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BILIPSCHITZ INVARIANCE OF THE FIRST TRANSVERSE CHARACTERISTIC MAP

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Abstract. Given a smooth S^1-foliated bundle, A. Connes has shown the existence of an additive morphism ϕ from the K-theory group of the foliation C*-algebra to the scalar field, which factorizes, via the assembly map, the Godbillon-Vey class, which is the first secondary characteristic class, of the classifying space. We prove the invariance of this map under a bilipschitz homeomorphism, extending a previous result for maps of class C^1 by H. Natsume.

1. Introduction. Let (W, F) be a codimension one smooth foliated manifold, transversely oriented. Although the transverse bundle is trivialisable, it encodes a nontrivial *secondary* characteristic class discovered by C. Godbillon and J. Vey in 1970 [GV]. This class denoted by $\mathcal{GV}(W, F)$ is naturally defined as an element of the de Rham cohomology group $H^3(V, \mathbb{R})$, and is the first of the secondary characteristic classes of the foliation [CC].

A natural question, often addressed, is the invariance of this class under a foliated homeomorphism (cf. survey by E. Ghys [G]). This question remains open today, but it has been proved under various regularity hypotheses on the homeomorphism (cf. end of section 1 for a discussion).

Moreover, a noncommutative version of this secondary class has been defined and discussed by A. Connes [C]. The Godbillon-Vey class can be viewed as a cohomology class of the classifying space BG of the holonomy groupoid G of the foliation. It should be mentioned that Godbillon-Vey class is also naturally a cohomology class of the classifying space of the Haefliger groupoid Γ_1 of germs of local diffeomorphisms of the line, and that our class comes naturally via the pull-back of the Godbillon-Vey class in $H^3(B\Gamma_1, \mathbb{R})$ by the classifying map $BG \to B\Gamma_1$. Thus let $\mathcal{GV}(W, F) \in H^3(BG, \mathbb{R})$ be the class so defined.

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There is the assembly map $\mu : K_*(BG) \to K_*(C^*(W,F))$, which is predicted, by Baum-Connes conjecture, to be an isomorphism, at least when there is no leaf with finite holonomy.

There exists an additive map ρ of the K-theory group of the C*-algebra of the foliation which factorizes, via μ, the homomorphism defined by $\mathcal{GV}(W,F)$. Precisely, for any $z \in K_*(BG)$, the following equality holds true:

$$\langle \mathrm{ch}(z), \mathcal{GV}(W,F)\rangle = \rho \circ \mu(z). \tag{1}$$

As we do not know if μ is an isomorphism in full generality, ρ appears more general at this stage. We call it the analytical Godbillon-Vey map. Cyclic cohomology plays an important role, and this map comes from a cyclic cocycle defined on a suitable Banach subalgebra of the C*-algebra of the foliation.

Moreover, by the work of A. Connes and H. Moscovici [CM], this class comes from the Hopf-cyclic cohomology of the universal Hopf algebra $\mathcal{H}_1$ which acts transversely (section 4).

A question very similar to the preceding and as much natural, is the invariance of the analytical Godbillon-Vey map under a foliated homeomorphism. H. Natsume (1987) has proved it to be true when the foliated homeomorphism is transversely of C^1 class [N] .

The purpose of this work is to give the proof of an affirmative answer when the foliated homeomorphism is transversely bilipschitz (Theorem 9.1).

For the sake of simplicity, we shall restrict this exposition to foliated circle bundles (subsection 2.2), and thus we shall consider a discrete group Γ acting by oriented diffeomorphisms on the one dimensional torus S^1. Then the analytical Godbillon-Vey map becomes a scalar valued additive map of $K_0(C(S^1) \rtimes \Gamma)$, and we compare two such maps coming from two actions conjugated by a homeomorphism.

However, the present method is slightly different from that of [N] and can be viewed as an alternative proof, as we shall use the auxiliary C*-algebra $C(S^1) \rtimes \Gamma \rtimes \mathbb{R}$, where $\mathbb{R}$ acts via the modulus of the action of Γ on the Lebesgue measure. The von Neumann crossed product $L^\infty(S^1) \rtimes \Gamma \rtimes \mathbb{R}$ is the continuous decomposition of the von Neumann algebra of the dynamical system, and its center is the flow of weights.

On the way, we obtain the invariance of the measure on the flow of weights coming from the analytical Godbillon-Vey map (section 10), and (Corollary 10.2) the invariance of the Duminy measure [G].

This last result was not known even for conjugation of class C^1, and seems to be difficult to prove directly, without using the analytical map.

2. Foliations and the Godbillon-Vey class

2.1. Foliated manifolds. Let W be a smooth manifold of dimension n, and $p, q \in N$ such that $p + q = n$. A foliation on W of codimension q is given by an atlas of local charts $(\mathcal{O}_i, \varphi_i)$, $i \in I$ where $\varphi_i : \mathcal{O}_i \to \mathbb{R}^q \times \mathbb{R}^p$ is a smooth embedding of the open subset $\mathcal{O}_i \subset W$, and such that for every $i, j \in I$, $\varphi_i \circ \varphi_j^{-1}$ defined on $\varphi_j(\mathcal{O}_i \cap \mathcal{O}_j)$ is of the form $(s, x) \to (f(s), g(s, x))$ where f, g are smooth.

The tangent bundle to the foliation is a sub-bundle $F \subset TW$ of dimension p of the vector bundle tangent to W. A classical result is:

THEOREM 2.1 (Frobenius). *A smooth vector sub-bundle $F \subset TW$ is the tangent bundle of a smooth foliation on W if and only if $C^\infty(W, F)$ is a Lie subalgebra of the Lie algebra of smooth vector fields on W.*

Recall that with the Lie bracket $(X, Y) \to [X, Y]$, $C^\infty(W, TW)$ is a Lie algebra.

2.2. Foliated bundles. A large family of foliations comes with the suspension construction of A. Haefliger [CC], which goes as follows:

Let W be a smooth manifold and α be a homomorphism of the fundamental group $\pi_1(W)$ to the group of diffeomorphisms of a smooth manifold V, and let $\tilde{W}$ be the universal covering of W.

Then $\pi_1(W)$ acts properly on the cartesian product $V \times \tilde{W}$, sending (z, v) to $(z\gamma, \gamma^{-1}v)$ for $\gamma \in \pi_1(W)$. The resulting quotient manifold W_α is a smooth locally trivial fibration over W with fiber V. The foliation on $V \times \tilde{W}$ the leaves of which are $\{z\} \times \tilde{W}$ descends on W_α to a foliation F transverse to the fibers of that fibration, and such that each leaf is a covering of W.

EXAMPLE 2.2. In the sequel, we will be mainly concerned with the case of suspension foliation with $V = S^1$, i.e. the one dimensional torus. For instance, take a torsion free fuchsian cocompact group $\Gamma \subset PSL(2, \mathbb{R})$, and let W be the quotient manifold $W = \Gamma \backslash PSL(2, \mathbb{R})$. Then W is the unit tangent bundle to the Riemann surface $\Gamma \backslash \mathcal{H}$, where $\mathcal{H} = PSL(2, \mathbb{R})/SO(2)$ is the Poincaré disk. The weak instable foliation of the geodesic flow on SW is topologically (and even C^2) conjugate to the suspension foliation coming from the natural injection $\pi_1(W/SO(2)) = \Gamma$ into the diffeomorphism group of S^1 [MS].

Now suppose that (W, F) is a smooth manifold with a codimension one foliation. Then $\tau = TW/F$ is a one dimensional real vector bundle, and if we assume, for simplicity, that τ is oriented, then there is a smooth differential form ω without zero with $\ker \omega = F$.

By Frobenius theorem, there exists a degree one differential form θ such that $d\omega = \theta \wedge \omega$. The differential form $\theta \wedge d\theta$ is closed and its cohomology class in the de Rham cohomology group $H^3_{DR}(W)$ is independent of the choice of ω and of θ.

This is the Godbillon-Vey class

$$\mathcal{GV}(W, F) \in H^3_{DR}(W). \tag{2}$$

REMARK 2.3. The nontriviality of Godbillon-Vey class has been shown by Roussarie [CC]. In the case of the weak unstable foliation on $W = \Gamma \backslash PSL(2, \mathbb{R})$ as in the example above, the Godbillon-Vey class in $H^3_{DR}(W, \mathbb{R})$ is proportional to the Euler characteristic of W. W. Thurston has shown the existence, for any $t \in \mathbb{R}$, of a dimension 2 foliation on S^3 with Godbillon-Vey class equal to t times the fundamental cohomology class of S^3.

A natural question is the topological invariance of the Godbillon-Vey class. Let (W_j, F_j) for $j = 1, 2$ be a foliated manifold. A (smooth, continuous) map $T : W_1 \to W_2$ is said to be foliated if the image by T of a leaf in W_1 is a leaf in W_2, and if T is smooth, it is equivalent to the condition $T_* F_1 \subset F_2$.

Such a map T induces a linear map of singular cohomology groups with real coefficients $T^* : H^*_{sing}(W_2, \mathbb{R}) \to H^*_{sing}(W_1, \mathbb{R})$. By de Rham's theorem, singular real cohomology groups are isomorphic to the de Rham cohomology groups of differential forms.

We get the following issue: if T is a foliated homeomorphism between two smooth foliated manifolds (W_1, F_1) and (W_2, F_2), with F_j of codimension one for $j = 1, 2$, then is the image of the Godbillon-Vey class of (W_2, F_2) in $H^3(W_2)$ by T^* equal to the Godbillon-Vey class of (W_1, F_1)?

In 1987, G. Raby and independently E. Ghys and T. Tsuboi proved invariance when T is of class C^1 [G], and in 1991, S. Hurder and A. Katok proved it when T is bilipschitz and $\dim W_1 = 3$ [HK].

In [H], the invariance has been established when T is a foliated continuous map and is transversely a homeomorphism absolutely continuous together with its inverse.

Recall that a homeomorphism $I \to J$ of bounded intervals of $\mathbb{R}$ has always bounded variation, so that its distributional derivative is a measure, and this homeomorphism is said to be absolutely continuous if that measure is absolutely continuous with respect to, or dominated by the Lebesgue class of measure.

3. The analytic Godbillon-Vey map. Besides the geometric Godbillon-Vey class, there is an analytic approach using the K-theory group of the C*-algebra of the foliation, elaborated by A. Connes [C].

From now on, we consider Γ a discrete at most countable group of oriented diffeomorphisms of $V = S^1$. There corresponds a topological groupoid $V \rtimes \Gamma$, which is as a set the cartesian product $V \times \Gamma$, with space of units equal to V, and with map $r(x, \gamma) = x, s(x, \gamma) = x\gamma$. Thus (x_1, γ_1) and (x_2, γ_2) are composable if $x_2 = x_1\gamma_1$ and then the product is equal to $(x_1, \gamma_1\gamma_2)$. The space $\mathcal{A}_0 = C_c(V \times \Gamma)$ is endowed with the structure of an involutive algebra given by convolution product and adjoint, given for $f^0, f^1 \in \mathcal{A}_0$:

$$f^0 * f^1(x, \gamma) = \sum_{\beta \in \Gamma} f^0(x, \beta) f^1(x\beta, \beta^{-1}\gamma), \qquad f^0(x, \gamma)^* = \bar{f}(x\gamma, \gamma^{-1}).$$

We shall denote by $C(V) \rtimes \Gamma$ the reduced crossed product corresponding to this action, or equivalently the reduced C*-algebra of the groupoid $V \rtimes \Gamma$ [R]. This is the completion of the involutive algebra $\mathcal{A}_0 = C_c(V \times \Gamma)$ for the family of representations $\pi_x : \mathcal{A}_0 \to l^2(\Gamma)$, $x \in V$:

$$\pi_x(f)\xi(\gamma) = \sum_{\beta \in \Gamma} f(x, \beta)\xi(\beta^{-1}\gamma).$$

Then there is an additive map $\rho_\omega : K_0(C(V) \rtimes \Gamma) \to \mathbb{C}$ which represents the Godbillon-Vey cohomology class, and we recall now how this map is defined.

Let V_Γ be the homotopy quotient, i.e. the quotient of the space $V \times E\Gamma$ by the diagonal action of Γ, where $E\Gamma$ is the infinite joint space of J. Milnor [MS]. The space $E\Gamma$ is a contractible space on which Γ acts freely and properly, and the quotient $B = \Gamma \backslash E\Gamma$ is the classifying space of Γ. Thus V_Γ is a locally trivial fibration over $B\Gamma$, with fiber V.

From now on, we shall assume the following hypothesis on the action of Γ:

(I) *For every open interval $U \subset V$ and $\gamma \in \Gamma$, if $\gamma_{|U} = Id_U$ then $\gamma = e$.*

Then, under this assumption, the classifying space of the groupoid of germs of local diffeomorphisms induced by the action of Γ is equal to V_Γ.

Let $d\theta$ be the Haar measure on the torus $V = S^1$ and $\delta : V \rtimes \Gamma \to \mathbb{R}_+^*$ be the modulus of the action with respect to $d\theta$, i.e. the Radon-Nikodym derivative given by $\delta(x,\gamma)d\theta = \gamma_* d\theta$ for $\gamma \in \Gamma$, and let $l(\gamma)(x) = \log \delta(x,\gamma)$. Recall the Bott-Thurston cocycle on Γ:

$$c(g_1, g_2) = \int_{S^1} \omega(g_1, g_2)$$

where $\omega(g_1, g_2) = dl(g_1 g_2) l(g_2) - l(g_1 g_2) dl(g_2)$ is a group 2-cocycle on Γ with values in the space of volume forms on V.

Then from the simplicial structure of V_Γ, we deduce from ω a well defined class $\mathcal{GV}(V,\Gamma)$ in the singular cohomology group $H^3(V_\Gamma, \mathbb{R})$ which represents Godbillon-Vey classes as follows.

Let W be a manifold and $f : W \to V_\Gamma$ a continuous map. Then to the composition of f with the projection $V_\Gamma \to B\Gamma$ corresponds a homomorphism $\alpha : \pi_1(W) \to \Gamma$. Let as before $p_\alpha : W_\alpha \to W$ be the suspension, with a codimension one foliation F_α. Then the following equality holds true:

$$p_\alpha^* \circ f^*(\mathcal{GV}(V,\Gamma)) = \mathcal{GV}(W_\alpha, F_\alpha). \tag{3}$$

There is a cyclic 2-cocycle defined on the algebraic crossed product $C_c(S^1 \times \Gamma)$ given by the formula, associated with ω:

$$(f^0, f^1, f^2) \to \sum_{\gamma_0\gamma_1\gamma_2 = e} \int_{S^1} f^0(x,\gamma_0) f^1(x\gamma_0, \gamma_1) f^2(x\gamma_0\gamma_1, \gamma_2)\omega(\gamma_1, \gamma_2).$$

This 2-cocycle is actually a 2-trace and gives rise to an additive map $\rho_\omega : K_0(C(V) \rtimes \Gamma) \to \mathbb{C}$ on the K-theory group of the C*-algebra [C].

This additive map is the *analytical Godbillon-Vey map* and is linked to the geometrical one via the geometric K-theory group and the assembly map on the K-homology of V_Γ:

$$\mu : K_1(V,\Gamma) \to K_0(C(V) \rtimes \Gamma). \tag{4}$$

For $z \in K_*(V,\Gamma)$, the following equality holds true:

$$\langle \mathrm{ch}(z), \mathcal{GV}(V,\Gamma)\rangle = \rho_\omega(\mu(z)). \tag{5}$$

More precisely, by the Baum-Douglas definition of K-homology, z is represented by a cycle (W, E, f), where W is a smooth spin manifold, $f : W \to V_\Gamma$ is a continuous map, $E \to W$ is a finite rank complex vector bundle. Then, previous equalities (3), (5) read

$$\rho_\omega(\mu(z)) = \langle p_\alpha^*(\hat{A}(W) \cup \mathrm{ch}(E)) \cup \mathcal{GV}(W_\alpha, F_\alpha), [W_\alpha]\rangle.$$

A natural problem is the invariance of this map under topological conjugation, which means that if T is a homeomorphism conjugating two smooth actions of Γ, then the two additive maps coming from the analytic Godbillon-Vey maps will correspond under the map induced by T between the C*-algebras of the actions.

The purpose of this article is to prove the following (theorem 9.1): let α_j for $j = 1, 2$ be a class C^∞ action of Γ on S^1 and $T : S^1 \to S^1$ a homeomorphism conjugating these two actions and ρ_j the analytical Godbillon-Vey map for α_j, $j = 1, 2$. Then it induces $T^* : K_0(C(S^1) \rtimes_{\alpha_2} \Gamma) \to K_0(C(S^1) \rtimes_{\alpha_1} \Gamma)$. Then if T is bilipschitz, it preserves the analytic Godbillon-Vey maps.

4. Transverse Hopf algebras. An interpretation of this class has been given by A. Connes and H. Moscovici [CM] using Hopf algebras.

The Hopf algebra $\mathcal{H}_1$ is the universal algebra generated by $\{X, Y, \delta_n; n = 1, 2, \cdots\}$ with the relations:

$$[Y, X] = X, \qquad [Y, \delta_n] = n\delta_n, \qquad [X, \delta_n] = \delta_{n+1}, \qquad [\delta_n, \delta_k] = 0,$$

and the coproduct $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ determined by the relations:

$$\Delta Y = Y \otimes 1 + 1 \otimes Y, \qquad \Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,$$
$$\Delta\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1,$$

and antipode $S : \mathcal{H}_1 \to \mathcal{H}_1$ determined by the relations:

$$S(Y) = -Y, \qquad S(X) = -X + \delta_1 Y, \qquad S(\delta_1) = \delta_1.$$

The periodic Hopf cyclic cohomology of this algebra has two generators, and one of them comes from a class δ_1 in the Hopf cyclic cohomology group $HC^1_{Hopf}(\mathcal{H}_1)$ and is related to the Godbillon-Vey map, as follows.

The homomorphism $\delta : V \times \Gamma \to \mathbb{R}_+^*$, gives rise to a right action of Γ on $V \times \mathbb{R}_+^*$ by $(x, \lambda)\gamma = (x\gamma, \delta(x, \gamma)\lambda)$.

The K-theory groups of the crossed product $C(V \times R_+^*) \rtimes \Gamma$ are isomorphic to K-theory groups of $C(V) \rtimes \Gamma$ by the Thom isomorphism:

$$\theta : K_0(C(V) \rtimes \Gamma) \to K_1(C(V \times R_+^*) \rtimes \Gamma).$$

There is a subalgebra $\mathcal{A} \subset C(V \times R_+^*) \rtimes \Gamma$ stable under holomorphic functional calculus, containing the algebraic crossed product $C_c^\infty(V \times R_+^*) \rtimes \Gamma$ and a morphism:

$$HC^*(\mathcal{H}_1) \to HC^*(\mathcal{A}).$$

Now the image of δ_1 in $HC^1(\mathcal{A})$ defines an additive map Π_{δ_1} of $K_1(\mathcal{A})$ and hence of $K_1(C(V \times R_+^*) \rtimes \Gamma)$ to the scalar field $\mathbb{C}$.

This latter is the analytical Godbillon-Vey map defined in the last section; thus for $x \in K_0(C(V) \rtimes \Gamma)$:

$$\Pi_{\delta_1}(\theta(x)) = \rho_\omega(x).$$

5. Dense subalgebras stable by holomorphic functional calculus. A unital subalgebra $\mathcal{A} \subset A$ of a unital algebra is a *full subalgebra* if it has the same unit as A and if every element of $\mathcal{A}$ invertible in A is already invertible in $\mathcal{A}$. The intersection of full subalgebras remains full. If $\mathcal{A}_0 \subset \mathcal{A}_1$ is full in $\mathcal{A}_1$ and $\mathcal{A}_1 \subset A$ is full, then $\mathcal{A}_0$ is full in A.

If A is a Banach algebra, then $\mathcal{A}$ is said to be stable by holomorphic functional calculus if $f(a) \in \mathcal{A}$ whenever $a \in \mathcal{A}$ and f is a holomorphic function defined in a neighbourhood of the spectrum of a. Then a Fréchet subalgebra $\mathcal{A} \subset A$ is full if and only if it is stable by holomorphic functional calculus [S, Lemma 1.2].

If $\mathcal{A}$ is a subalgebra dense in the Banach algebra A and stable by holomorphic functional calculus, then the natural morphism:

$$K_j(\mathcal{A}) \to K_j(A)$$

is an isomorphism for $j = 0, 1$ [B, Theorem A.2.1].

We give now two examples of such algebras.

5.1. Almost isometric vector bundles. Let Γ be a discrete group acting on a smooth manifold by diffeomorphism, and let $E \to W$ be a Γ-equivariant complex vector bundle with an hermitian metric not necessarily equivariant. With as before $r : V \times \Gamma \to V$ the target map of the groupoid, then on $\mathcal{E}_0 = C_c(W \times \Gamma, r^*E)$, there is an A-valued product given by:

$$\langle \xi, \eta \rangle (x, \gamma) = \sum_{\beta \in \Gamma} (\xi(x\beta, \beta^{-1}), \eta(x\beta, \beta^{-1}\gamma)).$$

There is also a right action of $\mathcal{A}_0 = C_c(V \times \Gamma)$ on $\mathcal{E}_0$ given by:

$$\xi * a(x, \gamma) = \sum_{\beta \in \Gamma} \xi(x, \beta) a(x\beta, \beta^{-1}\gamma).$$

It gives $\mathcal{E}_0$ the structure of a pre-hilbert module on A; let $\mathcal{E}$ be the Hilbert module on A obtained by completion.

Unless the hermitian metric is Γ-equivariant, there is no left action of A on $\mathcal{E}$. However, there is a homomorphism λ of $\mathcal{A}_0$ into $\mathcal{L}(\mathcal{E})$ (the algebra of adjointable morphisms of $\mathcal{E}$), given by, for $a \in \mathcal{A}_0$, $\xi \in \mathcal{E}_0$:

$$\lambda(a)\xi(x, \gamma) = \sum_{\beta \in \Gamma} a(x, \beta)\xi(x\beta, \beta^{-1}\gamma).$$

This homomorphism is closable and extends to a Banach subalgebra of A. If the hermitian metric is almost isometric, then the domain of the closure is a full subalgebra of A [C, Definition 3.3, Proposition 3.4].

5.2. Unbounded derivations. Let $\mathcal{A}_0$ be a dense subalgebra of a Banach algebra A, $\mathcal{E}$ a Banach right module over A and $\lambda : \mathcal{A}_0 \to \mathcal{L}_A(\mathcal{E})$, a closable left action, i.e. a closable morphism to the Banach algebra of continuous homomorphisms of $\mathcal{E}$ commuting with the right action, and $\delta : \mathcal{A}_0 \to \mathcal{E}$ a derivation, so that $\delta(ab) = \lambda(a)\delta(b) + \delta(a)b$ for $a, b \in \mathcal{A}_0$.

Lemma 5.1. *If the domain of the closure of λ is a full subalgebra of A, and if δ is closable, then the map $\pi : \mathcal{A}_0 \to \mathcal{L}_A(\mathcal{E}) \oplus \mathcal{E}$ given by $\pi(a) = \lambda(a) \oplus \delta(a)$ is closable and the domain $\mathcal{A}$ of its closure is a full subalgebra of A.*

Proof. The map π is closable as λ and δ are, and $\mathcal{A}$, the domain of the closure, is a subalgebra of A. Let $b \in \mathcal{A}$ invertible in A and let us show that $b^{-1} \in \mathcal{A}$. As $\mathcal{A}$ is contained in the domain of the closure of λ, which is full by hypothesis, then b^{-1} belongs also to that domain. Then for any $\epsilon > 0$, there exists $a \in \mathcal{A}_0$ such that $\|a - b^{-1}\| + \|\lambda(a) - \lambda(b^{-1})\| < \epsilon$, with the norms of A and $\mathcal{L}_A(\mathcal{E})$. With $v = 1 - ab$, and choosing ϵ sufficiently small, then $\max\{\|v\|, \|\lambda(v)\|\} < 1$. As $v \in \mathrm{dom}\delta$, we may apply δ to v^k and $\delta v^k = \sum_j \lambda(v^{j-1})(\delta v)v^{k-j}$. This shows that the series $\sum \delta v^k$ converges absolutely, and thus $(1 - v)^{-1} \in \mathcal{A}$, from which we deduce that $b^{-1} = (1 - v)^{-1}a$ is in $\mathcal{A}$. ∎

6. The cotangent C*-module. Let as before Γ be a discrete countable group acting on the right by diffeomorphisms on $V = S^1$, $\mathcal{A}_0 = C_c(V \times \Gamma)$ and $A = C(V) \rtimes \Gamma$ the reduced crossed product. Let $l = \log \delta$ be the logarithm of the modulus of the action of Γ with respect to the Lebesgue measure dx, i.e. $\gamma_* dx = \delta(x, \gamma)dx$; then l is a continuous homomorphism from the groupoid $V \rtimes \Gamma$ to $\mathbb{R}$ and $\mathbb{R}$ acts on A by

$$\alpha_t(f)(x,\gamma) = \delta^{it} f(x,\gamma).$$

Besides, we have an action of Γ on $W = V \times \mathbb{R}$ given by $(x,\lambda)\gamma = (x\gamma, \lambda + l(x,\gamma))$ for $(x,\lambda,\gamma) \in V \times \mathbb{R} \times \Gamma$. Let B be the crossed product $C(V \times \mathbb{R}) \rtimes \Gamma$ of this groupoid [1]. The Fourier transform on $\mathbb{R}$ implements an isomorphism of the crossed product $(C(V) \rtimes \Gamma) \rtimes_\alpha \mathbb{R}$ with B.

On W, there is a Γ-invariant smooth measure $\mu = \exp(\lambda) dx d\lambda$, where dx and $d\lambda$ are the Lebesgue measures on S^1 and $\mathbb{R}$. Thus there is a semi-finite trace τ on B characterized by $\tau(a^*a) = \int_W |a(x,\lambda,e)|^2 d\mu(x,\lambda)$ when $a \in C_c(W \times \Gamma)$.

Let $L^1(B,\tau)$ be the noncommutative L^1-space of τ, which is the completion of $\mathcal{B}_0 = C_c(W \times \Gamma)$ for the norm $\|a\|_1 = \tau(|a|)$. This is a Banach space and a bimodule over B.

On T^*W, the cotangent bundle, we choose the riemannian metric, for $X \in T^*_x V$ and $\Lambda \in T^*_\lambda \mathbb{R}$:

$$\|(X,\Lambda)\|^2_{(x,\lambda)} = \exp(-2\lambda) X^2 + \Lambda^2. \tag{6}$$

This metric is not invariant by Γ, but the action is almost isometric [C, Definition 3.3], as here $\gamma^* : T^*_{(x,\lambda)\gamma} W \to T^*_{(x,\lambda)} W$ is an upper triangular matrix, with diagonal entries equal to 1.

Recall (subsection 5.1) that we can form a B-Hilbert module $\mathcal{E}$ by completion of $\mathcal{E}_0 = C_c(W \times \Gamma, r^* T^* W)$. We denote the norm on $\mathcal{E}$ by:

$$\|\xi\|^{\mathcal{E}}_\infty = \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}_B.$$

We introduce now the analogue of the noncommutative L^1-space for $\mathcal{E}$: for $\xi \in \mathcal{E}_0$, let:

$$\|\xi\|^{\mathcal{E}}_1 = \tau(\langle \xi, \xi \rangle^{\frac{1}{2}}).$$

LEMMA 6.1. *For $\xi \in \mathcal{E}_0$, the following equality holds true:*

$$\|\xi\|^{\mathcal{E}}_1 = \sup\{\tau(\langle \eta, \xi \rangle); \eta \in \mathcal{E}_0, \|\eta\|^{\mathcal{E}}_\infty \leq 1\}.$$

Proof. Let ξ, η as in the lemma. By a well-known property of Hilbert modules, there is an inequality: $|\langle \eta, \xi \rangle| \leq \|\eta\|^{\mathcal{E}}_\infty \langle \xi, \xi \rangle^{\frac{1}{2}}$, which shows that the left hand side of the claimed equality is less than or equal to the right hand side.

Conversely, let $a_\varepsilon = (\varepsilon + \langle \xi, \xi \rangle^{\frac{1}{2}})^{-1} \in B$. Then, for any $\alpha > 0$, there exists $b_{\varepsilon,\alpha} \in \mathcal{A}_0$ such that $\|b_{\varepsilon,\alpha} - a_\varepsilon\|_B \leq \alpha$. Let $\eta_{\varepsilon,\alpha} = \xi * b_\alpha$, then when α, ε are sufficiently small, $\|\eta_{\varepsilon,\alpha}\|^{\mathcal{E}}_\infty \leq 1$ and $\lim_{\varepsilon,\alpha \to 0} \tau(\langle \eta_{\varepsilon,\alpha}, \xi \rangle) = \tau(\langle \xi, \xi \rangle^{\frac{1}{2}})$. ∎

As a consequence of this lemma, we get readily:

PROPOSITION 6.2. *The map $\xi \to \|\xi\|^{\mathcal{E}}_1$ on $\mathcal{E}_0$ is a norm, and the completion of $\mathcal{E}_0$, denoted by $L^1(\mathcal{E},\tau)$, is a Banach space and a right module over B. Moreover, for $\xi \in L^1(\mathcal{E},\tau)$, the following equality holds true:*

$$\|\xi\|^{\mathcal{E}}_1 = \sup\{\tau(\langle \eta, \xi \rangle); \eta \in \mathcal{E}, \|\eta\|_{\mathcal{E}} \leq 1\}.$$

[1] The reduced and maximal crossed products are equal.

Similarly to proposition 3.4 of [C], the action λ_0 of $\mathcal{B}_0$ on $\mathcal{E}_0$ is closable and the domain of its closure λ_1 is a full Banach subalgebra of B.

Let $E_\gamma : \mathcal{E}_0 \to C_c(W, T^*W)$ be the evaluation map at $\gamma \in \Gamma$; thus for $\xi \in \mathcal{E}_0$, $E_\gamma(\xi)(x,\lambda) = \xi(x,\lambda,\gamma)$. Let $L^1(W, T^*W)$ the Banach space of Lebesgue measurable sections ξ of T^*W, such that, with the euclidean norm (6) and the invariant measure above:

$$\int_W \|\xi(z)\|_z d\mu(z) < +\infty.$$

Lemma 6.3. *The linear map E_γ extends to a continuous norm one linear map $L^1(\mathcal{E},\tau) \to L^1(W, T^*W)$. Let ψ be a bounded Borel complex valued function on S^1. Then the linear form on $\mathcal{E}_0$, $\xi \to \int_W \xi(x,\lambda,e)\psi(x) \wedge d\lambda$ extends to a continuous linear form on $L^1(\mathcal{E},\tau)$.*

Proof. Let $\gamma \in \Gamma$, $z = (x,\lambda) \in W$, and $\eta^0, \eta^1 \in C_c(W)$ such that:

$$|\eta^0(z)|^2 \exp(-2\lambda) + |\eta^1(z)|^2 \leq 1$$

and let $\eta(z,\beta) \in \mathcal{E}_0$ equal to $\eta^0(z\gamma)dx + \eta^1(z\gamma)d\lambda$ if $\beta = \gamma$ and equal to 0 if $\beta \neq \gamma$. We evaluate $\|\langle \eta,\eta\rangle\|_B$ (e being the neutral element of Γ):

$$\langle \eta,\eta\rangle(z,\beta) = \begin{cases} |\eta^0(z)|^2 \exp(-2\lambda) + |\eta^1(z)|^2 & \text{if } \beta = e, \\ 0 & \text{if } \beta \neq e. \end{cases}$$

Thus $\|\langle \eta,\eta\rangle\|_B = \sup_z |\eta^0(z)|^2 \exp(-2\lambda) + |\eta^1(z)|^2 \leq 1$. For any $\xi \in \mathcal{E}_0$, $\langle \eta,\xi\rangle(z,e) = \eta^0(z)\exp(-2\lambda)\xi^0(z\gamma,\gamma) + \eta^1(z)\xi^1(z\gamma,\gamma)$. Then, as the measure is Γ-invariant, and with the following supremum taken over all couples (η^0,η^1) as above, the Cauchy-Schwarz inequality implies:

$$\| \sup_{\eta^0,\eta^1} \tau(\langle \eta,\xi\rangle)\| = \|E_\gamma(\xi)\|_1.$$

But the left hand side of the last equality is less than or equal to $\|\xi\|_1^{\mathcal{E}}$, by lemma 6.1.

For the last assertion, let $\xi = \xi^0 dx + \xi^1 d\lambda \in L^1(W, T^*W)$. Then as $T^*W = T^*V \oplus T^*\mathbb{R}$, we have $\|\xi\|_1 = \int_W \{\|\xi^0\|^2 + \|\xi_1\|^2 \exp(2\lambda)\}^{\frac{1}{2}} dx d\lambda$, and thus the linear form $\xi \to \int_W \psi E_e(\xi) \wedge d\lambda$ is well defined and continuous. ■

Let S_0 be the linear map densely defined on $\mathcal{E}_0$ for $\xi \in C_c^1(W \times \Gamma, r^*T^*W)$ with values in $L^1(\mathcal{E},\tau)$ such that $E_\gamma(S_0\xi) = dE_\gamma(\xi)$ (d is exterior derivative on W).

Proposition 6.4. *The map S_0 is closable.*

Proof. The map from $C_c^\infty(W) \to L^1(W, T^*W)$, $f \to df$ is closable in $C_0(W)$ and the domain of its closure is an algebra the elements of which are C^1-functions on W. Let $\xi_k \in C_c^\infty(W \times \Gamma)$ be a sequence which tends to zero in B and such that $S_0\xi_k$ converges in $L^1(\mathcal{E},\tau)$. For every $\gamma \in \Gamma$, $E_\gamma(\xi_k)$ converges to 0, and thus $\|S_0\xi_k\|_1^{\mathcal{E}}$ goes to zero too. ■

7. 1-traces. Let A be a C*-algebra and $\mathcal{A}_0 \subset A$ a dense involutive subalgebra. A cyclic 1-cocycle ϕ on $\mathcal{A}_0$ is a bilinear antisymmetric map and it gives rise to an additive morphism from $K_1(\mathcal{A}_0)$ to $\mathbb{C}$.

A 1-trace on $\mathcal{A}_0$ is a cyclic 1-cocycle ϕ such that there exists a constant $C > 0$ such that for $f^0, f^1 \in \mathcal{A}_0$, the following inequality holds (where the norm on the right side is the C*-norm):

$$|\phi(f^0, f^1)| \leq C\|f^0\|.$$

There is then a derivation $\kappa : \mathcal{A}_0 \to A^*$, the dual of A, such that:

$$\phi(f^0, f^1) = \langle f^0, \kappa(f^1)\rangle = -\langle \kappa(f^0), f^1\rangle.$$

The derivation $\kappa : \mathcal{A}_0 \to A^*$ considered as an unbounded operator $A \to A^*$ is closable and the domain of its closure is a dense subalgebra of A stable by holomorphic functional calculus. Thus ϕ extends to a one cyclic cocycle on $\mathcal{A}$ and thus to an additive map $K_1(\mathcal{A}) \cong K_1(A) \to \mathbb{C}$.

Remark 7.1. If $\mathcal{A}_1$ with $\mathcal{A}_0 \subset \mathcal{A}_1$ is also a full Banach subalgebra to which the 1-trace extends, then the K-theory map determined on $K_1(A)$ is the same. Indeed the intersection $\mathcal{A}_0 \cap \mathcal{A}_1$ is a full Banach algebra, and has the same K-theory as $\mathcal{A}, \mathcal{A}_1$ and A.

Here is an important example of a 1-trace. Let Γ be a discrete group acting on a locally compact metric space W by homeomorphisms. Let $\gamma \to \mu_\gamma$ be a 1-cocycle on Γ with values in the space of Radon measures on W. Thus, for $\gamma_1, \gamma_2 \in \Gamma$:

$$\gamma_2^* \mu_{\gamma_1} - \mu_{\gamma_1\gamma_2} + \mu_{\gamma_2} = 0$$

where $\gamma_2^*\mu_{\gamma_1}$ is the direct image of μ_{γ_1} by the homeomorphism γ_2^{-1}. Fixing $\gamma_1 = e$ to be the neutral element shows that $\mu_e = 0$, and then $\mu_{\gamma^{-1}} = -\gamma^*\mu_\gamma$.

Lemma 7.2. *The following formula, for $f^0, f^1 \in C_c(W \times \Gamma)$, defines a 1-cocycle on that algebra and a 1-trace on the crossed product algebra $C(W) \rtimes \Gamma$:*

$$\phi_\mu(f^0, f^1) = \sum_{\gamma\in\Gamma} \int_W f^0(x,\gamma) f^1(x\gamma, \gamma^{-1})\mu_{\gamma^{-1}}(x).$$

Proof. First ϕ_μ is a cyclic cochain. As ϕ_μ is clearly bilinear, it suffices to prove that $\phi_\mu(f^0, f^1) = -\phi_\mu(f^1, f^0)$. But, as $\mu(e) = 0$,

$$\begin{aligned}\phi_\mu(f^1, f^0) &= \sum_\gamma \int_W f^1(x,\gamma)) f^0(x\gamma, \gamma^{-1})\mu_{\gamma^{-1}}(x)\\ &= \sum_\gamma \int_W f^1(x\gamma^{-1},\gamma) f^0(x, \gamma^{-1})\gamma_*\mu_{\gamma^{-1}}(x)\\ &= \sum_\gamma \int_W -f^1(x\gamma,\gamma^{-1}) f^0(x, \gamma)\mu_{\gamma}(x)\\ &= -\phi_\mu(f^0, f^1).\end{aligned}$$

A similar computation shows that $b\phi_\mu = 0$, so that it is closed.

Then we prove that it is a 1-trace: let $f^0, f^1 \in C_c(W \times \Gamma)$ and let $Z \subset \Gamma$ be a finite subset which contains be the projection of the support of f^1 and such that $Z = Z^{-1}$; thus $f^1(x,\gamma) = 0$ for $\gamma \notin Z$.

Let μ be a measure on W with support equal to W. Then by the bipolar theorem, there exists for each $\gamma \in Z$ a sequence $a_n(x,\gamma)$ of Borel and μ-integrable functions, such that $a_n(.,\gamma)\mu$ converges weakly to μ_γ. Let $W_0 \subset W$ be a compact subset which contains the support of $f^1(.,\gamma)$ for every $\gamma \in Z$ and $K > 0$ be a majorant of $\int_{W_0} |a_n(x,\gamma)|\mu(x)$, for $n \geq 0, \gamma \in Z$. Then:

$$\begin{aligned}
|\phi_\mu(f^0, f^1)| &= \big|\lim_n \sum_{\gamma\in Z} \int_W f^0(x,\gamma) f^1(x\gamma, \gamma^{-1}) a_n(x,\gamma^{-1})\mu(x)\big| \\
&\leq \lim_n \int_W \|f^1\|_A \big(\sum_{\gamma\in\Gamma} |f^0(x,\gamma)a_n(x,\gamma^{-1})|^2\big)^{\frac{1}{2}} \mu(x) \\
&\leq \lim_n \|f^1\|_A \sum_{\gamma\in Z} \int_W |f^0(x,\gamma)||a_n(x,\gamma^{-1})|\mu(x) \\
&\leq \|f^1\|_A |Z| K \sup_{W\times\Gamma} |f^0(x,\gamma)|.
\end{aligned}$$

Here $|Z|$ is the cardinal of Z; thus $|\phi_\mu(f^0, f^1)| \leq K_1 \|f^0\|_A$. ■

8. A 1-trace which is a coboundary. Let as before $V = S^1$ as a smooth manifold and Γ be a discrete group of diffeomorphisms acting on the right, $A = C(V) \rtimes \Gamma$ the reduced crossed product, $\delta : V \rtimes \Gamma \to \mathbb{R}_+^*$ the modular homomorphism with respect to the Lebesgue measure, $W = V \times \mathbb{R}$ and $B = C_0(W) \rtimes \Gamma$ the crossed product.

Let ψ be a bounded Borel function such that $\gamma\psi - \psi$ is a Lipschitz function for any $\gamma \in \Gamma$. For $f^0, f^1 \in C_c^\infty(W \times \Gamma)$, let:

$$\phi_\psi(f^0, f^1) = \int_{W\times\Gamma} f^0(x,\lambda,\gamma) f^1(x\gamma, \lambda + l(x,\gamma), \gamma^{-1}) d(\gamma^{-1}\psi - \psi) \wedge d\lambda.$$

This is a 1-trace by lemma 7.2; it is a coboundary of a zero cochain on $C_c^\infty(V \times \Gamma)$, and we would like to show that it gives the zero map on K-theory. Actually, there is a full subalgebra on which it is a coboundary, this will be shown in the proof of the following claim:

Proposition 8.1. *The 1-trace ϕ_ψ implements the null map from $K_1(B)$ to $\mathbb{C}$.*

Proof. Let δ_ψ be the closable derivation with values in the dual space A^* associated to ϕ_ψ. Let $\mathcal{B} \subset B$ be the domain of the closure of the map $\lambda \oplus \delta_\psi \oplus S_0$ on $C_c^1(W \times \Gamma)$ where S_0 is defined in lemma 6.4. It is a full Banach subalgebra as in lemma 5.1.

As the canonical morphism $K_j(\mathcal{B}) \to K_j(B)$ is an isomorphism, it suffices to prove that ϕ_ψ is the zero map on the K-theory of $\mathcal{B}$. But on $\mathcal{B}$ with the graph norm, the following 0-cocycle is well defined and continuous, by lemma 6.3:

$$\kappa(f) = \int_W df(x,\lambda,e)\psi(x) d\lambda = -\int_W f(x,\lambda,e) d\psi \wedge d\lambda.$$

Actually, a more precise formulation is $\kappa(f) = \int_W Sf(x,\lambda,e)\psi(x)d\lambda$, where S is the restriction to $\mathcal{B}$ of the closure of the map S_0 of lemma 6.4.

Here $d\psi$ is a current, the distributional derivative of ψ. We compute $b\kappa(f^0, f^1) = \kappa(f^0 f^1) - \kappa(f^1 f^0)$ for $f^0, f^1 \in C_c^\infty(W \times \Gamma)$; then:

$$\begin{aligned}
-\kappa(f^1 f^0) &= \sum_\gamma \int_W f^1(x,\lambda,\gamma) df^0(x\gamma, \lambda + l(x,\gamma), \gamma^{-1})\psi(x) d\lambda \\
&\quad + \sum_\gamma \int_W df^1(x,\lambda,\gamma) f^0(x\gamma, \lambda + l(x,\gamma), \gamma^{-1})\psi(x) d\lambda.
\end{aligned}$$

As it is a finite sum, we may intertwine the summation with the integral, and the first term of the right hand side reads

$$\sum_\gamma \int_W f^1(x\gamma^{-1},\lambda,\gamma)df^0(x,-l(x,\gamma^{-1})+\lambda,\gamma^{-1})\gamma^{-1}\psi(x)d\lambda$$
$$= \sum_\gamma \int_W df^0(x,\lambda,\gamma)f^1(x\gamma,\lambda+l(x,\gamma),\gamma^{-1})\gamma^{-1}\psi(x)d\lambda$$

and similarly for the second term. As $\gamma\psi-\psi$ is Lipschitz, its distributional derivative is a bounded Borel function, and we get eventually the following equality:

$$\kappa(f^0f^1)-\kappa(f^1f^0)=\phi_\psi(f^0,f^1).$$

This equality extends by continuity to all of $\mathcal{B}$: for any $f^j\in\mathcal{B}$, $j=0,1$, there exist a sequence $f^j_k\in C_c^\infty(W\times\Gamma)$ such that f^j_k converges to f^j in B, $\lambda(f^j_k)$ converges to $\lambda(f^j)$ in $\mathcal{L}_A(L^1(\mathcal{E},\tau))$, Sf^j_k converges to Sf^j in $L^1(\mathcal{E},\tau)$, and $\delta_\psi f^j_k$ converges to $\delta_\psi f^j$ in A^*.

Then, for $k,l=0,1$, $\lim_k f^j_kf^l_k=f^jf^l$ in $\mathcal{B}$, and $\kappa(f^j_kf^l_k)\to\kappa(f^jf^l)$ and $\phi_\psi(f^0_k,f^1_k)\to\phi_\psi(f^0,f^1)$.

We have checked the equality $b\kappa=\phi_\psi$ on all $\mathcal{B}$, and this shows that the map ϕ_ψ on the K-theory of $\mathcal{B}$, and hence on that of A, is identically 0. ∎

9. Bilipschitz invariance. Let $V_1=V_2=S^1$, and α_j be a smooth action of Γ on V_j and $T:V_1\to V_2$ be a Γ-equivariant map. Let $\rho_j:K_0(C(V_j)\rtimes_{\alpha_j}\Gamma)\to\mathbb{C}$ be the analytic Godbillon-Vey map associated to the Bott-Thurston cocycle as in section 3.

THEOREM 9.1. *With these notations, if T is a bilipschitz homeomorphism, then the Godbillon-Vey map is preserved by T^*: for every $x\in K_0(C(V_1)\rtimes_{\alpha_1}\Gamma)$, the equality $\rho_2(T^*x)=\rho_1(x)$ holds true.*

In this section we give the proof of it.

9.1. The C*-algebra crossed product associated to a invariant class of measure. Let $L^\infty(V_j)$ be the von Neumann algebra of (equivalence classes) of essentially bounded measurable functions for the Lebesgue class and $L^\infty(V_j)\rtimes\Gamma$ the *reduced C* crossed product.* Then Γ acts on the C*-tensor product $L^\infty(V_j)\otimes C_0(\mathbb{R})$ by $\gamma f(x,\lambda)=f(x\gamma,l_j(x,\gamma)+\lambda)$, and let $L^\infty(V_j)\otimes C_0(\mathbb{R})\rtimes\Gamma$ the well defined C* crossed product.

The Thom isomorphism gives, with $W_j=V_j\times\mathbb{R}$:

$$\theta_j:K_0(C(V_j)\rtimes\Gamma)\to K_1(C_0(W_j)\rtimes\Gamma), \tag{7}$$
$$\bar\theta_j:K_0(L^\infty(V_j)\rtimes\Gamma)\to K_1(L^\infty(V_j)\otimes C_0(\mathbb{R})\rtimes\Gamma). \tag{8}$$

Let $\kappa_j:C(V_j)\rtimes\Gamma\to L^\infty(V_j)\rtimes\Gamma$ and $\lambda_j:C(W_j)\rtimes\Gamma\to L^\infty(V_j)\otimes C_0(\mathbb{R})\rtimes\Gamma)$ be the canonical injections. Then for $j=1,2$:

$$\lambda_j\circ\theta_j=\bar\theta_j\circ\kappa_j. \tag{9}$$

For $j=1,2$, $\gamma\in\Gamma\to dl_j(x,\gamma)\wedge d\lambda$ is a group 1-cocycle with values in the Γ-module of differential forms on $W_j=V_j\times\mathbb{R}$. Then by lemma 7.2, it is a 1-trace on $C_0(W_j)\rtimes\Gamma$ and determines a scalar valued additive map ϕ_j of $K_1(C_0(W_j)\rtimes\Gamma)$. Then, with ρ_j the additive map of $K_0(C(V_j) \rtimes \Gamma)$ given by the Bott-Thurston cocycle, then by [C], the two maps are related as follows:

$$\phi_j \circ \theta_j = \rho_j. \tag{10}$$

Viewing volume forms on W_j as Radon measures absolutely continuous with respect to the Lebesgue class of measure, the cocycle above is also a 1-trace on $L^\infty(V_j) \otimes C_0(\mathbb{R}) \rtimes \Gamma$, by lemma 7.2. Thus an additive map exists $\bar{\phi}_j : K_1(L^\infty(V_j) \otimes C_0(\mathbb{R}) \rtimes \Gamma) \to \mathbb{C}$, and the latter factorizes the former:

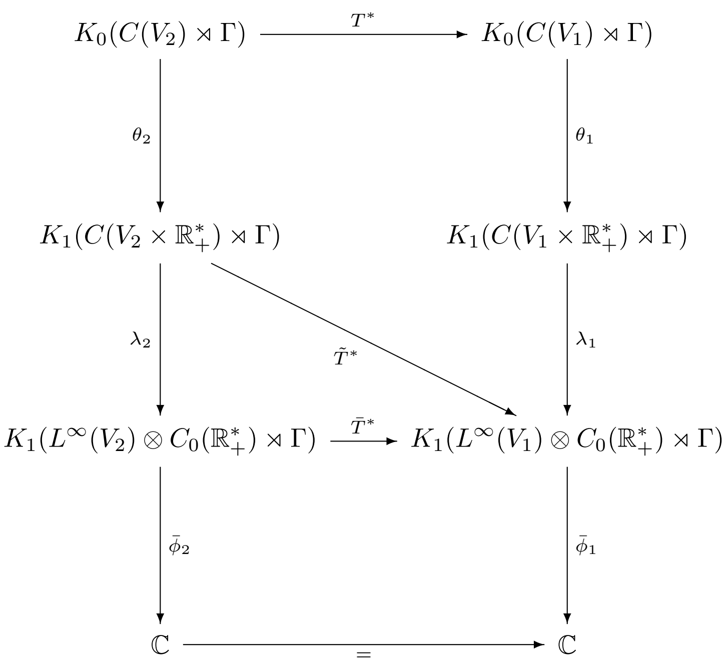
$$\phi_j = \bar{\phi}_j \circ \lambda_j.$$

Let μ_j be the Lebesgue measure on V_j. By Rademacher theorem, there exists a bounded Borel function ψ such that $T^*\mu_2 = \exp(\psi)\mu_1$. Let $\tilde{T} : W_1 \to W_2$ be given by $\tilde{T}(x, \lambda) = (Tx, \psi(x) + \lambda)$.

LEMMA 9.2. *The map $\tilde{T}$ implements $\tilde{T}^* : C_0(W_2) \rtimes \Gamma \to L^\infty(V_1) \otimes C_0(\mathbb{R}) \rtimes \Gamma$, and again $\tilde{T}^* : L^\infty(V_2) \otimes C_0(\mathbb{R}) \rtimes \Gamma) \to L^\infty(V_1) \otimes C_0(\mathbb{R}) \rtimes \Gamma$.*

Proof. The map $\tilde{T}$ implements a continuous homomorphism $L^\infty(V_2) \otimes C_0(\mathbb{R}) \to L^\infty(V_1) \otimes C_0(\mathbb{R})$. To see this, recall that $L^\infty(V_j) = C(K)$ where K is the spectrum of that C*-algebra. Then ψ is the restriction to V_1 of a continuous map $\bar{\psi}$ on K. Moreover, as T is absolutely continuous, it extends to a continuous map $\bar{T}$ of K. Thus we can view $\tilde{T}^*$ as the homomorphism of $C(K) \otimes C_0(\mathbb{R})$ implemented by $(v, \lambda) \to (\bar{T}v, \lambda + \bar{\psi}(v))$. As the action α_j of Γ is smooth, it preserves the Lebesgue class, and thus extends to a continuous action on $L^\infty(V_j)$. These remarks allow us to define $\tilde{T}^* : \bar{B}_2 \to \bar{B}_1$ and finally $\tilde{T}^* : B_2 \to \bar{B}_1$ is obtained by the composition with the canonical injection $C(V_2) \to L^\infty(V_2)$. ∎

9.2. End of the proof of theorem 9.1. Consider the following diagram:



It is a commutative diagram. We divide it into an upper trapezoid (above the diagonal with $\tilde{T}^*$) and a lower trapezoid (below the same diagonal).

The commutativity of the upper trapezoid is clear. With ψ the logarithm of the derivative of T as above, the equality $l_2(Tx,\gamma) = l_1(x,\gamma)+\gamma^{-1}\psi-\psi$ shows that $\gamma^{-1}\psi-\psi$ is a Lipschitz function on V_1 for every $\gamma \in \Gamma$.

The maps $\bar{\phi}_1 \circ \tilde{T}^*$ and ϕ_2 on $K_1(C(V_2) \rtimes \Gamma)$ are implemented by 1-traces which differ by the 1-trace associated to $d(\gamma^{-1}\psi - \psi)$. Thus they are equal by proposition 8.1.

The commutativity of the bottom trapezoid follows.

Theorem 9.1 is now a mere consequence of the commutativity of this diagram and of formula (10).

10. The measure on the flow of weights. Let as before Γ act on S^1 by oriented diffeomorphisms and $A = C(S^1) \rtimes \Gamma$ the reduced C*-crossed product and $B = C(S^1 \times \mathbb{R}) \rtimes \Gamma$. Recall that the flow of weights is the center $\mathcal{Z}$ of the von Neumman algebra crossed product $\bar{B} = L^\infty(S^1 \times \mathbb{R}) \rtimes \Gamma$, together with a dual action of $\hat{\mathbb{R}} \simeq \mathbb{R}$.

Let δ be the derivation coming with the 1-trace ϕ defined previously. As shown in [C], the domain of the bitranspose of the derivation δ restricted to the von Neumann crossed product $\bar{B}$ contains $\mathcal{Z}$, and, for every $z \in \mathcal{Z}$, $\delta(z) = 0$.

In particular, for every $z \in \mathcal{Z}$, the map on domδ, $(a^0, a^1) \to \phi(a^0, za^1)$ is a 1-trace and extends to a map $\phi_z : K_1(B) \to \mathbb{C}$.

Thus, by Thom isomorphism $\theta : K_0(A) \to K_1(B)$, there is a normal additive map ρ on $\mathcal{Z}$ with values the group of additive maps on $K_0(A)$. Let now α_j be a smooth action of Γ on S^1 for $j = 1, 2$, and $\mathcal{Z}_j$ the center of $L^\infty(S^1 \times \mathbb{R}) \rtimes_{\alpha_j} \Gamma$. Let $T : S^1 \to S^1$ be a bilipschitz homeomorphism of S^1 conjugating the two actions. Then, as T preserves the Lebesgue class of measure, it gives an isomorphism of $\mathcal{Z}_2$ with $\mathcal{Z}_1$. Let for $z \in \mathcal{Z}_j$ $\rho_{j,z}$ be the additive map of $K_0(C(S^1) \rtimes_{\alpha_j} \Gamma)$ previously defined.

THEOREM 10.1. *For every $z \in \mathcal{Z}_2$, the following equality between additive maps of $K_0(C(S^1) \rtimes \Gamma)$ holds true:*

$$\rho_{1,T^*z} \circ T^* = \rho_{2,z}.$$

Proof. Let ψ be a bounded measurable function on S^1 such that $\gamma\psi - \psi$ is lipschitz for every $\gamma \in \Gamma$ and z be a bounded Borel function on $S^1 \times \mathbb{R}$ invariant under the action of Γ. Then the group 1-cocycle $zd(\gamma^{-1}\psi - \psi) \wedge d\lambda$ implements the null map on $K_1(C(W) \rtimes \Gamma)$ in the same way as in proposition 8.1, and thus the claim follows as in the proof of theorem 9.1. ∎

10.1. The Duminy measure. Let F be a codimension one oriented smooth foliation on the dimension three oriented manifold M_0. If ω is a transverse 1-form defining F and θ a degree one form such that $d\omega = \theta \wedge \omega$, then G. Duminy, and later S. Hurder and J. Heitsch [G, CC], have shown that for every Borel subset saturated by F (i.e. union of leaves), the integral:

$$\int_{M_0} \chi_B \theta \wedge d\theta$$

where χ_B is the characteristic function of B, does not depend upon the choice of ω, θ. In particular, the Godbillon-Vey invariant is obtained with $B = M_0$. Thus we have a σ-additive measure ν on the σ-algebra of Borel saturated subsets of M_0.

Let Γ be a discrete group acting on S^1 by oriented diffeomorphisms and satisfying the hypothesis of section 3.

Let $\alpha : M \to B\Gamma$ be a continuous map of the oriented manifold M to the classifying space of Γ. Then, we get a homomorphism still denoted $\alpha : \pi_1(M) \to \Gamma$ and thus a foliated bundle (M_α, F_α). The holonomy groupoid of the foliation is Morita equivalent to $S^1 \rtimes \Gamma$, and the (reduced) C*-algebra of this foliation is Morita equivalent to the (reduced) crossed product $C(S^1) \rtimes_\alpha \Gamma$.

Every F_α-saturated Borel subset B_α corresponds to a Γ-invariant Borel subset B of S^1. Still, $B \times \mathbb{R}$ is a Γ-invariant Borel subset of $W = S^1 \times \mathbb{R}$, and thus its characteristic function is an element of $\mathcal{Z}$, the flow of weights of the action of Γ.

With the previous notations, the following equality holds true:

$$\int_{B_\alpha} d\nu = \phi_{B\times\mathbb{R}}(1). \tag{11}$$

Let α_j be two smooth actions of Γ on S^1, (M_j, F_j) the associated foliated bundles. Let $\tilde{T} : S^1 \to S^1$ be an orientation preserving homeomorphism such that $\tilde{T}\alpha_1(\gamma) : \alpha_2(\gamma)\tilde{T}$: then it defines a foliated homeomorphism $T : M_1 \to M_2$.

Thus as a consequence of equality (11), we have proven:

Corollary 10.2. *Let ν_j be the Duminy measure on (M_j, F_j). If $\tilde{T}$ is a bilipschitz conjugation between α_1 and α_2, then the following equality holds true:*

$$T_* \nu_1 = \nu_2.$$

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