Thanks to (ii), one can define the lifted weight $\psi_1 = \nu\circ\alpha^{-1}\circ T_{\mathfrak{a}}$, which is δ-invariant with respect to $\mathfrak{a}$ by 3.2. Moreover, using (i), one gets that

$$\psi_1[(\omega_{\xi,\eta} * id)(\widetilde{W}))^*((\omega_{\xi,\eta} * id)(\widetilde{W}))] = \hat\Phi[(\omega_{\xi,\eta} * id)(W))^*(\omega_{\xi,\eta} * id)(W))]$$

which gives the first formula of (iii); the formula $\sigma_t^{\psi_1}(\alpha(n)) = \alpha(\sigma_t^\nu(n))$ is clear by definition of ψ_1; as ψ_1 is δ-invariant, using ([E5], 8,8) and 2.4, we get that $\sigma_t^{\psi_1}(\hat\beta(n)) = \sigma_t^{\hat\Phi}(\hat\beta(n)) = \hat\beta(\gamma_{-t}(n))$, which finishes the proof of (iii).

Using (iii), we get the existence of an isometry u from H into H_{ψ_1} such that

$$u\Lambda_{\hat\Phi}((\omega_{\xi,\eta} * id)(W)) = \Lambda_{\psi_1}((\omega_{\xi,\eta} * id)(\widetilde{W})).$$

Let us write P for the projection on Imu; using 7.3, we get that $\pi_{\psi_1}(A_\Omega)P = P\pi_{\psi_1}(A_\Omega)P$, and therefore that $P \in \pi_{\psi_1}(A_\Omega)' = J_{\psi_1}\pi_{\psi_1}(A_\Omega)J_{\psi_1}$. Using Kaplansky's theorem, one can find a family ω_n in I_Φ, such that $\|(\omega_n * id)(\widetilde{W})\| \le 1$ and $J_{\psi_1}\pi_{\psi_1}[(\omega_n * id)(\widetilde{W})]J_{\psi_1}$ is weakly converging to $1-P$; then, we get

$$\pi_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})]J_{\psi_1}\Lambda_{\psi_1}[(\omega_n * id)(\widetilde{W})] = J_{\psi_1}\pi_{\psi_1}[(\omega_n * id)(\widetilde{W})]J_{\psi_1}\Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})]$$

is converging to 0, because $J_{\psi_1}\pi_{\psi_1}[(\omega_n * id)(\widetilde{W})]J_{\psi_1}$ is weakly converging to $1-P$; using now the weak density of the linear combinations of elements of the form $\pi_{\psi_1}[(\omega_{\xi,\eta}*id)(\widetilde{W})]$ in $\pi_{\psi_1}(A_\Omega)$, we get that $\Lambda_{\psi_1}[(\omega_n * id)(\widetilde{W})]$ is converging to 0; from which one gets that $\psi_1(J_{\psi_1}(1-P)J_{\psi_1}) = 0$ and that $P = 1$, which proves that u is a unitary.

Moreover, we have

$$\begin{aligned}
u\alpha(n)\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) &= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)((1 \underset{N^o}{_\alpha\otimes_{\hat{\beta}}} \alpha(n))W)] \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)(W(\alpha(n) \underset{N}{_\beta\otimes_\alpha} 1))] \\
&= u\Lambda_{\hat{\Phi}}((\omega_{\alpha(n)\xi,\eta} * id)(W)) = \Lambda_{\psi_1}((\omega_{\alpha(n)\xi,\eta} * id)(\widetilde{W})) \\
&= \Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W}(\alpha(n) \underset{N}{_\beta\otimes_\alpha} 1))] \\
&= \Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)((1 \underset{N^o}{_\alpha\otimes_{\hat{\beta}}} \alpha(n))\widetilde{W})] \\
&= \pi_{\psi_1}(\alpha(n))\Lambda_{\psi_1}((\omega_{\xi,\eta} * id)(\widetilde{W})).
\end{aligned}$$

Let us suppose now that $n \in N$ is analytic with respect to ν and let's use (ii). Then:

$$\begin{aligned}
u\hat{\beta}(n)\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) &= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)((1 \underset{N^o}{_\alpha\otimes_{\hat{\beta}}} \hat{\beta}(n))W)] \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)((\alpha(\sigma^\nu_{i/2}(n)) \underset{N^o}{_\alpha\otimes_{\hat{\beta}}} 1)W)] \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\alpha(\sigma_{-i/2}(n^*))\eta} * id)(W)] \\
&= \Lambda_{\psi_1}[(\omega_{\xi,\alpha(\sigma_{-i/2}(n^*))\eta} * id)(\widetilde{W})] \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)((\alpha(\sigma^\nu_{i/2}(n)) \underset{N^o}{_\alpha\otimes_{\hat{\beta}}} 1)\widetilde{W})] \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)((1 \underset{N^o}{_\alpha\otimes_{\hat{\beta}}} \hat{\beta}(n))\widetilde{W})] \\
&= \pi_{\psi_1}(\hat{\beta}(n))u\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W))
\end{aligned}$$

and

$$\begin{aligned}
u\beta(n)\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) &= uJ_{\hat{\Phi}}\alpha(n^*)J_{\hat{\Phi}}\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)(W)(1 \underset{N}{_\beta\otimes_\alpha} \alpha(\sigma_{-i/2}(n))] \\
&= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)(W)(\beta(n) \underset{N}{_\beta\otimes_\alpha} 1)] \\
&= u\Lambda_{\hat{\Phi}}[\omega_{\beta(n)\xi,\eta} * id)(W)] \\
&= \Lambda_{\psi_1}[\omega_{\beta(n)\xi,\eta} * id)(\widetilde{W})] \\
&= \Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})(\beta(n) \underset{N}{_\beta\otimes_\alpha} 1)] \\
&= \Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})(1 \underset{N}{_\beta\otimes_\alpha} \alpha(\sigma_{-i/2}(n))] \\
&= J_{\psi_1}\alpha(n^*)J_{\psi_1}\Lambda_{\psi_1}((\omega_{\xi,\eta} * id)(\widetilde{W})) \\
&= J_{\psi_1}\alpha(n^*)J_{\psi_1}u\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)).
\end{aligned}$$

If now we suppose that n is analytic with respect to γ, and use again (ii), we shall get

$$\begin{aligned} u\hat{\alpha}(n)\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) &= uJ_{\hat{\Phi}}\hat{\beta}(n^*)J_{\hat{\Phi}}\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) \\ &= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)(W)(1 \underset{N}{{}_\beta\otimes_\alpha} \hat{\beta}(\gamma_{i/2}(n))] \\ &= u\Lambda_{\hat{\Phi}}[(\omega_{\xi,\eta} * id)((\beta(\gamma_{i/2}(n) \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)W))] \\ &= u\Lambda_{\hat{\Phi}}[(\omega_{\beta(\gamma i/2(n))^*\xi,\eta} * id)(W))] \\ &= \Lambda_{\psi_1}[(\omega_{\beta(\gamma i/2(n))^*\xi,\eta} * id)(\widetilde{W}))] \\ &= \Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)((\beta(\gamma_{i/2}(n) \underset{N}{{}_\alpha\otimes_{\hat{\beta}}} 1)\widetilde{W}))] \\ &= \Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})(1 \underset{N}{{}_\beta\otimes_\alpha} \hat{\beta}(\gamma_{i/2}(n))] \\ &= J_{\psi_1}\hat{\beta}(n^*)J_{\psi_1}\Lambda_{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})] \\ &= J_{\psi_1}\hat{\beta}(n^*)J_{\psi_1}u\Lambda_{\hat{\Phi}}((\omega_{\xi,\eta} * id)(W)) \end{aligned}$$

which, by continuity, finishes the proof of (iv).

The weight $\nu \circ \alpha^{-1}$ satisfies the density condition if the subspace

$$D((H_{\psi_1})_{\pi_{\psi_1}\circ\hat{\beta}}, \nu^o) \cap D({}_{\pi_{\psi_1}\circ\alpha}H_{\psi_1}, \nu)$$

is dense in H_{ψ_1}. Using now (iv), we get that this subspace is the image by u of $D(H_{\hat{\beta}}, \nu^o) \cap D({}_\alpha H, \nu)$ which is dense in H_Φ by ([E4], 2.3), from which we get (v), using (iv) again.

In ([E5], 8.2), one gets that $\widehat{W^o} = \widehat{W}^c$ is the standard implementation of the action (β, Γ) of $\mathfrak{G}$ on M, associated to the δ-invariant weight Φ. So, W^c is the standard implementation of the action $(\hat{\beta}, \widehat{\Gamma})$ on $\widehat{M}$, associated to the $\hat{\delta}$-invariant weight $\widehat{\Phi}$. Which means that, for any orthogonal (α, ν)-basis of H, any ζ in $D({}_\alpha H, \nu) \cap \mathcal{D}(\hat{\delta}^{1/2})$ such that $\hat{\delta}^{1/2}$ belongs to $D(H_{\hat{\beta}}, \nu^o)$, any x in $\mathfrak{N}_{\widehat{\Phi}}$, we have

$$W^c(\Lambda_{\widehat{\Phi}}(x) \underset{\nu^o}{{}_{\hat{\alpha}}\otimes_{\hat{\beta}}} \hat{\delta}^{1/2}\zeta) = \sum_i \Lambda_{\widehat{\Phi}}[(id \underset{N}{{}_{\hat{\beta}}*_\alpha} \omega_{\zeta,e_i})\widehat{\Gamma}(x)] \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} e_i$$

and therefore in particular:

$$\begin{aligned} &W^c(\Lambda_{\widehat{\Phi}}(\omega_{\xi,\eta} * id)(W)) \underset{\nu^o}{{}_{\hat{\alpha}}\otimes_{\hat{\beta}}} \hat{\delta}^{1/2}\zeta) \\ &\qquad = \sum_i \Lambda_{\widehat{\Phi}}[(id \underset{N}{{}_{\hat{\beta}}*_\alpha} \omega_{\zeta,e_i})\widehat{\Gamma}((\omega_{\xi,\eta} * id)(W))] \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} e_i \\ &= \sum_i \Lambda_{\widehat{\Phi}}(\omega_{\xi,\eta} * id * \omega_{\zeta,e_i})[(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})(W \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)\sigma^{2,3}_{\beta,\alpha}(W \underset{N^o}{{}_{\hat{\beta}}\otimes_\alpha} 1)] \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} e_i. \end{aligned}$$

Using now (iv), we then get that $(u \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} 1)W^c(\Lambda_{\widehat{\Phi}}(\omega_{\xi,\eta} * id)(W)) \underset{\nu^o}{{}_{\hat{\alpha}}\otimes_{\hat{\beta}}} \hat{\delta}^{1/2}\zeta)$ is equal to

$$\sum_i \Lambda_{\psi_1}(\omega_{\xi,\eta} * id * \omega_{\zeta,e_i})[(1 \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} \sigma_{\nu^o})(W \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)\sigma^{2,3}_{\beta,\alpha}(\widetilde{W} \underset{N^o}{{}_{\hat{\beta}}\otimes_\alpha} 1)] \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} e_i$$

which, thanks to 2.4(i), is equal to

$$\sum_i \Lambda_{\psi_1}[(id \underset{N}{{}_{\hat{\beta}}*_\alpha} \omega_{\zeta,e_i})\mathfrak{a}((\omega_{\xi,\eta} * id)(\widetilde{W})) \underset{\nu}{{}_{\hat{\beta}}\otimes_\alpha} e_i$$

and therefore we have, where we denote by V_{ψ_1} the standard implementation of $(\hat{\beta}, \mathfrak{a})$ associated to the weight ψ_1:

$$(u \underset{N}{{}_{\hat\beta}\otimes_\alpha} 1)W^c(u^* \underset{N^o}{{}_{\hat\alpha}\otimes_{\hat\beta}} 1)(\Lambda_{\psi_1}(\omega_{\xi,\eta} * id)(\widetilde{W})) \underset{\nu^o}{{}_{\hat\alpha}\otimes_{\hat\beta}} \hat{\delta}^{1/2}\zeta)$$
$$= \sum_i \Lambda_{\psi_1}[(id \underset{N}{{}_{\hat\beta}*_\alpha} \omega_{\zeta,e_i})\mathfrak{a}((\omega_{\xi,\eta} * id)(\widetilde{W})) \underset{\nu}{{}_{\hat\beta}\otimes_\alpha} e_i$$
$$= V_{\psi_1}(\Lambda_{\psi_1}(\omega_{\xi,\eta} * id)(\widetilde{W})) \underset{\nu^o}{{}_{\hat\alpha}\otimes_{\hat\beta}} \hat{\delta}^{1/2}\zeta)$$

from which we get (vi), by density.

Let $\xi' \in D(H_\beta, \nu^o)$, $\eta' \in D({}_\alpha H, \nu)$, we get, with an orthogonal (β, ν^o)-basis $(e_i)_{i\in I}$ of H, that $(\omega_{\xi',\eta'} * id)(\widetilde{W})\Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta} * id)(W)]$ is equal to

$$\sum_i (\omega_{e_i,\eta'} * id)(W)(\omega_{\xi',e_i} \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)\pi'(\eta)^*\xi$$
$$= \sum_i (\omega_{e_i,\eta'} * id)(W)\pi'(\eta)^*(\omega_{\xi',e_i} \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)\xi$$
$$= \sum_i (\omega_{e_i,\eta'} * id)(W)\Lambda_{\widehat{\Phi}}(\omega_{(\omega_{\xi',e_i} \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)\xi,\eta} * id)(W)$$
$$= \sum_i \Lambda_{\widehat{\Phi}}[(\omega_{e_i,\eta'} * id)(W)(\omega_{(\omega_{\xi',e_i} \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)\xi,\eta} * id)(W)].$$

Let now $(f_j)_{j\in J}$ be an orthogonal (α, ν)-basis of H; we know that there exist $\xi_{i,j}$, η_j in H such that

$$W(e_i \underset{\nu}{{}_\beta\otimes_\alpha} (\omega_{\xi',e_i} \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)\xi) = \sum_j f_j \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi_{i,j},$$

$$W(\eta' \underset{\nu}{{}_\beta\otimes_\alpha} \eta) = \sum_j f_j \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \eta_j,$$

and then we get that

$$(\omega_{\xi',\eta'} * id)(\widetilde{W})\Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta} * id)(W)] = \sum_{i,j} \Lambda_{\widehat{\Phi}}[(\omega_{\xi_{i,j},\eta_j} * id)(W)]$$

which implies that

$$\sum_j f_j \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \sum_i \xi_{i,j} = W \sum_i (e_i \underset{\nu}{{}_\beta\otimes_\alpha} (\omega_{\xi',e_i} \underset{N}{{}_\beta*_\alpha} id)(\Omega^*)\xi)$$
$$= W\Omega^*(\xi' \underset{\nu}{{}_\beta\otimes_\alpha} \xi)$$

and finally

$$(\omega_{\xi',\eta'} * id)(\widetilde{W})\Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta} * id)(W)] = \sum_j \Lambda_{\widehat{\Phi}}[(\omega_{\xi'_j,\eta_j} * id)(W)]$$

where $\widetilde{W}(\xi' \underset{\nu}{{}_\beta\otimes_\alpha} \xi) = \sum_j f_j \underset{\nu^o}{{}_\alpha\otimes_{\hat\beta}} \xi'_j$. But, using again the calculation already made in

7.3(iii), we get that

$$\begin{aligned}u(\omega_{\xi',\eta'}*id)(\widetilde{W})\Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta}*id)(W) &= \sum_j \Lambda_{\psi_1}[(\omega_{\xi'_j,\eta_j}*id)(W)]\\ &= \Lambda_{\psi_1}[(\omega_{\xi',\eta'}*id)(\widetilde{W})(\omega_{\xi,\eta}*id)(\widetilde{W})]\\ &= \pi_{\psi_1}((\omega_{\xi',\eta'}*id)(\widetilde{W}))\Lambda_{\psi_1}[(\omega_{\xi,\eta}*id)(\widetilde{W})]\\ &= \pi_{\psi_1}((\omega_{\xi',\eta'}*id)(\widetilde{W}))u\Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta}*id)(W)\end{aligned}$$

from which, by density, we get (vii).

Moreover, (viii) is a direct application of 4.7 to (ii), which finishes the proof. ■

7.6. Theorem. *Let $\mathfrak{G}$ be a measured quantum groupoid, Ω a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$, A_Ω the von Neumann algebra on H defined in 7.3 and $(\hat{\beta},\mathfrak{a})$ the Galois action of $\widehat{\mathfrak{G}}$ on A_Ω defined in 2.4 whose invariant subalgebra $A_\Omega^{\mathfrak{a}}$ is equal to $\alpha(N)$ (7.5(ii)); let us write $\widetilde{W}=W\Omega^*$; moreover, the weight $\nu\circ\alpha^{-1}$ on $\alpha(N)$ has the Galois density property defined in 4.1, by 7.5(vi). Let us write $\psi_1=\nu\circ\alpha^{-1}\circ T_{\mathfrak{a}}$. Let u be the unitary from H onto H_{ψ_1} introduced in 7.5(iv).*

The canonical representation r of $A_\Omega^{\mathfrak{a}}$ on H_{ψ_1} is the restriction of π_{ψ_1} to $\alpha(N)$; using 7.5(iv), we get that the canonical antirepresentation s of $\alpha(N)$ (identified to N for simplification) on H_{ψ_1} is $s(n)=u\beta(n)u^*$ $(n\in N)$; for simplification again, we shall write α for $\pi_{\psi_1}\circ\alpha$ and $\hat{\beta}$ for $\pi_{\psi_1}\circ\hat{\beta}$. Then the Galois unitary $\tilde{G}$ is a unitary from $H_{\psi_1}\ {}_s\!\underset{\nu}{\otimes}_\alpha H_{\psi_1}$ onto $H\ {}_\alpha\!\underset{\nu^o}{\otimes}_{\hat{\beta}} H_{\psi_1}$. Then

$$(1\ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}}\ u^*)\widetilde{G}(u\ {}_\beta\!\underset{N}{\otimes}_\alpha\ u)=\widetilde{W}.$$

Proof. Let $\xi\in D({}_\alpha H,\nu)$, $\eta\in D(H_{\hat{\beta}},\nu^o)$; let $\xi'\in D(H_\beta,\nu^o)$ and $\eta'\in D({}_\alpha H,\nu)$ such that $\omega_{\xi',\eta'}$ belongs to I_Φ (in the sense of [E3], 4.1), which implies that $(\omega_{\xi',\eta'}*id)(W)$ belongs to $\mathfrak{N}_{\widehat{\Phi}}$, and using 7.5(iii), that $(\omega_{\xi',\eta'}*id)(\widetilde{W})$ belongs to $\mathfrak{N}_{\psi_1}$. We have then, using 7.5(iv), and 4.2(i):

$$\begin{aligned}(id*\omega_{\xi,\eta}\circ\pi_{\psi_1})(\tilde{G})u\Lambda_{\widehat{\Phi}}[(\omega_{\xi',\eta'}*id)(W)] &= (id*\omega_{\xi,\eta}\circ\pi_{\psi_1})(\tilde{G})\Lambda_{\psi_1}[(\omega_{\xi',\eta'}*id)(\tilde{W})]\\ &= \Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta}\ {}_{\hat{\beta}}\!\underset{N}{*}_\alpha\ id)\mathfrak{a}((\omega_{\xi',\eta'}*id)(\widetilde{W}))].\end{aligned}$$

Using now 2.4(i), we get that $(\omega_{\xi,\eta}\ {}_{\hat{\beta}}\!\underset{N}{*}_\alpha\ id)\mathfrak{a}((\omega_{\xi',\eta'}*id)(\widetilde{W}))$ is equal to

$$(\omega_{\xi',\eta'}*\omega_{\xi,\eta}*id)[(1\ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}}\ \sigma_{\nu^o})(W\ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}}\ 1)\sigma^{2,3}_{\beta,\alpha}(\widetilde{W}\ {}_{\hat{\beta}}\!\underset{N^o}{\otimes}_\alpha\ 1)].$$

Let now $(\xi_i)_{i\in I}$ be an orthogonal (α,ν)-basis of H; we get then that this last expression is equal to $\sum_i(\omega_{\xi',\eta'}*\omega_{\xi,\xi_i}*\omega_{\xi_i,\eta}*id)(W_{1,4}W_{1,3}\Omega^*_{1,2})$, where we use the leg numbering notation, for simplification. But we get then that it is equal to

$$\sum_i(\omega_{(id_\beta \underset{N}{*}_\alpha \omega_{\xi,\xi_i})(\Omega^*)\xi',\eta'}*\omega_{\xi_i,\eta}*id)[(1\ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}}\ \sigma_{\nu^o})(W\ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}}\ 1)\sigma^{2,3}_{\beta,\alpha}(W\ {}_{\hat{\beta}}\!\underset{N^o}{\otimes}_\alpha\ 1)]$$

which is $\sum_i(\omega_{\xi_i,\eta}\ {}_{\hat{\beta}}\!\underset{N}{*}_\alpha\ id)\hat{\Gamma}(\omega_{(id_\beta \underset{N}{*}_\alpha \omega_{\xi,\xi_i})(\Omega^*)\xi',\eta'}*id)(W))$. For any $i\in I$, the operator $(\omega_{\xi_i,\eta}\ {}_{\hat{\beta}}\!\underset{N}{*}_\alpha\ id)\hat{\Gamma}(\omega_{(id_\beta \underset{N}{*}_\alpha \omega_{\xi,\xi_i})(\Omega^*)\xi',\eta'}*id)(W)$ belongs to $\mathfrak{N}_{\widehat{\Phi}}$, and, by [E3], 3.10 (ii) applied to $\widehat{\mathfrak{G}}$, then, [E3], 4.6 and 4.1, we get that

$$\Lambda_{\widehat{\Phi}}[(\omega_{\xi_i,\eta}\ {}_{\hat{\beta}}*_{\alpha}\ id)\hat{\Gamma}(\omega_{(id_{\beta}*_{\alpha}\omega_{\xi,\xi_i})(\Omega^*)\xi',\eta'} * id)(W)]$$

$$= (\omega_{\xi_i,\eta} * id)(\widehat{W}^*)(id\ {}_{\beta}*_{\alpha}\ \omega_{\xi,\xi_i})(\Omega^*)\Lambda_{\widehat{\Phi}}(\omega_{\xi',\eta'} * id)(W))$$

$$= (id * \omega_{\xi_i,\eta})(W)(id\ {}_{\beta}*_{\alpha}\ \omega_{\xi,\xi_i})(\Omega^*)\Lambda_{\widehat{\Phi}}(\omega_{\xi',\eta'} * id)(W)$$

whose sum is weakly converging to $(id * \omega_{\xi,\eta})(\widetilde{W})\Lambda_{\widehat{\Phi}}[(\omega_{\xi',\eta'} * id)(W)]$. As the map $\Lambda_{\widehat{\Phi}}$ is closed, we get that

$$\Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta}\ {}_{\hat{\beta}}*_{\alpha}\ id)\mathfrak{a}((\omega_{\xi',\eta'} * id)(\widetilde{W}))] = (id * \omega_{\xi,\eta})(\widetilde{W})\Lambda_{\widehat{\Phi}}(\omega_{\xi',\eta'} * id)(W)$$

from which we deduce that

$$(id * \omega_{\xi,\eta} \circ \pi_{\psi_1})(\tilde{G}) = (id * \omega_{\xi,\eta})(\widetilde{W})$$

which gives the result, thanks to 7.5(viii). ■

7.7. Corollaries. *Let $\mathfrak{G}$ be a measured quantum groupoid, Ω a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$, A_Ω the von Neumann algebra on H defined in 7.3 and $(\hat{\beta}, \mathfrak{a})$ the Galois action of $\widehat{\mathfrak{G}}$ on A_Ω defined in 2.4 whose invariant subalgebra $A_\Omega^{\mathfrak{a}}$ is equal to $\alpha(N)$ (7.5(ii)); let us write $\widetilde{W} = W\Omega^*$; moreover, the weight $\nu \circ \alpha^{-1}$ on $\alpha(N)$ has the Galois density property defined in 4.1, by 7.5(vi). Let us write $\psi_1 = \nu \circ \alpha^{-1} \circ T_{\mathfrak{a}}$. Let u be the unitary from H onto H_{ψ_1} introduced in 7.5(iii).*

The canonical representation r of $A_\Omega^{\mathfrak{a}}$ on H_{ψ_1} is the restriction of π_{ψ_1} to $\alpha(N)$; using 7.5(iv), we get that the canonical antirepresentation s of $\alpha(N)$ (identified to N for simplification) on H_{ψ_1} is $s(n) = u\beta(n)u^*$ ($n \in N$); let ρ_t be the one-parameter group of automorphisms of $s(N)'$ and K^{it} its standard implementation defined in 4.4; for simplification again, we shall write α for $\pi_{\psi_1} \circ \alpha$ and $\hat{\beta}$ for $\pi_{\psi_1} \circ \hat{\beta}$. Then:

(i) For any $x \in A_\Omega$, we have

$$\mathfrak{a}(x) = \sigma_{\nu^o}\widetilde{W}\sigma_{\nu^o}(1\ {}_{\alpha}\underset{N}{\otimes}{}_{\beta}\ x)\sigma_\nu\widetilde{W}^*\sigma_\nu.$$

(ii) For any $y \in M'$, we have $\pi_{\mathfrak{a}}(1\ {}_{\hat{\beta}}\underset{N}{\otimes}{}_{\alpha}\ y) = uyu^*$.

(iii) For all $t \in \mathbb{R}$, we have

$$K^{it} = (u\ {}_{\beta}\underset{N}{\otimes}{}_{\alpha}\ u)\Omega(\hat{J}\hat{\delta}^{it}\hat{J}\ {}_{\beta}\underset{N}{\otimes}{}_{\alpha}\ \hat{\delta}^{it})\Omega^*(u^*\ {}_{\beta}\underset{N}{\otimes}{}_{\alpha}\ u^*).$$

(iv) For all $t \in \mathbb{R}$, we have $P_{A_\Omega}^{it} = \Delta_{\psi_1}^{it} uJ\delta^{it}Ju^*$.

(v) We have, for all $t \in \mathbb{R}$:

$$\widetilde{W}(u^*P_{A_\Omega}^{it}u\ {}_{\beta}\underset{N}{\otimes}{}_{\alpha}\ uP_{A_\Omega}^{it}u^*) = (P^{it}\ {}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\ uP_{A_\Omega}^{it}u^*)\widetilde{W}.$$

(vi) For any $\xi \in D(H_\beta, \nu^o)$ and $\eta \in D({}_\alpha H, \nu)$, we have

$$\tau_t^{A_\Omega}[(\omega_{\xi,\eta} * id)(\widetilde{W})] = (\omega_{u^*P_{A_\Omega}^{it}u\xi, P^{-it}\eta} * id)(\widetilde{W}) = (\omega_{u^*\Delta_{\psi_1}^{it}u\xi, \Delta_{\widehat{\Phi}}^{-it}\eta} * id)(\widetilde{W}).$$

(vii) For any $x \in \mathfrak{N}_{\psi_1} \cap \mathfrak{N}_{\psi_1}^*$, y, z in $\mathfrak{N}_{\widehat{\Phi}} \cap \mathfrak{N}_{\hat{T}}$, we have

$$(\omega_{u^*\Lambda_{\psi_1}(x),\hat{J}\Lambda_{\widehat{\Phi}}(y^*z)} * id)(\widetilde{W})^* = (\omega_{u^*\Lambda_{\psi_1}(x^*),\hat{J}\Lambda_{\widehat{\Phi}}(z^*y)} * id)(\widetilde{W}).$$

(viii) For any ζ_1, ζ_2 in $D(_\alpha H, \nu) \cap D(H_{\hat{\beta}}, \nu^o)$, $\xi \in \mathcal{D}(\Delta_{\psi_1}^{1/2})$, $\eta \in \mathcal{D}(\Delta_{\widehat{\Phi}}^{-1/2})$, we have

$$((id * \omega_{\zeta_1,\zeta_2})(\widetilde{W})u^*\xi|\eta) = ((id * \omega_{\zeta_2,\zeta_1})(\widetilde{W})^*\hat{J}\Delta_{\widehat{\Phi}}^{-1/2}\eta|u^*J_{\psi_1}\Delta_{\psi_1}^{1/2}\xi).$$

(ix) The operator $\Delta_{\widehat{\Phi}}^{1/2}(id * \omega_{\zeta_2,\zeta_1})(\widetilde{W})u^*\Delta_{\psi_1}^{-1/2}u$ is bounded, and we have

$$(id * \omega_{\zeta_1,\zeta_2})(\widetilde{W}) = \hat{J}\Delta_{\widehat{\Phi}}^{1/2}(id * \omega_{\zeta_2,\zeta_1})(\widetilde{W})u^*\Delta_{\psi_1}^{-1/2}uu^*J_{\psi_1}u.$$

(x) For any $\xi \in D(H_\beta, \nu^o) \cap \mathcal{D}(u\Delta_{\psi_1}^{1/2}u^*)$, and $\eta \in D(_\alpha H, \nu) \cap \mathcal{D}(\Delta_{\widehat{\Phi}}^{-1/2})$, we have

$$(\omega_{\xi,\eta} * id)(\widetilde{W})^* = (\omega_{u^*J_{\psi_1}\Delta_{\psi_1}^{1/2}u\xi,\hat{J}\Delta_{\widehat{\Phi}}^{-1/2}\eta} * id)(\widetilde{W}).$$

(xi) For all $t \in \mathbb{R}$, we have

$$\widetilde{W}(u^*\Delta_{\psi_1}^{it}u \underset{N}{{}_\beta\otimes_\alpha} u^*\Delta_{\psi_1}^{it}u) = [(\delta\Delta_{\widehat{\Phi}})^{it} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} u^*\Delta_{\psi_1}^{it}u]\widetilde{W}.$$

(xii) For all $t \in \mathbb{R}$, we have

$$\sigma_t^{\psi_1}[(\omega_{\xi,\eta} * id)(\widetilde{W})] = (\omega_{u^*\Delta_{\psi_1}^{it}u\xi,(\delta\Delta_{\widehat{\Phi}})^{-it}\eta} * id)(\widetilde{W}).$$

(xiii) If N is a finite sum of factors, there exists a normal semi-finite faithful operator weight T_Ω from M to $\alpha(N)$, (resp. T'_Ω from M to $\beta(N)$) such that

$$\mathfrak{G}_\Omega = (N, M, \alpha, \beta, \Gamma_\Omega, T_\Omega, T'_\Omega, \nu)$$

is a measured quantum groupoid.

Proof. Result (i) is just the application of 7.6 to 3.8(iv).

Let us apply 3.8(v) to the action $(\hat{\beta}, \mathfrak{a})$ of $\widehat{\mathfrak{G}}$ on A_Ω. We get that

$$\begin{aligned}\pi_{\mathfrak{a}}(1 \underset{N}{{}_{\hat{\beta}}\otimes_\alpha} y) \underset{N}{{}_s\otimes_\alpha} 1 &= \tilde{G}^*(y \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)\tilde{G} \\ &= (u \underset{N}{{}_\beta\otimes_\alpha} 1)\widetilde{W}^*(y \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)\widetilde{W}(u^* \underset{N}{{}_\beta\otimes_\alpha} 1) = uyu^* \underset{N}{{}_s\otimes_\alpha} 1\end{aligned}$$

from which we get (ii).

Applying now 7.6 to 4.4, and successively [E5], 3.11(iii) and [E5], 3.8 (vi) applied to $\widehat{\mathfrak{G}}$, one gets

$$K^{it} = \widetilde{G}^*(\hat{J}\hat{\delta}^{it}\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)\widetilde{G} = (u \underset{N}{{}_\beta\otimes_\alpha} u)\Omega W^*(\hat{J}\hat{\delta}^{it}\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)W\Omega^*(u^* \underset{N}{{}_\beta\otimes_\alpha} u^*)$$

which is equal to

$$(u \underset{N}{{}_\beta\otimes_\alpha} u)\Omega(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} J)W(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} J)(\hat{J}\hat{\delta}^{it}\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat{\beta}}} 1)\dots(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} J)W^*(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} J)\Omega^*(u^* \underset{N}{{}_\beta\otimes_\alpha} u^*)$$

and to

$$\begin{aligned}
&(u \underset{N}{{}_\beta\otimes_\alpha} u)\Omega(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} J)W(\hat{\delta}^{it} \underset{N}{{}_\beta\otimes_\alpha} 1)W^*(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} J)\Omega^*(u^* \underset{N}{{}_\beta\otimes_\alpha} u^*)\\
&= (u \underset{N}{{}_\beta\otimes_\alpha} u)\Omega(\hat{J} \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} J)(\hat{\delta}^{it} \underset{N^o}{{}_\alpha\otimes_{\hat\beta}} \hat{\delta}^{it})(\hat{J} \underset{N}{{}_\beta\otimes_\alpha} J)\Omega^*(u^* \underset{N}{{}_\beta\otimes_\alpha} u^*)\\
&= (u \underset{N}{{}_\beta\otimes_\alpha} u)\Omega(\hat{J}\hat{\delta}^{it}\hat{J} \underset{N}{{}_\beta\otimes_\alpha} \hat{\delta}^{it})\Omega^*(u^* \underset{N}{{}_\beta\otimes_\alpha} u^*)
\end{aligned}$$

which is (iii).

Applying (ii) to 4.8(ii), one gets (iv). Applying again (i) to 4.8(vi), one gets (v). Then, the first equality of (vi) is a direct corollary of (v), and the second equality is a corollary of (iv) and [E6], 3.10(vii). Result (vii) is a direct corollary of 4.2(iii) applied to 7.6. Then (ix) is an easy corollary from (viii), and (x) from (ix). Result (xi) is given by 3.8(vii) and ([E6], 3.11(ii)), applied to 7.6, and (xii) is a direct corollary of (xi).

If N is a finite sum of factors, we can apply 2.4(iv), and therefore we obtain, by 5.11, a measured quantum groupoid $\mathfrak{G}_1(\mathfrak{a})$, whose underlying Hopf bimodule has been defined in 5.4(iii). Using now (ii), we get that the von Neumann algebra is (up to u) equal to M; using 7.5(ii), we get that the basis is (up to α) equal to N, and, by 7.5(iv), that the imbedding of N into M are α and β. Using now 7.6, we get that the coproduct is Γ_Ω, as defined in 7.2, which finishes the proof. ∎

7.8. Proposition. *Let $\mathfrak{G}$ be a measured quantum groupoid, Ω a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$, A_Ω the von Neumann algebra on H defined in 7.3 and $(\hat\beta, \mathfrak{a})$ the action of $\widehat{\mathfrak{G}}$ on A_Ω defined in 2.4 whose invariant subalgebra $A_\Omega^{\mathfrak{a}}$ is equal to $\alpha(N)$ (7.5(ii)); let us write $\widetilde{W} = W\Omega^*$; moreover, the weight $\nu\circ\alpha^{-1}$ on $\alpha(N)$ has the Galois density property defined in 4.1, by 7.5(v). Let us write $\psi_1 = \nu\circ\alpha^{-1}\circ T_{\mathfrak{a}}$. Let u be the unitary from H onto H_{ψ_1} introduced in 7.5(iv), and let us write, for all $t\in\mathbb{R}$:*

$$v_t^\Omega = u^*\Delta_{\psi_1}^{it} u \Delta_{\widehat{\Phi}}^{-it}.$$

For all $t\in\mathbb{R}$, let us consider the 2-cocycle Ω_t introduced in 7.1, the algebra A_{Ω_t} associated, the action $(\hat\beta, \mathfrak{a}_t)$ of $\widehat{\mathfrak{G}}$ on A_{Ω_t}, whose invariant is also equal to $\alpha(N)$. Let us denote $\psi_{1,t}$ the weight $\nu\circ\alpha^{-1}\circ T_{\mathfrak{a}_t}$, and u_t the canonical unitary from H to $H_{\psi_{1,t}}$, which, by 7.5(vii) applied to Ω_t, implements $\pi_{\psi_{1,t}}$. Let us write $\widetilde{W_t} = W\Omega_t^*$. Then:

(i) v_t^Ω is a unitary in $M\cap\alpha(N)'\cap\beta(N)'$; moreover, the mapping $t\mapsto v_t^\Omega$ is a τ_t-cocycle.

(ii) We have

$$\widetilde{W}(v_t^\Omega \underset{N}{{}_\beta\otimes_\alpha} v_t^\Omega) = (1 \underset{N}{{}_\alpha\otimes_{\hat\beta}} v_t^\Omega)\widetilde{W_t}, \qquad \Gamma(v_t^\Omega)\Omega_t^* = \Omega^*(v_t^\Omega \underset{N}{{}_\beta\otimes_\alpha} v_t^\Omega).$$

(iii) The map $\mathfrak{I}_t : x\mapsto v_t^\Omega x (v_t^\Omega)^*$ is an isomorphism from A_{Ω_t} to A_Ω, and we have, for all $\xi\in D(H_\beta,\nu^o)$ and $\eta\in D({}_\alpha H,\nu)$:

$$\mathfrak{I}_t[(\omega_{\xi,\eta} * id)(\widetilde{W_t})] = (\omega_{v_t^\Omega\xi,\eta} * id)(\widetilde{W}).$$

(iv) We have $\psi_1\circ\mathfrak{I}_t = \psi_{1,t}$; then $uv_t^\Omega u_t^*$ is the standard implementation of $\mathfrak{I}_t$.

(v) We have $v_s^{\Omega_t} = \tau_t(v_s^\Omega)$.

(vi) If, for all $t\in\mathbb{R}$, we have $\Omega=\Omega_t$, then there exists a positive non-singular operator k_Ω affiliated to M, such that $\tau_t(k_\Omega)=k_\Omega$ and $v_t^\Omega=k_\Omega^{it}$.

Proof. By definition of ψ_1, we get that, for all $t\in\mathbb{R}$ and $n\in N$, we have $\sigma_t^{\psi_1}(\alpha(n))=\alpha(\sigma_t^\nu(n))=\sigma_t^{\widehat{\Phi}}(\alpha(n))$; therefore, we get that $v_t^\Omega\in\alpha(N)'$.

We have, using first [E5], 3.10 (iv) and 3.8(i), then 7.5(vii), [E5], 3.8(ii) and 3.10(vii):

$$\begin{aligned}u^*\Delta_{\psi_1}^{it}u\beta(n)u^*\Delta_{\psi_1}^{-it}u&=u^*\Delta_{\psi_1}^{it}u\hat{J}\alpha(n^*)\hat{J}u^*\Delta_{\psi_1}^{-it}u\\&=u^*\Delta_{\psi_1}^{it}uu^*J_{\psi_1}uX_\Omega\alpha(n^*)X_\Omega^*u^*J_{\psi_1}uu^*\Delta_{\psi_1}^{-it}u\\&=u^*J_{\psi_1}uu^*\Delta_{\psi_1}^{it}\alpha(n^*)u^*\Delta_{\psi_1}^{-it}uu^*J_{\psi_1}u=X_\Omega\hat{J}\alpha(\sigma_t^\nu(n^*))\hat{J}X_\Omega^*\\&=X_\Omega\beta(\sigma_t^\nu(n))X_\Omega^*=\beta(\sigma_t^\nu(n))=\tau_t(\beta(n))=\Delta_{\widehat{\Phi}}^{it}\beta(n)\Delta_{\widehat{\Phi}}^{-it}\end{aligned}$$

from which we get that $v_t^\Omega\in\beta(N)'$.

Using 7.5(vi), and [E5], 8.8(ii), we get that

$$W^c(u^*\Delta_{\psi_1}^{it}u\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}\hat\delta^{-it}P^{-it})=(u^*\Delta_{\psi_1}^{it}u\underset{N}{{}_{\hat\beta}\otimes_\alpha}\hat\delta^{-it}P^{-it})W^c$$

and, as we have also, applying [E5], 8.8(ii) to the weight $\widehat{\Phi}$:

$$W^c(\Delta_{\widehat{\Phi}}^{it}\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}\hat\delta^{-it}P^{-it})=(\Delta_{\widehat{\Phi}}^{it}\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}\hat\delta^{-it}P^{-it})W^c$$

we get that

$$W^c(v_t^\Omega\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}1)=(v_t^\Omega\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}1)W^c$$

from which one gets that v_t^Ω belongs to M, using [E5], 3.10(ii), applied to $\widehat{\mathfrak{G}^c}$.

We have, for s, t in $\mathbb{R}$:

$$v_{s+t}^\Omega=u^*\Delta_{\psi_1}^{i(s+t)}u\Delta_{\widehat{\Phi}}^{-i(s+t)}=u\Delta_{\psi_1}^{is}u\Delta_{\widehat{\Phi}}^{-is}\Delta_{\widehat{\Phi}}^{is}u\Delta_{\psi_1}^{it}\Delta_{\widehat{\Phi}}^{-it}\Delta_{\widehat{\Phi}}^{-is}=v_s^\Omega\tau_s(v_t^\Omega)$$

which finishes the proof of (i).

Using now 7.7(iii) and [E5], 3.10(vii), we get that $u^*P_{A_\Omega}^{it}u=v_t^\Omega P^{it}$; therefore, 7.7(iv) can be written

$$\widetilde{W}(v_t^\Omega\underset{N}{{}_\beta\otimes_\alpha}v_t^\Omega)(P^{it}\underset{N}{{}_\beta\otimes_\alpha}P^{it})=(P^{it}\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}v_t^\Omega P^{it})\widetilde{W}$$

or, using [E5], 3.8(vii):

$$\widetilde{W}(v_t^\Omega\underset{N}{{}_\beta\otimes_\alpha}v_t^\Omega)=(1\underset{N^o}{{}_\alpha\otimes_{\hat\beta}}v_t^\Omega)W(\tau_t\underset{N}{{}_\beta*_\alpha}\tau_t)(\Omega^*)$$

from which we get the first formula of (ii). The second formula of (ii) is just a straightforward corollary of the first formula. We then get that

$$(\omega_{v_t^\Omega\xi,\eta}*id)(\widetilde{W})=v_t^\Omega(\omega_{\xi,\eta}*id)(\widetilde{W}_t)(v_t^\Omega)*$$

from which we get (iii). Using now the definitions of $\mathfrak{a}$ and $\mathfrak{a}_t$, we get that $(\mathfrak{I}_t\underset{N}{{}_{\hat\beta}*_\alpha}id)\mathfrak{a}_t=\mathfrak{a}\circ\mathfrak{I}_t$, $T_{\mathfrak{a}_t}=T_{\mathfrak{a}}\circ\mathfrak{I}_t$ and $\psi_{1,t}=\psi_1\circ\mathfrak{I}_t$.

If we suppose now that $\omega_{\xi,\eta}$ belongs to I_Φ, we get

$$\begin{aligned}\Lambda_{\psi_1}(\mathfrak{I}_t[\omega_{\xi,\eta} * id)(\widetilde{W_t})]) &= \Lambda_{\psi_1}[(\omega_{v_t^\Omega\xi,\eta} * id)(\widetilde{W})] = u\Lambda_{\widehat{\Phi}}(\omega_{v_t^\Omega\xi,\eta} * id)(W)]\\ &= u\pi'(\eta)^* v_t^\Omega \xi = uv_t^\Omega \pi'(\eta)^*\xi = uv_t^\Omega \Lambda_{\widehat{\Phi}}[(\omega_{\xi,\eta} * id)(W)]\\ &= uv_t^\Omega u_t^* \Lambda_{\psi_{1,t}}[(\omega_{\xi,\eta} * id)(\widetilde{W_t})]\end{aligned}$$

which finishes the proof of (iv).

Using (iv), we get $uv_t^\Omega u_t^* \Delta_{\psi_{1,t}}^{is} = \Delta_{\psi_1}^{is} uv_t^\Omega u_t^*$, from which we infer

$$v_t^\Omega v_s^{\Omega_t} = v_t^\Omega u_t^* \Delta_{\psi_{1,t}}^{is} u_t \Delta_{\widehat{\Phi}}^{-is} = u^* \Delta_{\psi_1}^{is} u \Delta_{\widehat{\Phi}}^{-is} \Delta_{\widehat{\Phi}}^{is} v_t^\Omega \Delta_{\widehat{\Phi}}^{-is} = v_s^\Omega \tau_s(v_t^\Omega) = v_{s+t}^\Omega = v_t^\Omega \tau_t(v_s^\Omega)$$

from which we get (v).

Using (v), we get that, if $\Omega = \Omega_t$, v_t^Ω is invariant under τ_s, and is a one-parameter group of unitaries, which finishes the proof. ■

7.9. Theorem. *Let $\mathfrak{G}$ be a measured quantum groupoid, Ω a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$, A_Ω the von Neumann algebra on H defined in 7.3 and $(\hat{\beta}, \mathfrak{a})$ the action of $\widehat{\mathfrak{G}}$ on A_Ω defined in 2.4 whose invariant subalgebra $A_\Omega^{\mathfrak{a}}$ is equal to $\alpha(N)$ (7.5(ii)); let us write $\widetilde{W} = W\Omega^*$; moreover, the weight $\nu\circ\alpha^{-1}$ on $\alpha(N)$ has the Galois density property defined in 4.1, by 7.5(vi). Let us write $\psi_1 = \nu\circ\alpha^{-1}\circ T_{\mathfrak{a}}$. Let u be the unitary from H onto H_{ψ_1} introduced in 7.5(iii). Then the following are equivalent:*

(i) There exists a normal semi-finite faithful weight ϕ on A_Ω such that $(A_\Omega, \hat{\beta}, \mathfrak{a}, \phi, \nu)$ is a Galois system.

(ii) There exists a one-parameter group of unitaries δ_Ω^{it} on H, such that it is possible to define a one-parameter group of unitaries $uJ_{\psi_1}u^*\delta_\Omega^{it}uJ_{\psi_1}u^* \ {}_\beta\!\underset{N}{\otimes}_\alpha \delta_\Omega^{it}$, with natural values on elementary tensors, and such that

$$uJ_{\psi_1}u^*\delta_\Omega^{it}uJ_{\psi_1}u^* \ {}_\beta\!\underset{N}{\otimes}_\alpha \delta_\Omega^{it} = \Omega(\hat{J}\hat{\delta}^{it}\hat{J} \ {}_\beta\!\underset{N}{\otimes}_\alpha \hat{\delta}^{it})\Omega^*.$$

(iii) There exists a $\tau_{-s}\sigma_{-s}^{\Phi\circ R}$-cocycle $t \mapsto u_t^\Omega$ in $M\cap\beta(N)'$, such that

$$\Gamma(u_t^\Omega) = \Omega^*(u_t^\Omega \ {}_\beta\!\underset{N}{\otimes}_\alpha 1)(\tau_{-t}\sigma_{-t}^{\Phi\circ R} \ {}_\beta\!\underset{N}{*}_\alpha id)(\Omega)$$

and u_t^Ω is linked with the τ_s-cocycle v_t^Ω introduced in 7.8 by the formula, for all s, t in $\mathbb{R}$:

$$u_t^\Omega \tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega) = v_s^\Omega \tau_s(u_t^\Omega).$$

In that situation, $u^*\delta_\Omega u$ is the modulus of the action $(\hat{\beta}, \mathfrak{a})$, and we have $\delta_\Omega^{it} = u_t^\Omega\hat{\delta}^{it}$. Moreover, there exists then a normal semi-finite faithful operator weight T_Ω from M to $\alpha(N)$ (resp. T'_Ω from M to $\beta(N)$) such that

$$\mathfrak{G}_\Omega = (N, M, \alpha, \beta, \Gamma_\Omega, T_\Omega, T'_\Omega, \nu)$$

is a measured quantum groupoid.

Moreover, if N is a finite sum of factors, then any 2-cocycle satisfies these equivalent conditions.

Proof. The equivalence between (i) and (ii) is just an application of 4.10, thanks to 7.7(iii). We then get that $u^*\delta_\Omega u$ is the modulus of the action $(\hat{\beta}, \mathfrak{a})$ of $\widehat{\mathfrak{G}}$ on A_Ω, and using 7.5(vii), we get that δ_Ω is affiliated to A_Ω, and that there exists a one-parameter group of unitaries $\delta_\Omega^{it} {}_{\hat\beta}\otimes_\alpha{}_N \delta^{it}$ such that, for all $t\in\mathbb{R}$, we have $\mathfrak{a}(\delta_\Omega^{it}) = \delta_\Omega^{it} {}_{\hat\beta}\otimes_\alpha{}_N \hat\delta^{it}$. Let us write $u_t^\Omega = \delta_\Omega^{it}\hat\delta^{-it}$. Using 7.3(iv), we get that $u_t \in \beta(N)'$; moreover, using 2.4, and [E5], 3.12(v) and 3.8(vi), applied to $\widehat{\mathfrak{G}}$, we get that $W^c(u_t^\Omega {}_{\hat\alpha}\otimes_{\hat\beta}{}_{N^o} 1)(W^c)^* = u_t^\Omega {}_\beta\otimes_\alpha{}_N 1$, which gives that u_t^Ω belongs to M, thanks to [E5], 3.10(ii) applied to $\widehat{\mathfrak{G}}^c$. As, for any $x\in M$, and $t\in\mathbb{R}$, we have, using [E5], 3.11(ii), $\hat\delta^{it}x\hat\delta^{-it} = \tau_{-t}\sigma_{-t}^{\Phi\circ R}(x)$, we get that $t\mapsto u_t^\Omega$ is indeed a $\tau_{-s}\sigma_{-s}^{\Phi\circ R}$-cocycle.

Using now 7.7(i), we get that $\hat\delta^{it} {}_\alpha\otimes_{\hat\beta}{}_{N^o} \delta_\Omega^{it} = \widetilde{W}(\delta_\Omega^{it} {}_\beta\otimes_\alpha{}_N 1)\widetilde{W}^*$. And therefore

$$\begin{aligned}
1 {}_\alpha\otimes_\beta{}_{N^o} u_t^\Omega &= (\hat\delta^{it} {}_\alpha\otimes_{\hat\beta}{}_{N^o} \delta_\Omega^{it})(\hat\delta^{-it} {}_\alpha\otimes_{\hat\beta}{}_{N^o} \hat\delta^{-it}) = \widetilde{W}(\delta_\Omega^{it} {}_\beta\otimes_\alpha{}_N 1)\widetilde{W}^*(\hat\delta^{-it} {}_\alpha\otimes_{\hat\beta}{}_{N^o} \hat\delta^{-it})\\
&= \widetilde{W}(\delta_\Omega^{it} {}_\beta\otimes_\alpha{}_N 1)\Omega W^*(\hat\delta^{-it} {}_\alpha\otimes_{\hat\beta}{}_{N^o} \hat\delta^{-it}) = \widetilde{W}(\delta_\Omega^{it} {}_\beta\otimes_\alpha{}_N 1)\Omega(\hat\delta^{-it} {}_\beta\otimes_\alpha{}_N 1)W^*\\
&= \widetilde{W}(u_t^\Omega {}_\beta\otimes_\alpha{}_N 1)(\tau_{-t}\sigma_{-t}^{\Phi\circ R} {}_\beta *_\alpha{}_N id)(\Omega)W^*
\end{aligned}$$

and therefore

$$\Gamma(u_t^\Omega) = W^*(1 {}_\alpha\otimes_\beta{}_{N^o} u_t^\Omega)W = \Omega^*(u_t^\Omega {}_\beta\otimes_\alpha{}_N 1)(\tau_{-t}\sigma_{-t}^{\Phi\circ R} {}_\beta *_\alpha{}_N id)(\Omega)$$

which gives the first formula of (iii).

Moreover, using 4.9(iii), we get that $\sigma_t^{\psi_1}(\delta_\Omega^{it}) = \lambda^{ist}\delta_\Omega^{it}$. Using 7.8, and [E5], 3.8(vi) applied to $\widehat{\mathfrak{G}}$, we have

$$\begin{aligned}
\sigma_t^{\psi_1}(\delta_\Omega^{it}) &= v_s^\omega\widehat{\Delta}^{is}u_t^\Omega\hat\delta^{it}\widehat{\Delta}^{-is}(v_s^\Omega)^* = v_s^\Omega\tau_s(u_t^\Omega)\sigma_s^{\widehat\Phi}(\hat\delta^{it})(v_s^\Omega)^*\\
&= v_s^\Omega\tau_s(u_t^\Omega)\lambda^{ist}\hat\delta^{it}(v_s^\Omega)^* = \lambda^{ist}v_s^\Omega\tau_s(u_t^\Omega)\hat\delta^{it}(v_s^\Omega)^*
\end{aligned}$$

from which we get $u_t^\Omega\hat\delta^{it} = v_s^\Omega\tau_s(u_t^\Omega)\hat\delta^{it}(v_s^\Omega)^*$, which gives the the second formula of (iii).

Conversely, if we have (iii), we can define a one-parameter group of unitaries δ_Ω^{it} by writing $\delta_\Omega^{it} = u_t^\Omega\hat\delta^{it}$. Now, from the first formula of (iii), taking the same calculation upside down, we get that $\hat\delta^{it} {}_\alpha\otimes_{\hat\beta}{}_{N^o} \delta_\Omega^{it} = \widetilde{W}(\delta_\Omega^{it} {}_\beta\otimes_\alpha{}_N 1)\widetilde{W}^*$, which gives, by 7.3(iii), that δ_Ω is affiliated to A_Ω; so we have obtained that $\mathfrak{a}(\delta_\Omega^{it}) = \delta_\Omega^{it} {}_{\hat\beta}\otimes_\alpha{}_N \hat\delta^{it}$.

From the second formula of (iii), using again the same calculation upside down, we get that $\sigma_t^{\psi_1}(\delta_\Omega^{it}) = \lambda^{ist}\delta_\Omega^{it}$, which proves that λ is affiliated to A_Ω; by the definition of δ_Ω, we see that the operators δ_Ω and λ strongly commute. Therefore, by [V1], 5.1, there exists a normal semi-finite faithful weight ϕ on A_Ω such that $(D\phi : D\psi_1)_t = \lambda^{it^2/2}\delta_\Omega^{it}$.

Using now 7.5(iv) and [E5], 8.1, we get, for all $x\in\mathfrak{N}_\phi$ such that $x\delta_\Omega^{1/2}$ is bounded (its closure, denoted $\overline{x\delta_\Omega^{1/2}}$ belongs then to $\mathfrak{N}_{\psi_1}$, and we identify $\Lambda_\phi(x)$ with $\Lambda_{\psi_1}(\overline{x\delta_\Omega^{1/2}})$), for all η in $D(H_{\hat\beta},\nu^o)\cap\mathcal{D}(\delta_\Omega^{-1/2})$, such that $\delta_\Omega^{-1/2}\eta$ belongs to $D({}_\alpha H,\nu)$:

$$\begin{aligned}
\|\Lambda_\phi(x) {}_{\hat\alpha}\otimes_{\hat\beta}{}_{N^o} \eta\|^2 &= \|\Lambda_{\psi_1}(\overline{x\delta_\Omega^{1/2}}) {}_{\hat\alpha}\otimes_{\hat\beta}{}_{N^o} \eta\|^2 = (\psi_1 {}_{\hat\beta}*_\alpha{}_N \omega_{\delta_\Omega^{-1/2}\eta})\mathfrak{a}((\overline{x\delta_\Omega^{1/2}})^*\overline{x\delta_\Omega^{1/2}})\\
&= (\phi {}_{\hat\beta}*_\alpha{}_N \omega_\eta)\mathfrak{a}(x^*x)
\end{aligned}$$

which, by continuity, remains true for all $\eta \in D({}_\alpha H, \nu) \cap D(H_{\hat\beta}, \nu^o)$ and all $x \in \mathfrak{N}_\phi$, which proves that ϕ is invariant by $\mathfrak{a}$. But now, we are in the situation of 4.9, which gives that λ is affiliated to the center of A_Ω; we then have (i). ∎

7.10. Corollaries. *Let $\mathfrak{G}$ be a measured quantum groupoid, Ω a 2-cocycle for $\mathfrak{G}$; let W be the pseudo-multiplicative unitary associated to $\mathfrak{G}$. Then the following are equivalent:*

(i) Ω satisfies the equivalent conditions of 7.9.

(ii) For all $t \in \mathbb{R}$, Ω_t (resp. Ω'_t) satisfies the equivalent conditions of 7.9.

(iii) There is $t \in \mathbb{R}$ such that Ω_t (resp. Ω'_t) satisfies the equivalent conditions of 7.9.

Proof. We can easily check that we can write $\tau_s(u_t^\Omega) = u_t^{\Omega_s}$, and $\delta^{is} u_t^\Omega \delta^{-is} = u_t^{\Omega'_s}$, then 7.9 gives the result. ∎

7.11. Theorem. *Let $\mathfrak{G}$ be a measured quantum groupoid, and Ω a 2-cocycle for $\mathfrak{G}$; let us suppose that, for any $t \in \mathbb{R}$, we have $(\tau_t \sigma_{-t}^\Phi \underset{N}{{}_\beta *_\alpha} \tau_t \sigma_t^{\Phi\circ R})(\Omega) = \Omega$. Then, the cocycle Ω satisfies the equivalent conditions of 7.9. In particular, there exists a normal semi-finite faithful operator weight T_Ω from M to $\alpha(N)$ (resp. T'_Ω from M to $\beta(N)$) such that $\mathfrak{G}_\Omega = (N, M, \alpha, \beta, \Gamma_\Omega, T_\Omega, T'_\Omega, \nu)$ is a measured quantum groupoid. Moreover, we get, for all $t \in \mathbb{R}$, that $\tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega) = v_s^\Omega$ and $(\tau_{-t}\sigma_{-t}^{\Phi\circ R} \underset{N}{{}_\beta *_\alpha} id)(\Omega) = \Omega$.*

Proof. Using 7.7(iii), we get that

$$\Omega(\hat J \hat\delta^{it} \hat J \underset{N}{{}_\beta \otimes_\alpha} \hat\delta^{it})\Omega^* = \hat J \hat\delta^{it} \hat J \underset{N}{{}_\beta \otimes_\alpha} \hat\delta^{it}$$

from which, using 7.5(vii), we get that $\hat\delta^{it}$ belongs to A_Ω, and by 4.4(v), that $\mathfrak{a}(\hat\delta^{it}) = \hat\delta^{it} \underset{N}{{}_{\hat\beta} \otimes_\alpha} \hat\delta^{it}$. Using now [E5] 8.8(iii), one gets that, for any s, t in $\mathbb{R}$, we have

$$\mathfrak{a}(\sigma_s^{\psi_1}(\hat\delta^{it})) = \sigma_s^{\psi_1}(\hat\delta^{it}) \underset{N}{{}_{\hat\beta} \otimes_\alpha} \hat\delta^{it}$$

from which one gets that $\sigma_s^{\psi_1}(\hat\delta^{it})\hat\delta^{-it}$ belongs to $A_\Omega^{\mathfrak{a}} = \alpha(N)$ by 7.5.

More precisely, if $n \in N$, we get that

$$\begin{aligned}\sigma_s^{\psi_1}(\hat\delta^{it})\hat\delta^{-it}\alpha(n) = \sigma_s^{\psi_1}(\hat\delta^{it})\alpha(\sigma_t^\nu\gamma_t(n))\hat\delta^{-it} &= \sigma_s^{\psi_1}(\hat\delta^{it}\alpha(\sigma_{t-s}^\nu\gamma_t(n)))\hat\delta^{-it} \\ &= \sigma_t^{\psi_1}(\alpha(\sigma_{-s}^\nu(n))\hat\delta^{it})\hat\delta^{-it} = \alpha(n)\sigma_s^{\psi_1}(\hat\delta^{it})\hat\delta^{-it}\end{aligned}$$

and therefore we get that $\sigma_s^{\psi_1}(\hat\delta^{it})\hat\delta^{-it}$ belongs to $\alpha(Z(N))$.

But, on the other hand, using 7.8, we get that

$$\sigma_s^{\psi_1}(\hat\delta^{it})\hat\delta^{-it} = v_s^\Omega \sigma_s^{\widehat\Phi}(\hat\delta^{it})(v_s^\Omega)^*\hat\delta^{-it} = v_s^\Omega \lambda^{ist}\hat\delta^{it}(v_s^\Omega)^*\hat\delta^{-it} = \lambda^{ist} v_s^\Omega \hat\delta^{it}(v_s^\Omega)^*\hat\delta^{-it}$$

from which, using [E5], 8.11 (ii), we get

$$\sigma_s^{\psi_1}(\hat\delta^{it})\hat\delta^{-it} = \lambda^{ist} v_s^\Omega \tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega)^*$$

and, for all s, t in $\mathbb{R}$, $\tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega)(v_s^\Omega)^*$ belongs to $\alpha(Z(N))$. Therefore, there exists a one-parameter group of unitaries $t \mapsto \mu_s^{it}$ in $Z(N)$ such that $\tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega) = \alpha(\mu_s^{it})v_s^\Omega$; and therefore $\sigma_s^{\psi_1}(\hat\delta^{it}) = \lambda^{ist}\alpha(\mu_s^{-it})\hat\delta^{it}$. So, there exists a positive non-singular operator μ affiliated to $Z(N)$ such that $\sigma_s^{\psi_1}(\hat\delta^{it}) = \lambda^{ist}\alpha(\mu^{-ist})\hat\delta^{it}$ and $\tau_t\sigma_t^{\Phi\circ R}(v_s^\Omega) = \alpha(\mu^{ist})v_s^\Omega$. But now, as, for all $u \in \mathbb{R}$, we have $\tau_u\sigma_u^{\Phi\circ R}(\alpha(\mu^{ist})) = \alpha(\gamma_u(\mu^{ist}))$, we get that $\gamma_u(\mu) = \mu$, and therefore that $\hat{\delta}$ and $\lambda\alpha(\mu)$ strongly commute. Therefore, by [V1], 5.1, there exists a normal semi-finite faithful weight ϕ on A_Ω such that $(D\phi : D\psi_1)_t = (\lambda\alpha(\mu))^{it^2/2}\hat{\delta}^{it}$.

Using now 7.5(iv) and [E5], 8.1, as in 7.9 that ϕ is invariant by $\mathfrak{a}$. But now, we are in the situation of 4.9, which gives that $\mu = 1$, and proves that we are in the situation of 7.9, with, moreover, $u_t^\Omega = 1$; we then infer from 7.9 that $\tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega) = v_s(\Omega)$, $(\tau_{-t}\sigma_{-t}^{\Phi\circ R} \underset{N}{{}_\beta{*}_\alpha} id)(\Omega) = \Omega$. ∎

7.12. Theorem. *Let $\mathfrak{G}$ be a measured quantum groupoid, and Ω a 2-cocycle for $\mathfrak{G}$; let us suppose that, for any $t \in \mathbb{R}$, we have $(\tau_{-t}\sigma_{-t}^{\Phi\circ R} \underset{N}{{}_\beta{*}_\alpha} id)(\Omega) = \Omega$. Then, the cocycle Ω satisfies the equivalent conditions of 7.9. In particular, there exists a normal semi-finite faithful operator weight T_Ω from M to $\alpha(N)$ (resp. T'_Ω from M to $\beta(N)$) such that*

$$\mathfrak{G}_\Omega = (N, M, \alpha, \beta, \Gamma_\Omega, T_\Omega, T'_\Omega, \nu)$$

is a measured quantum groupoid. Moreover, we get, for all $t \in \mathbb{R}$, that $\tau_{-t}\sigma_{-t}^{\Phi\circ R}(v_s^\Omega) = v_s^\Omega$ and $(\tau_t\sigma_{-t}^{\Phi} \underset{N}{{}_\beta{*}_\alpha} \tau_t\sigma_t^{\Phi\circ R})(\Omega) = \Omega$.

Proof. The proof is similar to 7.11. ∎

8. Examples, at last. In this last chapter, we construct a general situation in which the deformations of a measured quantum groupoid by some 2-cocycles are still measured quantum groupoids.

8.1. Measured quantum groupoids associated to a matched pair of groupoids. In [Val6] was decribed a procedure for constructing measured quantum groupoids:

Let $\mathcal{G}$ be a locally compact groupoid, with $\mathcal{G}^{(0)}$ as set of units, and $r : \mathcal{G} \to \mathcal{G}^{(0)}$ (resp. $s : \mathcal{G} \to \mathcal{G}^{(0)}$) as range (resp. source) mapping, equipped with a Haar system $(\lambda^u)_{u\in\mathcal{G}^{(0)}}$ and a quasi-invariant measure ν on $\mathcal{G}^{(0)}$. Let us write $\mu = \int_{\mathcal{G}^{(0)}} \lambda^u \, d\nu(u)$.

Let $\mathcal{G}_1$, $\mathcal{G}_2$ two closed subgroupoids of $\mathcal{G}$, (with $r_1 = r_{|\mathcal{G}_1}$, etc.) equipped with their Haar systems $(\lambda_1^u)_{u\in\mathcal{G}^{(0)}}$, $(\lambda_2^u)_{u\in\mathcal{G}^{(0)}}$.

Then $(\mathcal{G}_1, \mathcal{G}_2)$ is called a matched pair of groupoids if:

(i) $\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{G}^{(0)}$.
(ii) The set $\mathcal{G}_1\mathcal{G}_2 = \{g_1g_2, g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2^{s(g_1)}\}$ is μ-conegligible in $\mathcal{G}$.
(iii) There exists a measure ν on $\mathcal{G}^{(0)}$ with is quasi-invariant for the three Haar systems.

Then, Vallin has constructed an action $(s_2, \mathfrak{a})$ of $\mathfrak{G}(\mathcal{G}_1)$ on $L^\infty(\mathcal{G}_2, \mu_2)$, and put a measured quantum groupoid structure on the crossed product.

Let $\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2) = (L^\infty(\mathcal{G}^{(0)}, \nu), L^\infty(\mathcal{G}_2, \mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1), m, s, \Gamma, T_L, T_R, \nu)$ be this measured quantum groupoid.

Moreover, there exists a right action $(r_1, \hat{\mathfrak{a}})$ of $\mathfrak{G}(\mathcal{G}_2)$ on $L^\infty(\mathcal{G}_1, \mu_1)$, which leads to a measured quantum groupoid structure on $L^\infty(\mathcal{G}_1, \mu_1) \ltimes_{\hat{\mathfrak{a}}} \mathfrak{G}(\mathcal{G}_2)$, we shall write $\mathfrak{G}(\mathcal{G}_2, \mathcal{G}_1)$; we have $\mathfrak{G}(\mathcal{G}_2, \mathcal{G}_1) = \widehat{\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)}$

This measured quantum groupoid $\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)$ has some properties:

(i) The scaling operator λ is equal to 1.
(ii) For any $f \in L^\infty(\mathcal{G}_2, \mu_2)$, $\mathfrak{a}(f)$ is invariant under σ_t^Φ ([Val6], 4.3.5).

(iii) For any $f \in L^\infty(\mathcal{G}_2, \mu_2)$, we have $R(\mathfrak{a}(f)) = \mathfrak{a}(\check{f})$, where R is the co-inverse of $\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)$, and $\check{f}(g_2) = f(g_2^{-1})$, for any $g_2 \in \mathcal{G}_2$. Therefore, using (i), we get that $\mathfrak{a}(f)$ is also invariant under $\sigma_t^{\Phi \circ R}$.

(iv) Using ([Val6], 4.1.1) and ([E5], 3.8(ii)), one can easily check that, for all $t \in \mathbb{R}$, and $f \in L^\infty(\mathcal{G}_2, \mu_2)$, we have $\tau_t(\mathfrak{a}(f)) = \mathfrak{a}(f)$. Namely we have, using (ii):

$$(\mathfrak{a} \underset{L^\infty(\mathcal{G}^{(0)},\nu)}{{}_{s_2}*_{r_2}} \mathfrak{a})\Gamma_{\mathcal{G}_2}(f) = \Gamma(\mathfrak{a}(f)) = \Gamma(\sigma_t^\Phi(\mathfrak{a}(f))) = (\tau_t \underset{L^\infty(\mathcal{G}^{(0)},\nu)}{{}_{s}*_{m}} \sigma_t^\Phi)\Gamma(\mathfrak{a}(f))$$
$$= (\tau_t \circ \mathfrak{a} \underset{L^\infty(\mathcal{G}^{(0)},\nu)}{{}_{s_2}*_{r_2}} \sigma_t^\Phi \circ \mathfrak{a})\Gamma_{\mathcal{G}_2}(f) = (\tau_t \circ \mathfrak{a} \underset{L^\infty(\mathcal{G}^{(0)},\nu)}{{}_{s_2}*_{r_2}} \mathfrak{a})\Gamma_{\mathcal{G}_2}(f)$$

from which we get the result. We refer to [Val6] for all details.

8.2. Theorem. *Let $\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)$ be the measured quantum groupoid constructed from a matched pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids. Let us use all notations of 8.1. Let Ω be a 2-cocycle for $\mathfrak{G}(\mathcal{G}_2)$, as defined in 7.1. Then:*

(i) $(\mathfrak{a} \underset{L^\infty(\mathcal{G}^{(0)},\nu)}{{}_{s_2}*_{r_2}} \mathfrak{a})(\Omega)$ is a 2-cocycle for $\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)$, we shall write $\Omega_\mathfrak{a}$ for simplification.

(ii) There exists a left-invariant operator-valued weight $T_{\Omega_\mathfrak{a}}$ and a right-invariant operator-valued weight $T'_{\Omega_\mathfrak{a}}$ such that

$$\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)_{\Omega_\mathfrak{a}} = (L^\infty(\mathcal{G}^{(0)}, \nu), L^\infty(\mathcal{G}_2, \mu_2) \rtimes_\mathfrak{a} \mathfrak{G}(\mathcal{G}_1), m, s, \Gamma_{\Omega_\mathfrak{a}}, T_{\Omega_\mathfrak{a}}, T'_{\Omega_\mathfrak{a}}, \nu)$$

is a measured quantum groupoid.

Proof. Using [Val6], 4.1.1, one gets (i). As, for all $t \in \mathbb{R}$, $\tau_t \sigma_{-t}^\Phi \circ \mathfrak{a} = \mathfrak{a}$, and $\tau_t \sigma_t^{\Phi \circ R} \circ \mathfrak{a} = \mathfrak{a}$, we get that this cocycle $\Omega_\mathfrak{a}$ satisfies the conditions of 7.11 or 7.12. So, we get (ii). ∎

8.3. Matched pair of groups acting on a space. As a particular case of 8.1, we can study, following ([Val6], 5.1) the case where G is a locally compact group acting (on the right) on a locally compact space X, and G_1, G_2 a matched pair of closed subgroups of G, in the sense of [BSV]. Then, we can define almost everywhere Borel functions p_1^G from G to G_1 and p_2^G from G to G_2, such that

$$g = p_1^G(g) p_2^G(g).$$

Following [VV], we can construct an action a_1 of G_1 on $L^\infty(G_2)$, and put on the crossed product $L^\infty(G_2) \rtimes_{a_1} G_1$ a structure of a locally compact quantum group we shall denote by $\mathbf{G}(G_1, G_2)$; let us denote by $\widetilde{\Gamma}$ the coproduct of this locally compact quantum group.

Let us denote now by $\mathcal{G}$ (resp. $\mathcal{G}_1$, resp. $\mathcal{G}_2$) the locally compact groupoid given by the action of G (resp. G_1, resp. G_2) on X. Then, it is easy to get that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two closed subgroupoids of $\mathcal{G}$, which are a matched pair of groupoids as defined in 8.1. So, there is an action $\mathfrak{a}$ of the measured quantum groupoid $\mathfrak{G}(\mathcal{G}_1)$ on $L^\infty(\mathcal{G}_2)$, and a measured quantum groupoid structure $\mathfrak{G}(\mathcal{G}_1, \mathcal{G}_2)$ on the crossed product $L^\infty(\mathcal{G}_2) \rtimes_\mathfrak{a} \mathfrak{G}(\mathcal{G}_1)$. The action $\mathfrak{a}$ can be identified with an action $\tilde{\mathfrak{a}}$ of G_1 on $L^\infty(X \times G_2)$ ([Val6], 5.1.2) and the crossed product $L^\infty(\mathcal{G}_2) \rtimes_\mathfrak{a} \mathfrak{G}(\mathcal{G}_1)$ can be identified with the crossed product $L^\infty(X \times G_2) \rtimes_{\tilde{\mathfrak{a}}} G_1$, which will be considered as bounded operators on $L^2(X \times G \times G)$ ([Val6], 5.1.1).

We can identify $L^2(X \times G_2) \underset{L^\infty(X)}{{}_{s_2}\otimes_{r_1}} L^\infty(X \times G_1)$ with $L^2(X \times G_2) \otimes L^2(G_1)$ ([Val6], 5.1.1); using these identifications, are given in ([Val6], 5.1.2) the formulae of the coproduct

Γ we can put on this crossed product. For any $f \in L^\infty(X \times G_2)$, $h \in L^\infty(X)$, $k \in L^\infty(G_1)$, we have

$$\Gamma(\mathfrak{a}(f))(x, g, g') = f(x.p_1^G(g), p_2^G(g)p_2^G(g')),$$
$$\Gamma(1 \underset{L^\infty(X)}{{}_{s_2}\otimes_{r_1}} \rho(h \otimes k)) = M(h)(1 \otimes \widetilde{\Gamma}(1 \otimes \rho_1(k))),$$

where $M(h)$ is the function $M(h)(x, g, g') = h(x.gp_2^G(g'))$.

Let's see now how this coproduct can be deformed by a 2-cocycle Ω_2 for $\mathfrak{G}(\mathcal{G}_2)$ to a new coproduct Γ_{Ω_2}. For simplification, we shall restrict to a 2-cocycle Ω_2 for G_2, which can be easily considered as a 2-cocycle for $\mathfrak{G}(\mathcal{G}_2)$. Using then [VV], we can put on the crossed product $L^\infty(G_2) \rtimes_{a_1} G_1$ another structure of locally compact quantum group we shall denote by $\mathbf{G}(G_1, G_2)_{\Omega_2}$, with a deformed coproduct we shall denote Γ_{Ω_2}.

By construction, we have

$$\Gamma_{\Omega_2}(\mathfrak{a}(f))(x, g, g') = \Gamma(\mathfrak{a}(f))(x, g, g') = f(x.p_1^G(g), p_2^G(g)p_2^G(g'))$$

and

$$\Gamma_{\Omega_2}(1 \underset{L^\infty(X)}{{}_{s_2}\otimes_{r_1}} \rho(h \otimes k)) = M(h)(1 \otimes \widetilde{\Gamma}_{\Omega_2}(1 \otimes \rho_1(k)))$$

8.4. Looking back to Kac-Paljutkin's examples. Following ([VV], 5.1.1), let's look at the particular case of 8.3 where G_2 is a normal subgroup of G; then G_1 acts on G_2 by (inner in G) automorphisms, the action of G_2 on G_1 is trivial, the map p_1^G is a homomorphism and G is the semi-direct product $G_2 \rtimes G_1$. Then we know that the old Kac-Paljutkin's examples can be obtained as locally compact quantum groups of the form $\mathbf{G}(G_1, G_2)_{\Omega_2}$.

(i) Taking $G_1 = \mathbb{Z}/2\mathbb{Z}$ acting on $G_2 = (\mathbb{Z}/2\mathbb{Z})^2$ by permutations, the cocycle Ω has been computed in ([BS], 8.26.1), in order to get that $\mathbf{G}(G_1, G_2)_{\Omega_2}$ is then the dimension 8 example constructed in [KP1]. Taking now an action of the semi-direct product $G = G_2 \rtimes G_1$ on a locally compact space X, we obtain, by 8.3 applied to this particular case, a measured quantum groupoid given by dimension 8 Kac-Paljutkin's example and a right action of $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ on a space X.

(ii) Taking $G_1 = \mathbb{R}$ acting on $G_2 = \mathbb{R}^2$ by $a_g(x) = exp(gK)(x)$ ($x \in \mathbb{R}^2$, K is a real 2×2 matrix). Then the cocycle has been computed in ([VV], 8.26.2) and leads to the infinite dimensional Kac-Paljutkin's example ([KP2]). So, starting from this example, and some right action of the Heisenberg group $H_3(\mathbb{R}) = \mathbb{R}^2 \rtimes_a \mathbb{R}$ on X, we get, by 8.3, another example of a measured quantum groupoid.

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