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COCYCLE CONDITION FOR MULTI-PULLBACKS OF ALGEBRAS

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Abstract. Take finitely many topological spaces and for each pair of these spaces choose a pair of corresponding closed subspaces that are identified by a homeomorphism. We note that this gluing procedure does not guarantee that the building pieces, or the gluings of some pieces, are embedded in the space obtained by putting together all given ingredients. Dually, we show that a certain sufficient condition, called the cocycle condition, is also necessary to guarantee sheaf-like properties of surjective multi-pullbacks of algebras with distributive lattices of ideals.

When constructing a topological space as the gluing of pieces, it is desirable that the parts are embedded into the described space. The gluing of three intervals $I_1 \cong I_2 \cong I_3 \cong [-1,1]$ into the space T_* described by Fig. 1 fails this property as the endpoints of I_2 and I_3 are glued into a single point.

There is, however, a more subtle way in which a gluing may fail to embed its parts into the whole space. To see this, consider another gluing of I_1, I_2 and I_3 depicted in Fig. 2(a) into the space $T_\circ$ pictured in Fig. 2(b). All the I_j's are embedded into $T_\circ$ but the partial gluing of I_2 and I_3 is not. Of course, one can define an alternative gluing procedure of I_j's into $T_\circ$ (see Fig. 2(c)) for which all partial gluings are embedded into $T_\circ$.

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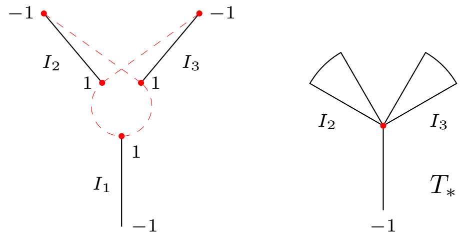
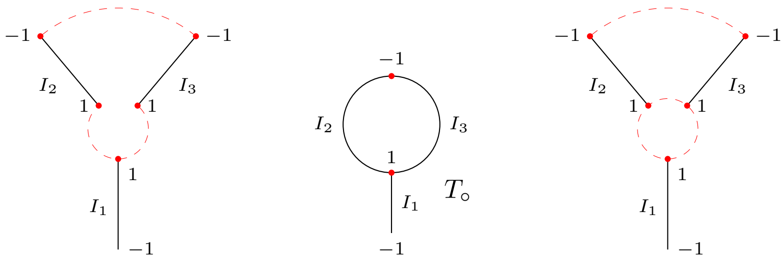


(a) (b)

Fig. 1



(a) (b) (c)

Fig. 2

Let us now consider the problem of gluing from the point of view of algebras. Let J be a finite set, and let

$$\{\pi^i_j : B_i \longrightarrow B_{ij} = B_{ji}\}_{i,j\in J,\, i\neq j} \tag{1}$$

be a family of algebra homomorphisms.

Definition 1 ([1, 5]). The *multi-pullback algebra* B^π of a family (1) of algebra homomorphisms is defined as

$$B^\pi := \left\{ (b_i)_i \in \prod_{i\in J} B_i \;\middle|\; \pi^i_j(b_i) = \pi^j_i(b_j),\ \forall\, i,j \in J,\, i\neq j \right\}.$$

Definition 2. A family (1) of algebra homomorphisms is called *distributive* if and only if all of them are surjective and for all $i \in J$ the kernels of π^i_j, $j \in J \setminus \{i\}$, generate a distributive lattice of ideals.

The multi-pullback algebra of a distributive family of homomorphisms is the main mathematical concept of this note, and plays a key role in [3, 2, 4]. In particular, it includes the multi-pullbacks of all finite families of surjective unital homomorphisms of C^*-algebras. In the case of commutative unital C^*-algebras, such a multi-pullback C^*-algebra can be identified with the algebra of all continuous functions on the compact Hausdorff space obtained by the gluing procedure described in the abstract applied to compact Hausdorff spaces.

Example 3. Consider the C^*-algebra $C(T_*)$ of all continuous functions on T_* as the multi-pullback C^*-algebra corresponding to the gluing depicted on Fig. 1(a). Here we take $B_i = C(I_i)$, $i = 1, 2, 3$, $B_{12} = B_{13} = \mathbb{C}$, $B_{23} = \mathbb{C} \oplus \mathbb{C}$, and define C^*-epimorphisms by the formulae

$$\pi^1_2 = \pi^2_1 = \pi^1_3 = \pi^3_1 : f \mapsto f(1), \quad \pi^2_3 : f \mapsto (f(-1), f(1)), \quad \pi^3_2 : f \mapsto (f(1), f(-1)).$$

The fact that I_2 is not embedded in T_* corresponds to the non-surjectivity of the canonical projection $B^\pi \to B_2$.

Example 4. Consider the C^*-algebra $C(T_\circ)$ of all continuous functions on $T_\circ$ as the multi-pullback C^*-algebra corresponding to the gluing depicted on Fig. 2(a). Here we take $B_i = C(I_i)$, $B_{ij} := \mathbb{C}$, $1 \le i, j \le 3$, $i \ne j$, and define C^*-epimorphisms by the formulae

$$\pi^1_2 = \pi^2_1 = \pi^1_3 = \pi^3_1 : f \mapsto f(1), \quad \pi^2_3 = \pi^3_2 : f \mapsto f(-1).$$

While the canonical projections $B^\pi \to B_i$ are all surjective, the canonical projection $B^\pi \to \{(b_2, b_3) \in B_2 \times B_3 \mid \pi^2_3(b_2) = \pi^3_2(b_3)\}$ is not. Indeed, a pair $b_2 := (t \mapsto t)$, $b_3 := (t \mapsto -1)$ satisfies $\pi^2_3(b_2) = \pi^3_2(b_3)$, but there is no function $b_1 \in B_1$ such that $(b_1, b_2, b_3) \in B^\pi$. This corresponds to the fact that the gluing of I_2 and I_3 is not embedded in $T_\circ$.

Example 5. We can present the C^*-algebra $C(T_\circ)$ of all continuous functions on $T_\circ$ pictured in Fig. 2(b) by using different multi-pullbacks: one corresponding to the gluing depicted in Fig. 2(a) (see Example 4) and one corresponding to the gluing depicted in Fig. 2(c). For the latter case, we take the B_i's, B_{12}, B_{13}, π^1_2, π^2_1, π^1_3, π^3_1 as in Example 4, but we put $B_{23} := \mathbb{C} \oplus \mathbb{C}$ and $\pi^2_3 = \pi^3_2 : f \mapsto (f(-1), f(1))$. Now not only the canonical projections $B^\pi \to B_i$ are all surjective, but also, for all distinct i, j, k and all $b_i \in B_i$, $b_j \in B_j$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$, there exists $b_k \in B_k$ such that $\pi^i_k(b_i) = \pi^k_i(b_k)$ and $\pi^j_k(b_j) = \pi^k_j(b_k)$.

It turns out that the cocycle condition defined below is a perfect tool to understand the differences between the above examples. To define the cocycle condition, for any distinct i, j, k we put $B^i_{jk} := B_i/(\ker \pi^i_j + \ker \pi^i_k)$ and take $[\cdot]^i_{jk} : B_i \to B^i_{jk}$ to be the canonical surjections. Next, we introduce the family of maps

$$\pi^{ij}_k : B^i_{jk} \longrightarrow B_{ij}/\pi^i_j(\ker \pi^i_k), \qquad [b_i]^i_{jk} \longmapsto \pi^i_j(b_i) + \pi^i_j(\ker \pi^i_k). \tag{2}$$

They are isomorphisms when π^i_j's are epimorphisms. Now we are ready for:

Definition 6 ([1, in Proposition 9]). We say that a family (1) of surjective algebra homomorphisms satisfies the *cocycle condition* if and only if, for all distinct $i, j, k \in J$,

1. $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$,
2. the isomorphisms $\phi^{ij}_k := (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k : B^j_{ik} \to B^i_{jk}$ satisfy $\phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$.

It was proven in [1] that, if a distributive family of π^i_j's satisfies the cocycle condition, then the canonical projections $B^\pi \to B_i$ are all surjective. In particular, the multi-pullback from Example 3 cannot satisfy the cocycle condition. The multi-pullback presentation of $C(T_\circ)$ from Example 4 demonstrates, however, that the cocycle condition is not necessary for the canonical projections $B^\pi \to B_i$ to be surjective. Indeed, they are all clearly surjective in this case, but π^i_j's do not satisfy the cocycle condition because $\pi_2^1(\ker\pi_3^1)=\{0\}$ whereas $\pi_1^2(\ker\pi_3^2)=\mathbb{C}$. On the other hand, the cocycle condition is satisfied by an alternative multi-pullback presentation of $C(T_\circ)$ given in Example 5. This suggests that the cocycle condition is related to the possibility of extending partial multi-pullbacks. Thus we arrive at the main result of this note:

THEOREM 7. *The following statements about a distributive family* (1) *of algebra homomorphisms are equivalent:*

1. *The family* (1) *satisfies the cocycle condition.*
2. *For any* $K\subsetneq J$, $k\in J\setminus K$ *and* $(b_l)_{l\in K}\in\prod_{l\in K}B_l$ *such that* $\pi^i_j(b_i)=\pi^j_i(b_j)$ *for all distinct* $i,j\in K$, *there exists* $b_k\in B_k$ *such that* $\pi^l_k(b_l)=\pi^k_l(b_k)$ *for all* $l\in K$.
3. *For all distinct* $i,j,k\in J$ *and all* $b_i\in B_i$, $b_j\in B_j$ *such that* $\pi^i_j(b_i)=\pi^j_i(b_j)$, *there exists* $b_k\in B_k$ *such that also* $\pi^i_k(b_i)=\pi^k_i(b_k)$ *and* $\pi^j_k(b_j)=\pi^k_j(b_k)$.

Proof. The proof of (1) ⇒ (2) is essentially identical with the proof of [1, Proposition 9], and (3) is obviously a special case of (2). In order to prove (3) ⇒ (1) and close the loop of implications, assume that for any distinct $i,j,k\in J$ and for arbitrary elements $b_i\in B_i$ and $b_j\in B_j$ such that $\pi^i_j(b_i)=\pi^j_i(b_j)$ there exists $b_k\in B_k$ such that also $\pi^i_k(b_i)=\pi^k_i(b_k)$ and $\pi^j_k(b_j)=\pi^k_j(b_k)$. Specializing this condition for $b_j=0$ yields that for any $b_i\in\ker\pi^i_j$ there exists a $b_k\in\ker\pi^k_j$ such that $\pi^i_k(b_i)=\pi^k_i(b_k)$, that is $\pi^i_k(\ker\pi^i_j)\subseteq\pi^k_i(\ker\pi^k_j)$. Exchanging i and k we obtain the set equality. This proves Condition (1) defining the cocycle condition.

To prove the second condition observe that, for all distinct $i,j,k\in J$ and any $b_i\in B_i$, $b_j\in B_j$,

$$[b_i]^i_{jk}=\phi^{ij}_k([b_j]^j_{ik})\ \Leftrightarrow\ \pi^{ji}_k([b_j]^j_{ik})=\pi^{ij}_k([b_i]^i_{jk})\ \Leftrightarrow\ \pi^i_j(b_i)-\pi^j_i(b_j)\in\pi^i_j(\ker\pi^i_k).\tag{3}$$

Now let us pick any distinct $i,j,k\in J$ and any $b_j\in B_j$. Since π^k_j is surjective, there exists $b_k\in B_k$ such that $\pi^k_j(b_k)=\pi^j_k(b_j)$, so that $[b_k]^k_{ji}=\phi^{kj}_i([b_j]^j_{ik})$ by (3). Furthermore, by assumption, there exists $b_i\in B_i$ such that $\pi^i_k(b_i)=\pi^k_i(b_k)$ and $\pi^i_j(b_i)=\pi^j_i(b_j)$. Therefore, again by (3), we obtain

$$[b_i]^i_{jk}=\phi^{ik}_j([b_k]^k_{ji})=\phi^{ik}_j(\phi^{kj}_i([b_j]^j_{ik}))\quad\text{and}\quad[b_i]^i_{jk}=\phi^{ij}_k([b_j]^j_{ik}).\tag{4}$$

Plugging in the second equality to the first one, we get $\phi^{ij}_k([b_j]^j_{ik})=\phi^{ik}_j(\phi^{kj}_i([b_j]^j_{ik}))$ for any $[b_j]^j_{ik}\in B^j_{ik}$, as needed. ∎

Finally, let us remark that the fact that in Example 5 we could remedy the lack of the cocycle condition in Example 4 is not a coincidence. Indeed, following [1, Proposition 8 and Remark 2], one sees that, if B^π is the multi-pullback of an appropriate family (1), then B^π can also be presented as the multi-pullback of a family satisfying the cocycle condition even if the original family failed to do so. More precisely:

PROPOSITION 8. *If* B^π *is the multi-pullback of a family* (1) *such that the canonical projections* $B^\pi\to B_i$ *are all surjective and their kernels generate a distributive lattice of ideals, then the family defined via the canonical surjections*

$$\{\pi'^i_j: B_i\cong B^\pi/\ker(B^\pi\to B_i)\longrightarrow B^\pi/(\ker(B^\pi\to B_i)+\ker(B^\pi\to B_j))\}_{i,j\in J,\,i\neq j}\tag{5}$$

satisfies the cocycle condition and its multi-pullback is isomorphic to B^π.

The aforementioned example is a special case of this general claim because C^*-ideals always generate a distributive lattice.

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