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POSITIVE LINEAR MAPS OF MATRIX ALGEBRAS

W. A. MAJEWSKI

Institute of Theoretical Physics and Astrophysics, Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail: fizwam@univ.gda.pl

Dedicated to Professor S. L. Woronowicz, on the occasion of his 70-th birthday

Abstract. A characterization of the structure of positive maps is presented. This sheds some more light on the old open problem studied both in Quantum Information and Operator Algebras. Our arguments are based on the concept of exposed points, links between tensor products and mapping spaces and convex analysis.

1. Introduction. Physicists are accustomed to pass freely from the Heisenberg picture to the Schrödinger picture and vice versa. As it is well known, the Heisenberg picture deals with observables while the Schrödinger picture concentrates on states, and both pictures are fitted in a dual pair. Time evolutions provided by the pictures are equivalent, and to formulate laws of dynamics within the Heisenberg picture which are compatible with the duality, dynamical maps should be described by positive, continuous, unital maps.

Consequently, the concept of continuous, unital, positive maps is at the heart of mathematical foundations of Quantum Theory, and therefore a characterization of the structure of this set is of paramount importance for both Quantum Mechanics and Quantum Information. The important point to note here is our assumption on continuity and normalization of positive maps. Both assumptions are indispensable when one studies dynamical maps (in Quantum Mechanics) or in an analysis of states (in Quantum Informations where one considers a composition of a given state with a positive map).

The intention of our lecture is twofold. First, we give a brief exposition of our recent results on the structure of the set of positive maps. Thus we will frequently quote our results from [15]. Second, we want to indicate that the given characterization is working nicely, i.e. to show by examples how the general theory can be applied to very concrete models. In that way we will get a better understanding of the dramatic difference between

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two dimensional and three dimensional cases. It is worth reminding that in 3D case, the so called non-decomposable maps appear [9], [27]); see also [8], [7], [26], and [16].

Let us outline the plan of our lecture. Since we will use the algebraic approach to Quantum Physics we will include, for convenient reference, the introductory Section 2. In particular, consequences of the Grothendieck result (see [12]) for maps on algebras will be presented. Our characterization of the set of positive maps will be reviewed in Section 3. Section 4 contains examples indicating the origin of non-decomposable maps in three dimensional case.

The detailed results appeared in [15], so we will frequently omit the proofs and put emphasis on the basic ideas.

2. Preliminaries. In this section we set up notation, terminology and review some of the standard facts on positive maps. Let $\mathcal{A}$ be a C^*-algebra. $\mathcal{A}^+$ will denote the set of all positive elements of $\mathcal{A}$. If $\mathcal{A}$ is a unital C^*-algebra then a *state* on $\mathcal{A}$ is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) \geq 0$ for every $A \in \mathcal{A}^+$ and $\phi(\mathbb{I}) = \mathbb{I}$ where $\mathbb{I}$ is the unit of $\mathcal{A}$. We denote the set of all states on $\mathcal{A}$ by $\mathcal{S}_{\mathcal{A}}$.

A linear map $T : \mathcal{A}_1 \to \mathcal{A}_2$ between C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ will be called *positive* if $T(\mathcal{A}_1^+) \subset \mathcal{A}_2^+$. If $k \in \mathbf{N}$, then one can consider a map $T_k : M_k(\mathbb{C}) \otimes \mathcal{A}_1 \to M_k(\mathbb{C}) \otimes \mathcal{A}_2$ where $M_k(\mathbb{C})$ denotes the algebra of $k\times k$-matrices with complex entries and $T_k = \mathbb{I}_{M_k} \otimes T$. We say that T is *k-positive* if the map T_k is positive. Finally, the map T is said to be *completely positive* when T is k-positive for every $k \in \mathbb{N}$. For any Hilbert space $\mathcal{H}$, we denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. The canonical form of any completely positive map $T : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is (see [21] and [10])

$$T(a) = W^*\pi(a)W, \qquad a \in \mathcal{A}, \tag{1}$$

where $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-morphism while $W : \mathcal{K} \to \mathcal{H}$ is a linear bounded map.

Let us fix an orthonormal basis $\{e_i\}_{i=1}^n$ in the space $\mathcal{H}$, where $n = \dim \mathcal{H}$. By $\tau_{\mathcal{H}}$, we denote the transposition map on $\mathcal{B}(\mathcal{H})$, associated with the base $\{e_i\}$. Let us note that for every finite dimensional Hilbert space $\mathcal{H}$ the transposition $\tau_{\mathcal{H}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive but not completely positive (in fact it is not even 2-positive). We will write $\tau_{\mathcal{H}}(x) \equiv x^t$.

A positive map $T : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is called *decomposable* if it can be written in the form (cf. [23, 22])

$$T(a) = W^*\pi(a)W, \qquad a \in \mathcal{A}, \tag{2}$$

where $\pi(\cdot)$ is a Jordan morphism of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, while $W : \mathcal{K} \to \mathcal{H}$ is a linear bounded map. Those positive maps, which are not decomposable, are called *non-decomposable* maps.

By $\mathcal{P}$, $\mathcal{P}_{\mathrm{C}}$ and $\mathcal{P}_{\mathrm{D}}$ we will denote respectively the set of all positive, completely positive and decomposable maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$. Note that

$$\mathcal{P}_{\mathrm{C}} \subset \mathcal{P}_{\mathrm{D}} \subset \mathcal{P} \equiv \mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K}))^+$$

and the inclusions are proper unless $\dim \mathcal{H} \leq 2$ and $\dim \mathcal{K} \leq 3$ or $\dim \mathcal{H} \leq 3$ and $\dim \mathcal{K} \leq 2$.

In the sequel, we will be interested in the set of unital, continuous, positive maps, which will be denoted by $\mathcal{P}_1$. Moreover, for simplicity, *we will assume that* $\dim \mathcal{H} = n < \infty$.

The set $\mathcal{P}_1$ is convex and compact. Consequently, due to the Krein-Milman theorem, the subset of extreme points of $\mathcal{P}_1$ is a dense subset. However, the program of finding all extremal points in $\mathcal{P}_1$, i.e. all extremal positive normalized maps, seems to be too difficult. Therefore, we turn to a special subset of extremal positive maps. To this end we remind the concept of exposed points. Namely:

DEFINITION 2.1. Let C be a convex set in a Banach space X. A point $x \in C$ is an exposed point of C ($x \in \mathrm{Exp}\{C\}$) if there is $f \in X^*$ (dual of X) such that f attains its maximum on C at x and only at x.

In other words, we wish to have $\langle f, x\rangle > \langle f, y\rangle$ for $y \in C \setminus \{x\}$ (where $\langle f, x\rangle \equiv f(x)$). Thus, the concept of exposed point is related to a variational principle. In general, one has $\mathrm{Ext}\{C\} \supseteq \mathrm{Exp}\{C\}$ but there are simple examples of 2-dimensional convex compact sets such that the inclusion $\mathrm{Ext}\{C\} \supset \mathrm{Exp}\{C\}$ is proper (see [11]).

Our interest in exposed points stems from the following result (see [25], [14] and [11])

PROPOSITION 2.2. *Every norm-compact convex set C in a Banach space X is the closed convex hull of its exposed points.*

To proceed with the analysis of exposed points of $\mathcal{P}_1$ we recall some elementary facts from the theory of tensor product of Banach spaces. We put a special emphasis on the Grothendieck result on the projective tensor product. Denote by $\mathfrak{B}(X \times Y)$ the Banach space of bounded bilinear mappings B from $X \times Y$ into the field of scalars with the norm given by $\|B\| = \sup\{|B(x,y)|; \|x\| \le 1, \|y\| \le 1\}$. Note (for all details see [20]) that there is an operator $L_B \in \mathcal{L}(X, Y^*)$ associated with each bounded bilinear form $B \in \mathfrak{B}(X \times Y)$. It is defined by $\langle y, L_B(x)\rangle = B(x,y)$. The mapping $B \mapsto L_B$ is an isometric isomorphism between the spaces $\mathfrak{B}(X \times Y)$ and $\mathcal{L}(X, Y^*)$. Hence, there is an identification

$$(X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*), \tag{3}$$

such that the action of an operator $S : X \to Y^*$ as a linear functional on $X \otimes_\pi Y$ is given by

$$\Big\langle \sum_{i=1}^n x_i \otimes y_i, S \Big\rangle = \sum_{i=1}^n \langle y_i, S x_i\rangle. \tag{4}$$

Note that the identification (3) and the relation (4) determine the linear duality which is required for the definition of exposed points of $\mathcal{L}^+(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$.

$\mathcal{B}(\mathcal{H})$ equipped with the trace norm will be denoted by $\mathfrak{T}$ (we have assumed that $\dim \mathcal{H} = n < \infty$!). Finally, we denote by $\mathcal{B}(\mathcal{H}) \odot \mathfrak{T}$ the algebraic tensor product of $\mathcal{B}(\mathcal{H})$ and $\mathfrak{T}$ and $\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T}$ means its Banach space closure under the projective norm defined by

$$\pi(x) = \inf\Big\{ \sum_{i=1}^n \|a_i\| \, \|b_i\|_1 : x = \sum_{i=1}^n a_i \otimes b_i, \ a_i \in \mathcal{B}(\mathcal{H}), \ b_i \in \mathfrak{T} \Big\}, \tag{5}$$

where $\|\cdot\|_1$ stands for the trace norm. Now, we can quote (see [22]).

Lemma 2.3. *There is an isometric isomorphism* $\phi \mapsto \tilde{\phi}$ *between* $\mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ *and* $(\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T})^*$ *given by*

$$(\tilde{\phi})\Big(\sum_{i=1}^n a_i \otimes b_i\Big) = \sum_{i=1}^n \operatorname{Tr}(\phi(a_i) b_i^t), \tag{6}$$

where $\sum_{i=1}^n a_i \otimes b_i \in \mathfrak{A} \odot \mathfrak{T}$. *Furthermore,* $\phi \in \mathcal{L}^+(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ *if and only if* $\tilde{\phi}$ *is positive on* $\mathcal{B}(\mathcal{H})^+ \otimes_\pi \mathfrak{T}^+$.

Remark 2.4. As we are interested in continuous positive maps, due to the identification (3), we are forced to consider the projective tensor product. This is the origin of the basic difference between our and standard approaches to the characterization of positive (continuous) maps.

3. Positive maps and exposed maps. Lemma 2.3 establishes the following isomorphism $\phi \mapsto \tilde{\phi}$ between $\mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})) \equiv \mathcal{L}(\mathcal{B}(\mathcal{H}))$ and $(\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T})^*$ such that $\phi \in \mathcal{L}^+(\mathcal{B}(\mathcal{H}))$ if and only if $\tilde{\phi}$ is positive on $\mathcal{B}(\mathcal{H})^+ \otimes_\pi \mathfrak{T}^+$. Further, note that identifying the real algebraic tensor product $\mathcal{B}(\mathcal{H})_h \odot \mathfrak{T}_h$ of self-adjoint parts of $\mathcal{B}(\mathcal{H})$ and $\mathfrak{T}$ respectively, with a real subspace of $\mathcal{B}(\mathcal{H}) \odot \mathfrak{T}$, one has $\mathcal{B}(\mathcal{H})_h \odot \mathfrak{T}_h = (\mathcal{B}(\mathcal{H}) \odot \mathfrak{T})_h$. Obviously, this can be extended for the corresponding closures. From now on, we will use these identifications and we will study certain subsets of real tensor product spaces.

The next easy observation says that the discussed isomorphism sends the set $\operatorname{Exp}\{\mathcal{L}^+(\mathcal{B}(\mathcal{H}))\}$ onto the set $\operatorname{Exp}\{(\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T})^{*,+}\}$, where $(\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T})^{*,+}$ stands for functionals on $\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T}$ which are positive on $\mathcal{B}(\mathcal{H})^+ \otimes_\pi \mathfrak{T}^+$.

Therefore, our task can be reduced to a study of exposed points of the last set. Let us elaborate upon this point. Any (linear, bounded) functional in $(\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T})^{*,+}$ is of the form

$$\varphi(x \otimes y) = \operatorname{Tr} \varrho_\varphi \, x \otimes y, \tag{7}$$

with ϱ_φ being a "density" matrix satisfying the following positivity condition (frequently called "block-positivity", and denoted "bp" for short)

$$\varrho_\varphi \geq_{bp} 0 \quad \Leftrightarrow \quad (f \otimes g, \varrho_\varphi f \otimes g) \geq 0 \tag{8}$$

for any $f, g \in \mathcal{H}$.

To take into account that the isomorphism given in Lemma 2.3 is also isometric, note that $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ is equipped with the Banach space operator norm $\|\cdot\|$. On the other hand, formula (5) defines the cross - norm, which is not smaller than max $\mathbf{C}^*$-norm. To be more precise, we need

Definition 3.1 (see [15]).

$$\alpha(\varrho_\varphi) = \sup_{0 \neq a \in \mathcal{B}(\mathcal{H}) \otimes_\pi \mathcal{B}(\mathcal{H})} \frac{|\operatorname{Tr} \varrho_\varphi a|}{\pi(a)}. \tag{9}$$

Definition 3.2 (see [15]). The set of bp normalized density matrices is defined as

$$\mathfrak{D} = \{\varrho_\phi : \alpha(\varrho_\phi) = 1,\ \varrho_\phi = \varrho_\phi^*,\ \varrho_\phi \geq_{bp} 0,\ \operatorname{Tr} \varrho_\phi = n\}. \tag{10}$$

Our first characterization of $\mathcal{P}_1$ is

Proposition 3.3 (see [15]). *1. Lemma 2.3 gives an isometric isomorphism between the convex set of unital positive maps $\mathcal{P}_1$ and the set of bp normalized density matrices $\mathfrak{D}$. This isomorphism sends exposed points of $\mathcal{P}_1$ onto exposed points of $\mathfrak{D}$.*

2. $\mathfrak{D}$ is a convex, compact set.

3. The convex hull of $Exp\mathfrak{D}$ is dense in $\mathfrak{D}$.

Remark 3.4 (see [15]). Geometrically speaking, we are using the correspondence between two "flat" subsets of balls in $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ and in the set of all self-adjoint bp-density matrices on $(\mathcal{B}(\mathcal{H}) \otimes_\pi \mathfrak{T})_h$, respectively. The interest of this remark follows from the fact that the balls considered are not so nicely shaped as the closed ball of real Euclidean 2– or 3 space (see Chapter 5 in [18] for geometrical details).

As the next step, we wish to look more closely at the structure and properties of $\mathfrak{D}$ before turning to an analysis of its exposed points. We begin with the following result which is of interest for Quantum Information:

Proposition 3.5 (see [15]). *$\mathfrak{D}$ is globally invariant with respect to the following operations:*

1. *local operations, LO for short, i.e. maps implemented by unitary operators $U : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ of the form $U = U_1 \otimes U_2$ where $U_i : \mathcal{H} \to \mathcal{H}$ is unitary, $i = 1, 2$;*
2. *partial transpositions $\tau_p = id_{\mathcal{H}} \otimes \tau : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ where τ stands for transposition.*

Remark 3.6. It is worth pointing out that τ_p is an isometry with respect to the α-norm, but τ_p is not an isometry with respect to the (standard) operator norm. To see this it is enough to note that $\tau_p(W) = n|f >< f|$, where W is a "typical" symmetry (see Example 3.10. 3 for more details). In other words, α-norm seems to be more natural for description of $\mathcal{P}_1$.

To provide the reader with interesting examples of elements of $\mathfrak{D}$ we will need

Definition 3.7 (see [4]). A self-adjoint unitary operator s is called a symmetry, i.e. $s = s^*$ and $s^2 = I$.

A self-adjoint operator s is called a partial symmetry (equivalently, e-symmetry) if s^2 is a (orthogonal) projector e.

It is an easy observation (see [4]) that each e-symmetry has the following canonical form: $s = p - q$, where p, q are orthogonal projectors, i.e. $p = \frac{1}{2}(e + s)$, $q = \frac{1}{2}(e - s)$, and $pq = 0 = qp$.

To show that a certain element is in $\mathfrak{D}$ one needs to calculate its α-norm. To do this in an easy way, we need

Lemma 3.8 (see [15]).

$$\alpha(\sigma) = \max\{|\operatorname{Tr} \sigma \cdot s \otimes p \,| : s \in \mathcal{S}(\mathcal{H}), p \in Proj^1(\mathcal{H})\}$$

where $\mathcal{S}(\mathcal{H})$ denotes the set of all symmetries in $\mathcal{B}(\mathcal{H})_h$ while $Proj^1(\mathcal{H})$ stands for the set $\{\pm|f >< f| : f \in \mathcal{H}, \|f\| = 1\}$.

Corollary 3.9 (cf. [15]). *1. Let P be a projector (on $\mathcal{H}\otimes\mathcal{H}$). Then $\alpha(P) = \|\mathrm{Tr}_{\mathcal{H}} P\|$ where $\mathrm{Tr}_{\mathcal{H}}$ stands for the partial trace.*

2. Let W be a symmetry (on $\mathcal{H}\otimes\mathcal{H}$). Then $\alpha(W) = \max\{\|\mathrm{Tr}_{\mathcal{H}}(Ws\otimes 1)\| : s \in \mathcal{S}(\mathcal{H})\}$.

In particular, this enables us to give:

Example 3.10 (see [15]). 1. Any projector of the form $P = p \otimes I$ where p is a one dimensional projector on $\mathcal{H}$, I is the identity on $\mathcal{H}$.

2. Let $\{e_i\}$ be a basis in $\mathcal{H}$. Define $f \in \mathcal{H}\otimes\mathcal{H}$ by

$$f = \frac{1}{\sqrt{n}}|e_1 \otimes e_1 + e_2 \otimes e_2 + \ldots + e_n \otimes e_n > \tag{11}$$

Such an f is called (in Quantum Information) a fully entangled state. Obviously, $n|f >< f| \in \mathfrak{D}$.

3. Define

$$W = \sum_{i,j} E_{ij} \otimes E_{ji} \tag{12}$$

where $E_{ij} \equiv |e_i >< e_j|$. W is a bp symmetry with $\mathrm{Tr}\, W = n$. Moreover, $\tau_p(W) = n|f >< f|$. Hence, $W \in \mathfrak{D}$.

One of the big "mysteries" of the structure of positive maps is the appearance of non-decomposable maps for nD (n-dimensional) cases with $n \geq 3$. To understand this phenomenon, as the first step (the rest will be given in the next section) we present

Example 3.11. 1. Assume $2D$ case and let s be a symmetry in $\mathfrak{D}$. Then $s = p - q$ and $\mathrm{Tr}\, s = 2$. As $p + q = I$, then $\mathrm{Tr}(p + q) = 4$. Hence $\mathrm{Tr}\, p = 3$ and $\mathrm{Tr}\, q = 1$. Consequently, q is a one dimensional orthoprojector. This indicates that for 2D case, symmetries should be of a very simple form. Indeed (cf. [15]), up to the transformation implemented by $U = U_1 \otimes U_2$, there is room for a symmetry of type W only. It is worth pointing out that this symmetry leads to the transposition. We end our comments on $\mathfrak{D}$ for 2D case with a remark that there is no "room" for any non-trivial e-symmetry.

2. $3D$ case. Let s be a symmetry in $\mathfrak{D}$, Then $s = p - q$ and $\mathrm{Tr}\, s = 3$. As $p + q = I$, then $\mathrm{Tr}(p + q) = 9$. Hence $\mathrm{Tr}\, p = 6$ and $\mathrm{Tr}\, q = 3$. Although, now symmetries seems to be more nontrivial, one can show [17] that again up to the transformation implemented by $U = U_1 \otimes U_2$, there is a room for a symmetry of type W only. However, now partial symmetries appear, i.e. there are nontrivial partial symmetries in $\mathfrak{D}$. In other words, the geometry of $\mathfrak{D}$ is more complicated. We will come back to this point in the next section.

Having the description of $\mathfrak{D}$ we wish to look more closely at the characterization of its exposed points. To this end we will use the analytic approach with some emphasis on the underlying geometry. For the reader's convenience we begin by recalling selected definitions appearing in the convex analysis of real Banach space X (see [1], Section II.5 in [3], Chapter 5 in [18], and Chapters 5 and 6 in [19]).

We denote by S_X (X_1) the unit sphere (ball) of X.

Definition 3.12. A point x of S_X is said to be

1. an exposed point of X_1 if $\{x\}$ is an exposed face of X_1,
2. a rotund point of X_1 if every $y \in S_X$ with $\|\frac{x+y}{2}\| = 1$ satisfies $x = y$.
3. a smooth point of X_1 if there is exactly one element f of S_{X^*} such that $f(x) = 1$.

The sets of rotund points (smooth points) of X_1 will be denoted by $rot(X_1)$ ($smo(X_1)$).

If each point of S_X is smooth (rotund) then the space X is said to be smooth (rotund). The following classic result says that these two concepts are dual to each other (see [18]): *A reflexive Banach space is rotund (smooth) if and only if its dual space is smooth (rotund).*

To present a characterization of exposed points of S_X we need one more definition.

Definition 3.13 (see [2]). A point $x \in S_X$ is defined to be a strongly non-smooth point of X_1 if for every $y \in S_{X\setminus\{x\}}$ with $[x,y] \subseteq S_X$, x is not a smooth point of Y_1, where $Y = span\{x,y\}$ and $[x,y] = \{v = \lambda x + (1-\lambda)y, \quad \lambda \in [0,1]\}$.

Obviously, Y_1 stands for the unit ball in Y. The set of all strongly non-smooth points of X_1 will be denoted by $nsmo(X_1)$.

The general characterization of exposed points of a unit ball of a real Banach space was obtained by Aizupuru and Garcia-Pacheco. They proved

Theorem 3.14 (see [2]). *Let X be a real, separable Banach space. Then one has*

$$Exp(X_1) = rot(X_1) \cup nsmo(X_1)$$

Using this result we have proved

Theorem 3.15 ([15]). *An exposed point of $\mathfrak{B}_1^{(+)}$ is also an exposed point of $\mathfrak{B}_1$. Conversely, a bp positive, strongly non-smooth, non-rotund $\varrho \in \mathfrak{B}_1$ is an exposed point of $\mathfrak{B}_1^{(+)}$, where $\mathfrak{B}_1 \equiv \{\sigma \in \mathcal{B}(\mathcal{H}) \otimes_\alpha \mathcal{B}(\mathcal{H});\ \sigma^* = \sigma,\ \alpha(\sigma) \leq 1\}$, and $\mathfrak{B}_1^{(+)} \equiv \{\varrho_\varphi : \varrho_\varphi \in \mathfrak{B}_1,\ \varrho_\varphi \geq_{bp} 0\}$.*

Corollary 3.16. *Strongly non-smooth, non-rotund, bp-positive points σ of unit sphere (with respect to the norm α) having the normalization $\mathrm{Tr}\,\sigma = n$ are exposed points of $\mathfrak{D}$.*

Remark 3.17 (see [15]). We recall that for any linear positive map ϕ from a C^*-algebra $\mathfrak{A}$ to C^*-algebra $\mathfrak{B}$ one has $\|\phi\| = \|\phi(1)\|$ (cf. [13], Lemma 8.2.2). Further, note that a "density matrix" σ in $\mathfrak{B}_0 = \{\sigma \in \mathfrak{B}_1^{(+)}; \alpha(\sigma) = 1\}$ corresponds to a linear positive map of norm one (cf. Lemma 2.3). Therefore, Theorem 3.15 gives *the full characterization* of exposed positive linear maps of norm one, exposed both for the set $\mathfrak{B}_1$ as well as for $\mathfrak{B}_1^{(+)}$. For additional comments see [15].

4. Examples; 2D and 3D cases. Having a characterization of $\mathfrak{D}$, so equivalently, a characterization of $\mathcal{P}_1$ we wish to investigate the difference between maps on $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ in more detail. Our motivation is again twofold. First, we want to show that our characterization is working in concrete models. Second, it is natural to investigate what is an essential difference between 2D and 3D cases.

We begin with the following result:

PROPOSITION 4.1 (see [17]). *The bp-density matrix $\rho_\phi \in \mathfrak{D}$ corresponding to the regular extreme normalised unital map (i.e. maps with the property that their restriction to the diagonal subalgebra is still extreme) $\phi \in \mathcal{P}_1$ can be written in one of the following block forms in some matrix representation:*

$$\rho_\phi = \begin{pmatrix} |y_1 >< y_1| & c_0|y_1 >< y_2| + c|y_1 >< y_1| \\ c_0|y_2 >< y_1| + \bar{c}|y_1 >< y_2| & |y_2 >< y_2| \end{pmatrix} \text{ or } \rho_\phi = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix},$$

where $c_0 \geq 0$, $c \in \mathbb{C}$ and $\{y_1, y_2\}$ is some basis in $\mathbb{C}^2$.

The proof relies on α-normalization, the Arveson characterization of extreme points of the set of unital completely positive maps from an abelian C^*-algebra $C(X)$ to $M_n(\mathbb{C})$ (see [6]), and uses bp-positivity condition. Having this, one can reproduce, for regular maps, the well known Størmer result on characterization of positive maps in 2D case (see [24]). In other words, the subset of $\mathfrak{D}$ corresponding to regular maps is generated by bp-symmetries, projectors of the type: $2|f >< f|$ with f being a fully entangled vector (2 since we are considering 2D case!), $p \otimes \mathbb{I}$, where p is a 1-dimensional projector in $\mathbb{C}^2$. The important point to note here is that $\rho \in \mathfrak{D}$ corresponding to a homomorphism is of the form $n|f >< f|$ (n stands for the dimensionality of the model) while a symmetry $W \in \mathfrak{D}$ corresponds to an antihomomorphism. Moreover, W, $\tau_p(W)$ as well as $U_1 \otimes U_2 W U_1 \otimes U_2$, ($U_i$ are unitaries) are exposed points in $\mathfrak{D}$ (cf. [15]). Note that this indicates the important role of Example 3.10.

Passing to 3D case, partial symmetries are appearing. This destroys the simple picture we had for 2D case. To give examples of bp-partial symmetries we observe

$$s = \sum_{i,j=1}^{2} |e_i >< e_j| \otimes |e_j >< e_i| + |e_3 >< e_3| \otimes |e_3 >< e_3|$$

and

$$s_0 = \sum_{i,j=1}^{2} |e_i >< e_j| \otimes |e_j >< e_i| + \frac{1}{2}|e_1 \otimes e_3 + e_2 \otimes e_3 >< e_1 \otimes e_3 + e_2 \otimes e_3|$$

are in $\mathfrak{D}$, where $\{e_1, e_2, e_3\}$ is a basis in $\mathbb{C}^3$. The map ϕ corresponding to s has the form

$$\phi(E_{i,j}) = E_{j,i} \quad \text{for} \quad i,j \in \{1,2\}, \quad \phi(E_{3,3}) = E_{3,3}, \quad 0 \quad \text{otherwise}$$

while

$$\phi(E_{i,j}) = E_{j,i} + \frac{1}{4}E_{3,3} \quad \text{for} \quad i,j \in \{1,2\}, \quad \phi(E_{3,3}) = \frac{1}{2}E_{3,3}, \quad 0 \quad \text{otherwise}$$

corresponds to s_0.

The last map seems to be of special interest as its restriction to "diagonal" variables $E_{i,i}$ does not satisfy the Arveson criterion. The existence of such elements of $\mathfrak{D}$ (so also the corresponding maps) is an obstacle to get a version of Proposition 4.1 for 3D case. It is worth pointing out that the famous generalization of Choi map, see [26] and [7], has the same property, i.e. also this map does not satisfy the Arveson criterion. Further, in

4D case, one can provide an even more suggestive example of an element in $\mathfrak{D}$:

$$s' = \sum_{i,j=1}^{2} |e_i >< e_j| \otimes |e_j >< e_i| + 2|e_3 \otimes e_3 + e_4 \otimes e_4 >< e_3 \otimes e_3 + e_4 \otimes e_4|,$$

where $\{e_1, e_2, e_3, e_4\}$ is a basis in $\mathbb{C}^4$. Note that neither s' nor $\tau_p(s')$ are positive (they are only bp-positive).

Although it would be desirable to get an explicit characterization of all exposed points of $\mathfrak{D}$ for low dimensional case, we have not been able to do this. We were able to show only (cf. [15]) that symmetries, e-symmetries, certain projectors are exposed points of $\mathfrak{D} \cap \mathfrak{B}_1^{\|\cdot\|}$, where $\mathfrak{B}_1^{\|\cdot\|}$ is the unit ball with respect to the operator norm.

We wish to close this section with a remark that a more detailed analysis of low dimensional cases will be presented in [17].

5. Final remarks. The given characterization of normalized bp density matrices $\mathfrak{D}$, so also the description of positive normalized maps, utilizes the concepts of exposed faces, exposed points, certain projections, symmetries, and partial symmetries. But, it is to be expected. Namely, these concepts proved to be very useful in the analysis of the question which compact convex sets can arise as the state space of unital $\mathbf{C}^*$or W^* algebras (see [4] and [5]). The answer to this question, in "physical" terms, gives the proof of the statement that Schrödinger and Heisenberg pictures are fully equivalent. On the other hand, the concept of positive, continuous, unital maps stems from the duality of these pictures. Therefore, it is natural to expect that essential ingredients of the description of the duality are also important for the characterization of $\mathcal{P}_1$.

Finally, we would like to say that although finite dimensional case was assumed, sometimes, we deliberately used more sophisticated notation—the purpose of that is to indicate a possibility for generalizations.

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