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THE K-THEORY OF THE TRIPLE-TOEPLITZ DEFORMATION OF THE COMPLEX PROJECTIVE PLANE

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Abstract. We consider a family $\pi^i_j : B_i \to B_{ij} = B_{ji}$, $i, j \in \{1, 2, 3\}$, $i \neq j$, of C^*-epimorphisms assuming that it satisfies the cocycle condition. Then we show how to compute the K-groups of the multi-pullback C^*-algebra of such a family, and exemplify it in the case of the triple-Toeplitz deformation of $\mathbb{C}P^2$.

Introduction. Starting from the affine covering of a projective space, a new type of noncommutative deformations of complex projective spaces was introduced in [4]. Therein, the complex projective space $\mathbb{C}P^n$ is presented as a natural gluing of polydiscs, dualized to the multi-pullback C^*-algebra, and deformed to a multi-pullback of tensor powers of Toeplitz algebras. The case of $n = 1$ was analyzed in detail in [7], and called the mirror quantum sphere. In particular, its K-groups were easily determined.

The goal of this note is to determine the K-groups in the case $n = 2$, which requires some additional tools. The C^*-algebra of the mirror quantum sphere is simply a pullback C^*-algebra, so that its K-theory is immediately computable by the Mayer–Vietoris six-term exact sequence. The C^*-algebra of the triple-Toeplitz deformation of $\mathbb{C}P^2$ is a triple-pullback C^*-algebra, and it turns out that, in order to apply (three times) the Mayer–Vietoris six-term exact sequence, we need to check the cocycle condition.

We begin by general considerations allowing us to combine the cocycle condition, the distributivity of C^*-ideals, and the Mayer–Vietoris six-term exact sequence into a certain general computational method. Then we use it to establish the K-groups of the aforementioned quantum $\mathbb{C}P^2$.

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To focus attention and for the sake of simplicity, we start by considering the category of vector spaces. Let J be a finite set, and let

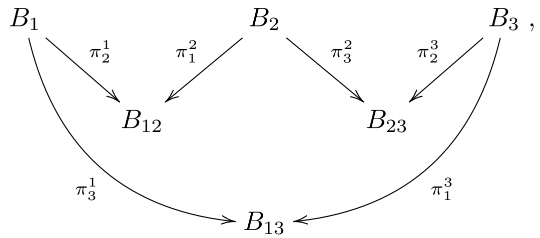
$$\{\pi^i_j : B_i \longrightarrow B_{ij} = B_{ji}\}_{i,j\in J,\, i\neq j} \tag{1}$$

be a family of homomorphisms. In this category, the multi-pullback of a family (1) can be defined as follows.

Definition 0.1 ([9, 3]). The *multi-pullback* B^π of a family (1) is defined as

$$B^\pi := \Big\{(b_i)_i \in \prod_{i\in J} B_i \;\Big|\; \pi^i_j(b_i) = \pi^j_i(b_j)\ \forall\, i,j \in J,\ i \neq j\Big\}.$$

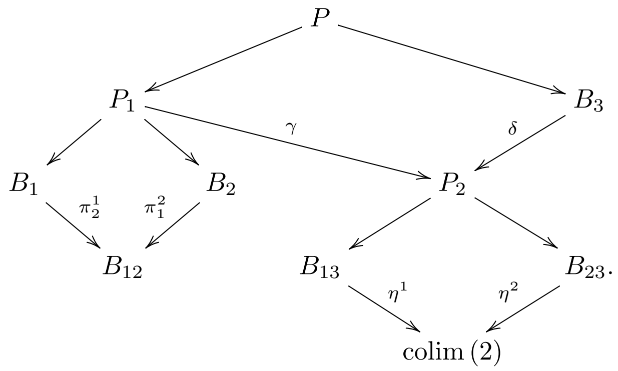
If $J = \{1, 2, 3\}$, a family (1) is depicted by the diagram



(2)

and its multi-pullback B^π can be interpreted as the limit of this diagram. (Recall that the limit (colimit) of a diagram is a certain universal object together with morphisms from it to (to it from) all objects in the diagram.) Furthermore, one can easily transform the triple-pullback B^π into an iterated pullback:

Lemma 0.2. *Let B^π be the multi-pullback of a family* (1) *for $J = \{1, 2, 3\}$. Then the canonical identification of vector spaces $V^3 \to V^2 \times V$ yields an isomorphism from B^π to the pullback vector space P of the top sub-diagram of the diagram*



(3)

Here all three square sub-diagrams are pullback diagrams, $\gamma(b_1, b_2) := (\pi^1_3(b_1), \pi^2_3(b_2))$, $\delta(b_3) := (\pi^3_1(b_3), \pi^3_2(b_3))$, and η^1, η^2 come from the colimit of the diagram (2).

Proof. By construction, any element of P is a pair $((b_1, b_2), b_3) \in (B_1 \times B_2) \times B_3$ such that $\pi^1_2(b_1) = \pi^2_1(b_2)$ and $(\pi^1_3(b_1), \pi^2_3(b_2)) =: \gamma((b_1, b_2)) = \delta(b_3) := (\pi^3_1(b_3), \pi^3_2(b_3))$. Hence the re-bracketing map from B^π to P is an isomorphism, as claimed. ∎

We can still remain in the category of vector spaces to define the second key concept of this note, namely the cocycle condition. First, we assume that all maps of a family (1) are surjective. Then, for any distinct i, j, k, we put $B^i_{jk} := B_i/(\ker \pi^i_j + \ker \pi^i_k)$ and take $[\,\cdot\,]^i_{jk} : B_i \to B^i_{jk}$ to be the canonical surjections. Next, we introduce the family of isomorphisms

$$\pi^{ij}_k : B^i_{jk} \longrightarrow B_{ij}/\pi^i_j(\ker \pi^i_k), \qquad [b_i]^i_{jk} \longmapsto \pi^i_j(b_i) + \pi^i_j(\ker \pi^i_k). \tag{4}$$

Now we are ready for:

Definition 0.3 ([3, in Proposition 9]). We say that a family (1) of epimorphisms of vector spaces satisfies the *cocycle condition* if and only if, for all distinct $i, j, k \in J$,

1. $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$,
2. the isomorphisms $\phi^{ij}_k := (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k : B^j_{ik} \to B^i_{jk}$ satisfy $\phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$.

1. A method for computing the K-groups of triple-pullback C^*-algebras. To avoid redundant assumptions, we split this section into an algebraic and C^*-algebraic part. The latter appears as the special case of the former.

1.1. Algebras with distributive lattices of ideals. From now on we specialize the category of vector spaces to the category of unital algebras and algebra homomorphisms. For the sake of brevity, we say that a family (1) of algebra homomorphisms is *distributive* if and only if all of them are surjective and for all $i \in J$ the kernels of π^i_j, $j \in J \setminus \{i\}$, generate a distributive lattice of ideals. Much of what we do in this subsection is re-casting [5, Corollary 4.3]. However, since our focus is on triple-pullback algebras, we provide simple direct arguments to spare the reader the language of sheaves. First, we slightly extend [3, Proposition 9]:

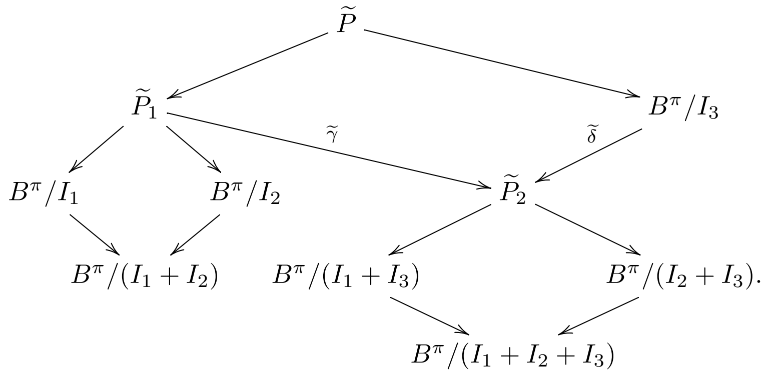
Lemma 1.1. *Assume that a family* (1) *of algebra homomorphisms is distributive and satisfies the cocycle condition. Denote by π_i, $i \in J$, the restriction of the i-th canonical projection to the multi-pullback B^π of the family* (1)*. Then $B_i \cong B^\pi/\ker \pi_i$ for all $i \in J$ and $B_{ij} \cong B^\pi/(\ker \pi_i + \ker \pi_j)$ for all distinct $i, j \in J$.*

Proof. The existence of isomorphisms $B_i \cong B^\pi/\ker \pi_i$, $i \in J$, is simply a re-statement of [3, Proposition 9]. To show the existence of the second family of isomorphisms, we apply [8, Theorem 7(2)] to conclude that, for any distinct $i, j \in J$ and any $b_i \in B_i$, $b_j \in B_j$, such that $\pi^i_j(b_i) = \pi^j_i(b_j)$, there exists an element $b \in B^\pi$ such that $\pi_i(b) = b_i$ and $\pi_j(b) = b_j$. This allows us to prove that the kernels of algebra epimorphisms $\pi_{ij} := \pi^i_j \circ \pi_i = \pi^j_i \circ \pi_j$ are $\ker \pi_i + \ker \pi_j$. Indeed, if $b \in \ker \pi_{ij}$, then $\pi^i_j(\pi_i(b)) = 0$ and there exists $b' \in B^\pi$ such that $\pi_i(b') = \pi_i(b)$ and $\pi_j(b') = 0$. Therefore, since $b - b' \in \ker \pi_i$ and $b' \in \ker \pi_j$, we infer that $b \in \ker \pi_i + \ker \pi_j$, as needed. The inclusion $\ker \pi_i + \ker \pi_j \subseteq \ker \pi_{ij}$ is obvious. ■

Combining the above lemma with [8, Proposition 8], we obtain:

Lemma 1.2. *Assume that a family* (1) *of algebra homomorphisms is distributive and satisfies the cocycle condition. Assume also that the kernels of the restrictions of the canonical projections to the multipullback B^π of* (1) *generate a distributive lattice of* *ideals. Then the algebra* B^π *is isomorphic to the multi-pullback algebra of the family of canonical surjections* $B^\pi/\ker\pi_i \to B^\pi/(\ker\pi_i + \ker\pi_j)$, $i, j \in J$, $i \neq j$.

Now we specialize multi-pullbacks to triple-pullbacks, and consider a special case of the iterated pullback diagram of Lemma 0.2:



(5)

Here $I_i := \ker\pi_i$, $i \in \{1,2,3\}$, $\widetilde{\gamma}(a,b) := (a+I_3, b+I_3)$, $\widetilde{\delta}(c) := (c+I_1, c+I_2)$, and all three square sub-diagrams are pullback diagrams. To further abbreviate the notation, we will use $B^\pi_i := B^\pi/I_i$ and $B^\pi_{ij} := B^\pi/(I_i+I_j)$ for all distinct $i, j \in \{1,2,3\}$, and $B^\pi_{123} := B^\pi/(I_1+I_2+I_3)$.

PROPOSITION 1.3. *Assume that a family* (1) *of algebra homomorphisms is distributive and satisfies the cocycle condition. Assume also that the kernels of the restrictions of the canonical projections to the multi-pullback* B^π *of* (1) *generate a distributive lattice of ideals. Take* $J = \{1,2,3\}$. *Then the iterated pullback algebra* $\widetilde{P}$ *of Diagram* (5) *is isomorphic to* B^π, *and all homomorphisms in this diagram are surjective.*

Proof. First we take advantage of Lemma 1.2 to transform the family (1) into its canonical form. Then we apply Lemma 0.2 to conclude that the pullback algebra $\widetilde{P}$ of the iterated pullback diagram (5) is isomorphic to the triple-pullback algebra B^π by the re-bracketing isomorphism. Thus we can replace $\widetilde{P}$ by B^π in the diagram (5).

Since all square sub-diagrams are pullback diagrams and canonical quotient maps are surjective, to prove the surjectivity of all homomorphisms in the diagram (5) it suffices to show the surjectivity of $\widetilde{\gamma}$ and $\widetilde{\delta}$. The latter map is surjective by [5, Lemma 2.1]. It requires a little bit more work to prove the surjectivity of $\widetilde{\gamma}$, but our argument is again based on [5, Lemma 2.1].

Let $(b,c) \in \widetilde{P}_2$. Take $a \in B^\pi_{12}$ that is mapped to the same element in B^π_{123} as b and c. It follows from [5, Lemma 2.1] that there exists an element $\alpha \in B^\pi_1$ such that $\alpha + I_2 = a$ and $\alpha + I_3 = b$. Much in the same way, we show that there exists an element $\beta \in B^\pi_2$ satisfying $\beta + I_1 = a$ and $\beta + I_3 = c$. By construction, $(\alpha,\beta) \in \widetilde{P}_1$ and $\widetilde{\gamma}((\alpha,\beta)) = (b,c)$. ■

Finally, since for all distinct $i, j \in \{1,2,3\}$ the identifications $B_i \cong B^\pi_i$ and $B_{ij} \cong B^\pi_{ij}$ are such that together with π^i_j's and canonical quotient maps they form commutative square diagrams, we immediately conclude:

COROLLARY 1.4. *Assume that a family* (1) *of algebra homomorphisms is distributive and satisfies the cocycle condition. Assume also that the kernels of the restrictions of the canonical projections to the multi-pullback B^π of the family* (1) *generate a distributive lattice of ideals. Take $J = \{1,2,3\}$. Then in Diagram* (3) *we can take η^1 and η^2 to be defined as $B_{13} \overset{\eta^1}{\to} B^\pi_{123} \overset{\eta^2}{\leftarrow} B_{23}$, $\eta^i(b) := \widetilde{b} + I_1 + I_2 + I_3$, where $\widetilde{b}$ is such that $\pi^i_3(\pi_i(\widetilde{b})) = b$, $i \in \{1,2\}$, and all homomorphisms in this diagram are surjective.*

1.2. The case of C^*-algebras. Let us assume from now on that all our algebras are unital C^*-algebras, and morphisms are C^*-homomorphisms. Due to the property of C^*-ideals that $I \cap J = IJ$, their kernels always generate a distributive lattice of ideals, so that we are in the special case of the preceding section. On the other hand, recall that for the pullback C^*-algebra A of any pair of C^*-homomorphisms $A_1 \overset{\alpha^1}{\to} A_{12} \overset{\alpha^2}{\leftarrow} A_2$ of which at least one is surjective, there is the Mayer–Vietoris six-term exact sequence (e.g., see [2, Theorem 21.2.2] [1, Section 1.3], [10]):

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A_1) \oplus K_0(A_2) & \xrightarrow{\alpha^1_* - \alpha^2_*} & K_0(A_{12}) \\ \uparrow & & & & \downarrow \\ K_1(A_{12}) & \xleftarrow{\alpha^1_* - \alpha^2_*} & K_1(A_1) \oplus K_1(A_2) & \longleftarrow & K_1(A). \end{array} \tag{6}$$

Now we can combine Lemma 0.2 with Corollary 1.4 and apply three times the above Mayer–Vietoris six-term exact sequence to infer:

COROLLARY 1.5. *Assume that a family* (1) *is a family of surjective unital homomorphisms of C^*-algebras and $J = \{1,2,3\}$. Then, if this family satisfies the cocycle condition, there are three six-term exact sequences:*

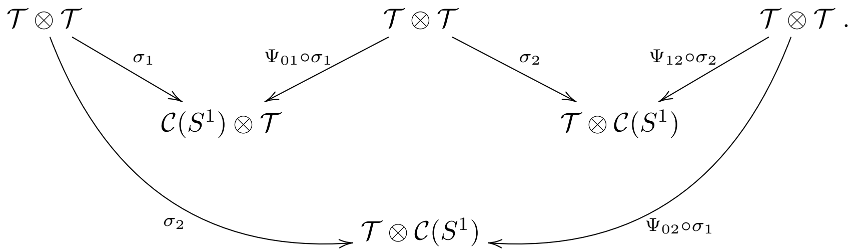
$$\begin{array}{ccccc} K_0(P_1) & \longrightarrow & K_0(B_1) \oplus K_0(B_2) & \xrightarrow{\pi^1_{2*} - \pi^2_{1*}} & K_0(B_{12}) \\ \uparrow & & & & \downarrow \\ K_1(B_{12}) & \xleftarrow{\pi^1_{2*} - \pi^2_{1*}} & K_1(B_1) \oplus K_1(B_2) & \longleftarrow & K_1(P_1), \end{array}$$

$$\begin{array}{ccccc} K_0(P_2) & \longrightarrow & K_0(B_{13}) \oplus K_0(B_{23}) & \xrightarrow{\eta^1_* - \eta^2_*} & K_0(B^\pi_{123}) \\ \uparrow & & & & \downarrow \\ K_1(B^\pi_{123}) & \xleftarrow{\eta^1_* - \eta^2_*} & K_1(B_{13}) \oplus K_1(B_{23}) & \longleftarrow & K_1(P_2), \end{array}$$

$$\begin{array}{ccccc} K_0(B^\pi) & \longrightarrow & K_0(P_1) \oplus K_0(B_3) & \xrightarrow{\gamma_* - \delta_*} & K_0(P_2) \\ \uparrow & & & & \downarrow \\ K_1(P_2) & \xleftarrow{\gamma_* - \delta_*} & K_1(P_1) \oplus K_1(B_3) & \longleftarrow & K_1(B^\pi). \end{array}$$

2. The triple-Toeplitz deformation of $\mathbb{C}P^2$

2.1. C^*-algebra. We consider the case $n = 2$ of the multi-Toeplitz deformations [4, Section 2] of the complex projective spaces. The C^*-algebra of our quantum projective plane is given as the triple-pullback of the following diagram:



(7)

Here $\mathcal{T}$ is the Toeplitz algebra, $\sigma : \mathcal{T} \to \mathcal{C}(S^1)$ is the symbol map, $\sigma_1 := \sigma \otimes \mathrm{id}$, $\sigma_2 := \mathrm{id}\otimes\sigma$, and

$$\mathcal{C}(S^1)\otimes\mathcal{T} \ni u\otimes z \xrightarrow{\Psi_{01}} S(z^{(1)}u)\otimes z^{(0)} \in \mathcal{C}(S^1)\otimes\mathcal{T}, \tag{8}$$

$$\mathcal{C}(S^1)\otimes\mathcal{T} \ni u\otimes z \xrightarrow{\Psi_{02}} z^{(0)}\otimes S(z^{(1)}u) \in \mathcal{T}\otimes\mathcal{C}(S^1), \tag{9}$$

$$\mathcal{T}\otimes\mathcal{C}(S^1) \ni z\otimes u \xrightarrow{\Psi_{12}} z^{(0)}\otimes S(z^{(1)}u) \in \mathcal{T}\otimes\mathcal{C}(S^1), \tag{10}$$

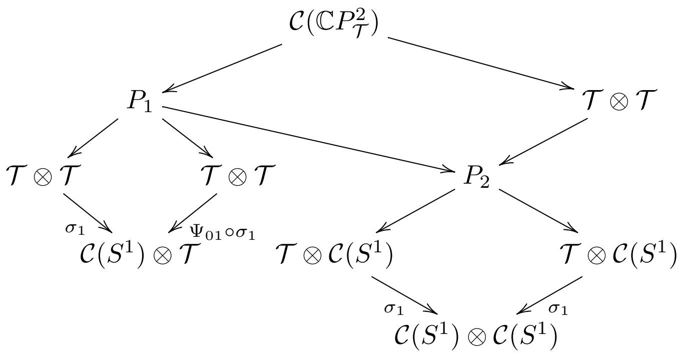
where $\mathcal{T} \ni z \mapsto z^{(0)}\otimes z^{(1)} \in \mathcal{T}\otimes\mathcal{C}(S^1)$ is the coaction dual to the gauge action on $\mathcal{T}$, and $S(f)(g) := f(g^{-1})$.

2.2. K-theory. The main result of this note is the following:

THEOREM 2.1. *The K-groups of the triple-Toeplitz deformation of $\mathbb{C}P^2$ are:*

$$K_0\big(\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})\big) = \mathbb{Z}^3, \qquad K_1\big(\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})\big) = 0.$$

Proof. Since the family (7) satisfies the cocycle condition by [4, Lemma 3.2], we can apply Corollary 1.5 to compute the K-groups of its triple-pullback C^*-algebra $\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})$. First, we present $\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})$ as the iterated pullback C^*-algebra of the diagram



(11)

with all arrows surjective. We know that

$$\begin{aligned} &K_0(\mathcal{T}^{\otimes 2}) = \mathbb{Z}, \quad K_0(\mathcal{C}(S^1)) = \mathbb{Z}, \quad K_0(\mathcal{T}\otimes\mathcal{C}(S^1)) = \mathbb{Z}, \\ &K_1(\mathcal{T}^{\otimes 2}) = 0, \quad K_1(\mathcal{C}(S^1)) = \mathbb{Z}, \quad K_1(\mathcal{T}\otimes\mathcal{C}(S^1)) = \mathbb{Z}, \end{aligned} \tag{12}$$

and that the generators of K_0 are $[1 \otimes 1] \in K_0(\mathcal{T}^{\otimes 2})$ and $[1 \otimes 1] \in K_0(\mathcal{T} \otimes \mathcal{C}(S^1))$. Now the first diagram of Corollary 1.5 becomes

$$\begin{array}{ccccc} K_0(P_1) & \longrightarrow & K_0(\mathcal{T}^{\otimes 2}) \oplus K_0(\mathcal{T}^{\otimes 2}) & \longrightarrow & K_0(\mathcal{T} \otimes \mathcal{C}(S^1)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T} \otimes \mathcal{C}(S^1)) & \longleftarrow & K_1(\mathcal{T}^{\otimes 2}) \oplus K_1(\mathcal{T}^{\otimes 2}) & \longleftarrow & K_1(P_1). \end{array} \tag{13}$$

After plugging in (12), we obtain

$$\begin{array}{ccccc} K_0(P_1) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(m,n)\mapsto m-n} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & K_1(P_1). \end{array} \tag{14}$$

This yields $K_0(P_1) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(P_1) = 0$ because the dotted arrow is onto.

Next, the second diagram of Corollary 1.5 becomes

$$\begin{array}{ccccc} K_0(P_2) & \longrightarrow & K_0(\mathcal{T} \otimes \mathcal{C}(S^1)) \oplus K_0(\mathcal{C}(S^1) \otimes \mathcal{T}) & \longrightarrow & K_0(\mathcal{C}(S^1) \otimes \mathcal{C}(S^1)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}(S^1) \otimes \mathcal{C}(S^1)) & \longleftarrow & K_1(\mathcal{T} \otimes \mathcal{C}(S^1)) \oplus K_1(\mathcal{C}(S^1) \otimes \mathcal{T}) & \longleftarrow & K_1(P_2). \end{array} \tag{15}$$

This is a special case of an exact sequence studied in [1, Section 4]. On the other hand, using a different method, it was already determined in [6, Section 3] that $K_0(P_2) = \mathbb{Z}$ (generated by $1 \in P_2$) and $K_1(P_2) = \mathbb{Z}$.

Finally, the last diagram of Corollary 1.5 becomes

$$\begin{array}{ccccc} K_0(\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})) & \longrightarrow & K_0(P_1) \oplus K_0(\mathcal{T}^{\otimes 2}) & \longrightarrow & K_0(P_2) \\ \uparrow & & & & \downarrow \\ K_1(P_2) & \longleftarrow & K_1(P_1) \oplus K_1(\mathcal{T}^{\otimes 2}) & \longleftarrow & K_1(\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})). \end{array} \tag{16}$$

Equivalently, we can write it as

$$\begin{array}{ccccc} K_0(\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})) & \longrightarrow & (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & K_1(\mathcal{C}(\mathbb{C}P^2_{\mathcal{T}})). \end{array} \tag{17}$$

The dotted map is of the form $(m, n, l) \mapsto km + k'n - l$. In particular, it is onto, so that $K_1(\mathbb{C}P^2_{\mathcal{T}}) = 0$. Furthermore, the kernel of this map is $\mathbb{Z}^2$. Combining this with the fact that the surjective homomorphisms onto free modules split, we infer that $K_0(\mathbb{C}P^2_{\mathcal{T}}) = \mathbb{Z}^3$. ∎

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References

[1] P. F. Baum, P. M. Hajac, R. Matthes and W. Szymański, *The K-theory of Heegaard-type quantum 3-spheres*, K-Theory 35 (2005), 159–186.

[2] B. Blackadar, *K-theory for operator algebras*, Math. Sci. Res. Inst. Publ. 5, Cambridge Univ. Press, Cambridge, 1998.

[3] D. Calow and R. Matthes, *Covering and gluing of algebras and differential algebras*, J. Geom. Phys. 32 (2000), 364–396.

[4] P. M. Hajac, A. Kaygun and B. Zieliński, *Quantum complex projective spaces from Toeplitz cubes*, J. Noncommut. Geom. 6 (2012), 603–621.

[5] P. M. Hajac, U. Krähmer, R. Matthes and B. Zieliński, *Piecewise principal comodule algebras*, J. Noncommut. Geom. 5 (2011), 591–614.

[6] P. M. Hajac, R. Matthes and W. Szymański, *A locally trivial noncommutative Hopf fibration*, Algebr. Represent. Theory 9 (2006), 121–146.

[7] P. M. Hajac, R. Matthes and W. Szymański, *Noncommutative index theory for mirror quantum spheres*, C. R. Math. Acad. Sci. Paris 343 (2006), 731–736.

[8] P. M. Hajac and B. Zieliński, *Cocycle condition for multi-pullbacks of algebras*, this volume, 239–243.

[9] G. K. Pedersen, *Pullback and pushout constructions in C^*-algebra theory*, J. Funct. Anal. 167 (1999), 243–344.

[10] C. Schochet, *Topological methods for C^*-algebras III: Axiomatic homology*, Pacific J. Math. 114 (1984), 399–445.