ON $\mathbb{Z}/2\mathbb{Z}$-EXTENSIONS OF POINTED FUSION CATEGORIES

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Abstract. We give a classification of $\mathbb{Z}/2\mathbb{Z}$-graded fusion categories whose 0-component is a pointed fusion category. A number of concrete examples are considered.

1. Introduction

1.1. A fusion category over an algebraically closed field k of characteristic zero is a k-linear semisimple rigid tensor category $\mathcal{C}$ with finitely many simple objects (the unit object 1 is supposed to be simple) and finite dimensional spaces of morphisms. Our main reference on fusion categories is [4]. Throughout this paper we work with $k = \mathbb{C}$ although many results hold for general k. If there is no ambiguity, we use the same notation for an equivalence class and for its representative.

A fusion category $\mathcal{C}$ is said to be graded by a finite group G if $\mathcal{C} = \oplus_{g \in G} c(g)$, where $c(g)$ are full abelian subcategories of $\mathcal{C}$ such that $c(g)^* = c(g^{-1})$ and the tensor product maps $c(g) \times c(h)$ to $c(gh)$, for all $g, h \in G$ (we call $\mathcal{C}$ a G-extension of its fusion subcategory $c(e)$, where e is the unit of G). A fusion category is said to be pointed if all its simple objects are invertible with respect to the tensor product. Such a category is equivalent to the category Vec_S^ω whose simple objects are elements of a finite group S with tensor product $s \otimes t = st$, the unit object $1 = 1_S$, the duality $s^* = {}^*s = s^{-1}$, and the associativity isomorphisms defined by $\omega \in H^3(S, \mathbb{C}^\times)$.

Our aim is classification of $\mathbb{Z}/2\mathbb{Z}$-extensions $\mathcal{C} = c(0) \oplus c(1)$ of $c(0) = Vec_S^\omega$. Tambara and Yamagami [23] showed that for any such an extension with $c(1)$ containing only one simple object, S must be abelian and equipped with a symmetric non-degenerate bicharacter χ, and $\omega = 1$. The general case is much more complicated. We show that S must contain a normal abelian subgroup A, so $S \cong A \underset{\rho}{\rtimes} S/A$, a twisted semidirect product with an action of S/A on A and $\rho \in Z^2(S/A, A)$; see, for instance, [3] (the simplest example: $S = \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \underset{\rho}{\rtimes} \mathbb{Z}/2\mathbb{Z}$ with trivial action and nontrivial ρ). Then, A must be equipped with a symmetric non-degenerate bicharacter χ and with an equivalence class of 2-cochains μ^r invariant with respect to $Aut(S)$ and such that $\partial^2\mu^r = \omega$, modulo 2-cocycles on A which can be extended to 2-cocycles on S^{op}, the opposite group to S.

The above extensions of 2-cocycles are classified by couples (ψ, ν), where maps $\psi \in Z^1(S/A, Fun(A \times A, \mathbb{C}^\times))$ and $\nu \in C^2(S/A, \mathbb{C}^\times)$ satisfy some relations. If S is a usual semidirect product, i.e., $\rho = 1$, this was explained in [12]; in this partial case we show that there is only one class μ^r, so it can be omitted. Finally, the Grothendieck rings of $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_S^ω are classified by couples (ε, δ), where $\varepsilon \in Aut(S/A), \delta \in S/A$ are such that $\varepsilon^2 = Ad(\delta)$ and $\varepsilon(\delta) = \delta$.

Thus, our main result, Theorem 3.16, claims that $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_S^ω are classified, up to equivalence, by collections $(A, \chi, \mu^r, \tau, \varepsilon, \delta, \psi, \nu)$, where $\tau = \pm|A|^{-1/2}$, like in [23]. It also describes their structure and allows to construct a number of new examples of fusion categories.

We have to mention that Liptrap [14] earlier obtained some classification of $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_S^ω (analyzing, as in [23], the solutions of the system of 16 pentagon equations for the associativity isomorphisms of $\mathcal{C}$), but in terms that are difficult to apply to the construction of concrete examples. On the other hand, we need new examples of fusion categories in order to construct new families of finite index and finite depth II_1-subfactors as follows: 1) For a given concrete fusion category $\mathcal{C}$, the Hayashi's reconstruction theorem [25] (see also [24]) allows to construct a canonical weak Hopf algebra (a quantum groupoid) H [1], [18] whose representation category is equivalent to $\mathcal{C}$. 2) From a given H, one can construct a subfactor whose bimodule category is equivalent to $\mathcal{C}$, and compute its index, principal and dual graphs, and the lattice of intermediate subfactors; see [19], [20]. If $\mathcal{C}$ is Tambara-Yamagami category, this was done in [15] and gave a family of subfactors of index $(n + \sqrt{n})^2/d$ $(n, d \in \mathbb{N}, d|n)$. We will describe in a separate paper a much larger family of subfactors coming from the fusion categories constructed in the present paper.

1.2. The classification of G-extensions of fusion categories given in [5] implies, in the case when $G = \mathbb{Z}/2\mathbb{Z}$, that any $\mathbb{Z}/2\mathbb{Z}$-extension $\mathcal{C} = c(0) \oplus c(1)$ determines the following data (for all needed definitions see [5] and the references therein):

(1) A group homomorphism $c : \mathbb{Z}/2\mathbb{Z} \to \pi_1 = BrPic(c(0))$ $(0 \mapsto \overline{c(0)}, 1 \mapsto \overline{c(1)})$, where the elements of the *Brauer-Picard group* π_1 are the equivalence classes of invertible $c(0)$-bimodule categories and the operation in π_1 is the relative tensor product $\boxtimes_{c(0)}$. In fact, c is defined by the choice of an invertible $c(0)$-bimodule category $c(1)$ such that $c(1) \cong c(1)^{op}$.

(2) A collection of $c(0)$-bimodule equivalences $M_{g,h} : c(g) \boxtimes_{c(0)} c(h) \cong c(gh)$ $(g, h \in \mathbb{Z}/2\mathbb{Z})$ such that the following functors are isomorphic to Id:

$$T_{f,g,h} : M_{fg,h}(M_{f,g} \boxtimes_{c(0)} Id_{c(h)})(Id_{c(f)} \boxtimes_{c(0)} M_{g,h}^{-1})M_{f,gh}^{-1} : c(fgh) \to c(fgh).$$

(3) Natural isomorphisms $\alpha_{f,g,h} : M_{f,gh}(Id_{c(f)} \boxtimes_{c(0)} M_{g,h}) \cong M_{fg,h}(M_{f,g} \boxtimes_{c(0)} Id_{c(h)})$ satisfying the pentagon equations

$$M_{f,gh,k}(id_{c(f)} \boxtimes_{c(0)} \alpha_{g,h,k})$$
$$\times \alpha_{f,gh,k}(Id_{c(f)} \boxtimes_{c(0)} M_{g,h} \boxtimes_{c(0)} Id_{c(k)})M_{fgh,k}(\alpha_{f,g,h} \boxtimes_{c(0)} id_{c(k)})$$

$$= \alpha_{f,g,hk}(Id_{c(f)} \boxtimes_{c(0)} Id_{c(g)} \boxtimes_{c(0)} M_{h,k})\alpha_{fg,h,k}(M_{f,g} \boxtimes_{c(0)} Id_{c(h)} \boxtimes_{c(0)} Id_{c(k)}).$$

Vice versa, given $c(0)$, c and $M_{g,h}$ as above, the $c(0)$-bimodule category $\mathcal{C} = c(0) \oplus c(1)$ can be equipped with a $c(0)$-bimodule tensor product which is associative if and only if certain cohomological obstruction $O_3(c)$ vanishes or, equivalently, if and only if the functors $T_{f,g,h}$ are isomorphic to Id. If this is the case, the above tensor product admits two possible families of associativity isomorphisms satisfying the pentagon equations.

1.3. The paper is organized as follows: Section 2 contains preliminary results on induction and extension of cocycles from a subgroup of a finite group and also on invertible bimodule categories over $c(0) = Vec_S^\omega$. In Section 3 we give a classification of $\mathbb{Z}/2\mathbb{Z}$-extensions of $c(0) = Vec_S^\omega$. In order to do this, we obtain the following intermediate results:

- Description of homomorphisms $c : \mathbb{Z}/2\mathbb{Z} \to BrPic(c(0))$, or equivalently of invertible $c(0)$-bimodule categories $c(1)$ such that $c(1) \cong c(1)^{op}$.
- Classification of fusion rings of possible $\mathbb{Z}/2\mathbb{Z}$-extensions of $c(0)$.
- Explicit calculation of the $c(0)$-bimodule equivalences $M_{f,g}$.
- Explicit calculation of the functors $T_{f,g,h}$ which allows not only to deduce that they are isomorphic to Id (i.e., that the cohomological obstruction $O_3(c)$ vanishes), but also to calculate explicitly the natural isomorphisms $\alpha_{f,g,h}$ satisfying the pentagon equations (there are exactly 2 families of them).

Section 4 is devoted to examples: we compute the number of non-equivalent $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_S^ω, where S is either an abelian group of order $2p$, or dihedral group D_p (p is prime), or the alternate group A_4.

Note that our results imply the classification, up to (categorical) Morita equivalence, of $\mathbb{Z}/2\mathbb{Z}$-extensions of *group-theoretical categories*, i.e., Morita equivalent to pointed ones [4]. Indeed, [6], Lemma 3.4 implies that for any such extension there is a Morita equivalent $\mathbb{Z}/2\mathbb{Z}$-extension of a pointed category.

2. Preliminaries

2.1. Some cohomological constructions

2.1.1. Basic definitions, induction of cocycles. Let $C(S, P) = \{C^n(S, P)\}_{n \geq 0}$ be a cochain complex of a finite group S with coefficients in a left or right S-module P [3].

Namely, $C^n(S, P) = Fun(S^n, P)$ is a set of normalized n-cochains (i.e., equal to 0 if at least one of arguments equals to 1_S), $S^n = S \times \cdots \times S$ (n factors). If P is a left S-module with an action $s \cdot p$ ($s \in S, p \in P$), then the coboundary operator $\partial^n : C^n(S, P) \to C^{n+1}(S, P)$ is defined by

$$(\partial^n f)(s_1, \ldots, s_n, s_{n+1}) = s_1 \cdot f(s_2, \ldots, s_{n+1}) + \sum_{i=1}^n (-1)^i f(s_1, \ldots, s_{i-1}, s_i s_{i+1}, \ldots, s_{n+1})$$
$$+ (-1)^{n+1} f(s_1, \ldots, s_n). \tag{1}$$

Similarly, if P is a right S-module with an action $p \cdot s$, then the coboundary operator $\underline{\partial}^n : C^n(S, P) \to C^{n+1}(S, P)$ is defined by

$$(\underline{\partial}^n f)(s_1, \ldots, s_n, s_{n+1}) = f(s_2, \ldots, s_{n+1}) + \sum_{i=1}^n (-1)^i f(s_1, \ldots, s_{i-1}, s_i s_{i+1}, \ldots, s_{n+1})$$
$$+ (-1)^{n+1} f(s_1, \ldots, s_n) \cdot s_{n+1}. \tag{2}$$

Let $Z^n(S, P) = Ker(\partial^n)$ (resp., $\underline{Z}^n(S, P) = Ker(\underline{\partial}^n)$) be the set of n-cocycles, and also $B^n(S, P) = Im(\partial^{n-1})$ (resp., $\underline{B}^n(S, P) = Im(\underline{\partial}^{n-1})$) the set of n-coboundaries, and $H^n(S, P) = Z^n(S, P)/B^n(S, P)$ (resp., $\underline{H}^n(S, P) = \underline{Z}^n(S, P)/\underline{B}^n(S, P)$) the n-th cohomology group of S with coefficients in P.

Any left G-module is also a right G-module with the action $(s, M) \mapsto s^{-1} \cdot M$. Let $\sigma_n : C^n(S, P) \to C^n(S, P)$ be the map defined by $\sigma_n(f)(g_1, \ldots, g_n) = -f(g_n^{-1}, \ldots g_1^{-1})$, we have easily $\sigma_{n+1} \partial^n \sigma_n = \underline{\partial}^n$, so σ_n is an isomorphism $\underline{Z}^n(S, P) \to Z^n(S, P)$ and passes to an isomorphism $\underline{H}^n(S, P) \to H^n(S, P)$.

Given a subgroup A of S, we denote by $p : S \to S/A$ the usual surjection $p(s) = sA$, for all $s \in S$, and $\mathbf{1} = p(1_S)$. Let us choose a representative $u(M)$ in any coset $M \in S/A$, in particular, $u(\mathbf{1}) = 1_S$. S acts on S/A via $s \cdot M = p(su(M))$ and also on the set $\{u(M) | M \in S/A\}$ via $s \cdot u(M) = u(s \cdot M)$. Then, for all $s \in S, M \in S/A$, there exists an element $\kappa_{M,s} \in A$ such that $su(M) = u(s \cdot M)\kappa_{M,s}$. One can check that $\kappa_{M,s_1 s_2} = \kappa_{s_2 \cdot M, s_1} \kappa_{M, s_2}$.

Let $C = Fun(S/A, \mathbb{C}^\times)$ be the coinduced right S-module with the natural action $f(M) \cdot s = f(s \cdot M)$, for all $s \in S, M \in S/A$ (here $\mathbb{C}^\times$ is viewed as a trivial right S-module). By Shapiro's lemma (see [3]) the groups $\underline{H}^n(S, C)$ and $\underline{H}^n(A, \mathbb{C}^\times)$ are isomorphic. Explicitly, [16], Lemmas 2.1 and 2.2 show that for $n = 1$ this isomorphism is induced by the maps

$$\varphi_1 : \underline{Z}^1(A, \mathbb{C}^\times) \to \underline{Z}^1(S, C) : (\varphi_1(\rho)(s))(M) = \rho(\kappa_{M,s}),$$
$$\varphi_1^{-1} : \underline{Z}^1(S, C) \to \underline{Z}^1(A, \mathbb{C}^\times) : \; \varphi_1^{-1}(\beta)(a) = \beta(a)(\mathbf{1}), \tag{3}$$

and for $n = 2$, respectively, by the maps

$$\varphi_2 : \underline{Z}^2(A, \mathbb{C}^\times) \to \underline{Z}^2(S, C) : (\varphi_2(\mu)(s_1, s_2))(M) = \mu(\kappa_{s_2 \cdot M, s_1}, \kappa_{M, s_2}),$$
$$\varphi_2^{-1} : \underline{Z}^2(S, C) \to \underline{Z}^2(A, \mathbb{C}^\times) : \; \varphi_2^{-1}(\gamma)(a_1, a_2) = \gamma(a_1, a_2)(\mathbf{1}). \tag{4}$$

2.1.2. Extension of 2-cocycles. If $A \lhd S$ is abelian, then S is isomorphic (see [3], IV, 3)) to the twisted semi-direct product $A \underset{\rho}{\rtimes} T$ defined by:

$$(a, t) \cdot (a', t') = ((a \cdot {}^t a')\rho(t, t'), tt') \quad \forall a, a' \in A, \ t, t' \in T,$$

where $T = S/A$ acts on A by inner automorphisms (i.e., ${}^t a = u(t)au(t)^{-1}$, $u(t) \in S$), $\rho \in Z^2(T, A)$ is given by $\rho(t, t') = u(t)u(t')u(tt')^{-1}$, the isomorphism between $A \underset{\rho}{\rtimes} T$ and S is defined by $(a, t) \mapsto au(t)$. Moreover, the map $(a, t)^{op} \mapsto ({}^{t^{-1}}a, t)$ is an isomorphism between S^{op} and $A \underset{\rho^{op}}{\rtimes} T^{op}$, where $\rho^{op}(t, t') = {}^{(t't)^{-1}}\rho(t', t) = u(t't)^{-1}u(t')u(t) \in Z^2(T^{op}, A)$ and the action of T^{op} on A is given by ${}^{t^{op}}a = {}^{t^{-1}}a$.

In the case of usual semidirect product (i.e., $\rho = 1$) Karpilovski [12] explained how to extend $\sigma \in H^2(A, \mathbb{C}^\times)$ to $\mu \in H^2(S, \mathbb{C}^\times)$. We generalize this construction to twisted semidirect products using essentially the same arguments (however, slightly more complicated because of the presence of ρ). Like in [12], Lemma 2.2.3, one can show that:

1) Any $\mu \in Z^2(S, \mathbb{C}^\times)$ is cohomologous to μ' that is *normal*, i.e., $\mu'((a, e), (e, t)) = 1$, for all $a \in A, t \in T$, and has the same restriction $\mu_{T,T}$ on $(e, T) \times (e, T)$. Note that (e, T) is not a subgroup of S, in general, because the products in S and in T are related by the formula $(e, t) \cdot_S (e, t)' = (\rho(t, t'), tt')$.

2) Any normal $\mu \in Z^2(S, \mathbb{C}^\times)$ is completely determined by its restrictions $\mu_{T,T}$, $\mu_{A,A} := \mu|_{(A,e) \times (A,e)}$, and $\mu_{T,A} := \mu|_{(e,T) \times (A,e)}$ by the following formula in which we identify (a, t) with at and (a', t') with $a't'$ ($\forall a, a' \in A, t, t' \in T$):

$$\mu(at, a't') = \mu_{T,T}(t, t')\mu_{T,A}(t, a')\mu_{A,A}(a, {}^t a')\mu_{AA}(a({}^t a'), \rho(t, t')). \tag{5}$$

PROPOSITION 2.1. *A 2-cocycle $\sigma \in Z^2(A, \mathbb{C}^\times)$ can be extended to a normal 2-cocycle $\mu \in Z^2(S, \mathbb{C}^\times)$ if and only if:*

1) σ is cohomologically S/A-invariant, i.e., there exists $\mu_{T,A} \in C^1(S/A, Fun(A, \mathbb{C}^\times))$ such that $\dfrac{\sigma}{{}^t\sigma} = \partial_A^1 \mu_{T,A}(t, \cdot)$, where $t \in T, {}^t\sigma(a, b) := \sigma({}^t a, {}^t b)$.

2) $\mu_{T,A} \in Z^1(T, Fun(A, \mathbb{C}^\times))$, where ${}^{t'}\mu_{T,A}(t, a) := \mu_{T,A}(t, {}^{t'}a), \ \forall \ t, t' \in T, a \in A.$

3) The 3-cochain on T

$$\zeta(t, t', t'') := \mu_{T,A}(t, \rho(t', t''))\frac{\sigma({}^t\rho(t', t''), \rho(t, t't''))}{\sigma(\rho(t, t'), \rho(tt', t''))}$$

is a 3-coboundary, i.e., can be presented as $\partial^2 \mu_{T,T}$ for some $\mu_{T,T} \in C^2(T, \mathbb{C}^\times)$.

Proof. 1) Condition 1) is necessary due to [12], Proposition 1.5.8 (iii).

2) If a normal extension μ of σ exists, write for it the 2-cocycle equality restricted to $(e, T) \times (e, T) \times (e, T)$, in terms of the product in S. Passing then to the product in T and using (5), we get the necessity of condition 3).

3) By direct calculations, exactly like in [12], Lemma 2.2.4, one has:

$$\mu_{T,A}(tt', a) = \mu_{T,A}(t', a)\mu_{T,A}(t, {}^{t'}a)$$

with the product tt' in T. This means that $\mu_{T,A} \in Z^1(T, Fun(A, \mathbb{C}^\times))$.

Vice versa, it is straightforward to check that the relation (5) defines, under conditions 1), 2), 3), a normal extension of σ. ∎

REMARK 2.2. 1) In a slightly more general situation when 2-cocycles μ and σ are replaced by 2-cochains such that $\partial^2\mu = \partial^2 h$ for some $h \in C^2(S, \mathbb{C}^\times)$ and $\partial^2\sigma = \partial^2 h|_{A \times A \times A}$, respectively, one can use Proposition 2.1 applied to 2-cocycle $\dfrac{\sigma}{h_{|A \times A}}$ which describes when this 2-cocycle can be extended to a normal 2-cocycle $\dfrac{\mu}{h}$ or, equivalently, when σ can be extended to μ.

2) Let $\sigma = \partial^1\eta$ be a cohomologically S/A-invariant 2-coboundary on A and μ its normal extension described by $\mu_{T,A}$ and $\mu_{T,T}$ as above. Then one can deduce from (5) that μ is a 2-coboundary if and only if $\mu_{T,A} = \partial^0\tilde\eta$ and $\mu_{T,T} = \dfrac{\partial^1 f_T}{\eta \circ \rho}$, where $f_T \in C^1(T, \mathbb{C}^\times)$ and $\tilde\eta(t, \cdot) \equiv \eta$ is a constant function.

2.2. Invertible bimodule categories over $c(0) = Vec_S^\omega$

2.2.1. Indecomposable left $c(0)$-module categories.
These categories are indexed (see [22]) by conjugacy classes of pairs (A, μ), where A is a subgroup of S and a 2-cochain $\mu \in C^2(A, \mathbb{C}^\times)$ satisfies $\partial^2\mu = \omega|_{A \times A \times A}$ (so $\mu|_{A \times A \times A} = 1$ in $H^3(A, \mathbb{C}^\times)$). A 2-cochain μ satisfying the relation $\partial^2\mu = \omega$ will be called an ω-2-cocycle.

If $\mathcal{M}(A, \mu)$ is such a category, then group S acts transitively on the left on the set $Irr(\mathcal{M}(A, \mu)) = S/A$ of its simple objects; its associativity isomorphisms are defined by a 2-cochain $\tilde\mu(s, t, M) \in C^2(S, C)$ induced from μ and such that

$$\tilde\mu(t, u, M)\tilde\mu(s, tu, M) = \omega(s, t, u)\tilde\mu(s, t, u \cdot M)\tilde\mu(st, u, M). \tag{6}$$

In its turn, $\mu(a, b) = \tilde\mu(a, b, \mathbf{1})$ for all $a, b \in A$.

REMARK 2.3. If $\omega = 1$, the induction above is given explicitly by the map φ_2 from subsection 2.1. In general, there is no canonical way for such an induction. One can proceed as follows. Fix $\mu_0 \in C^2(A, \mathbb{C}^\times)$ such that $\partial^2\mu_0 = \omega|_{A \times A \times A}$ and $\tilde\mu_0 \in C^2(S, C)$ such that $\partial^2\tilde\mu_0 = \omega$. Then, for any other $\mu \in C^2(A, \mathbb{C}^\times)$ such that $\partial^2\mu = \omega|_{A \times A \times A}$, we can put $\tilde\mu = \tilde\mu_0\varphi_2(\mu/\mu_0)$.

Two pairs, (A, μ) and (A', μ'), give rise to equivalent $c(0)$-module categories if and only if $A' = sAs^{-1}$ for some $s \in S$ and μ is cohomologous to the s-conjugate $(\mu')^s$ of μ', that is, they differ by a 2-coboundary. Let

$\Omega_{A,\omega} :=$ the set of equivalence classes of $\{\mu \in C^2(A, \mathbb{C}^\times) | \partial^2\mu = \omega|_{A \times A \times A}\}$.

Note that $\Omega_{A,\omega}$ is a torsor over $H^2(A, \mathbb{C}^\times)$, in particular, $\Omega_{A,1} = H^2(A, \mathbb{C}^\times)$. For any $s \in S$ and $\mu \in \Omega_{A,\omega}$, let us define

$$\mu \triangleleft s := \mu^s \times \Upsilon_s|_{A \times A}, \qquad \text{where}$$

$$\Upsilon_s(t, u) := \frac{\omega(sts^{-1}, sus^{-1}, s)\omega(s, t, u)}{\omega(sts^{-1}, s, u)}, \quad \mu^s(t, u) := \mu(sts^{-1}, sus^{-1}),$$

for all $s, t, u \in S$. Let $(\Omega_{A,\omega})^S$ be the set of S-invariant elements of $\Omega_{A,\omega}$, i.e.,

$$(\Omega_{A,\omega})^S := \{\mu \in \Omega_{A,\omega} | \mu^s \times \Upsilon_s|_{A \times A} = \mu \text{ in } H^2(A, \mathbb{C}^\times), \text{ for all } s \in S\}.$$

If $\omega = 1$, we have the usual definition of the class of S-invariant 2-cocycles on A.

2.2.2. Invertible $c(0)$-bimodule categories. By definition, a $c(0)$-bimodule category $\mathcal{M}$ is a left $Vec^{\omega \otimes \omega^{op}}_{S \times S^{op}}$-module category, where S^{op} is the group opposite to S and also $\omega^{op}(s^{op}, t^{op}, u^{op}) = \omega^{-1}(s^{-1}, t^{-1}, u^{-1})$, for all $s, t, u \in S$. The action of $(s, t^{op}) \in S \times S^{op}$ on $M \in Irr(\mathcal{M})$ defines left and right actions of S by $(s, t^{op}) \cdot M := (s \cdot M) \cdot t$, so $\mathcal{M}$ can be viewed as both left and right $c(0)$-module category. Note that right indecomposable $c(0)$-module categories are also parameterized by the classes of equivalence of pairs (A, μ), their associativity isomorphisms are defined by 2-cochains $\tilde{\mu}(s, t, M) \in C^2(S, C)$ induced from μ and satisfying

$$\tilde{\mu}(M \cdot s, t, u))\tilde{\mu}(M, s, tu) = \omega(s, t, u)\tilde{\mu}(M, s, t)\tilde{\mu}(M, st, u). \tag{7}$$

If $\mathcal{M}$ is *invertible* (i.e., $\mathcal{M}^{op} \boxtimes_{c(0)} \mathcal{M} \cong \mathcal{M} \boxtimes_{c(0)} \mathcal{M}^{op} \cong c(0)$, where $\mathcal{M}^{op}$ is the $c(0)$-bimodule category opposite to $\mathcal{M}$ and $\boxtimes_{c(0)}$ is the relative tensor product; see [5]), then it is indecomposable as both left and right $c(0)$-module category (see [5], Corollary 4.4), so it is indecomposable as a left $Vec^{\omega \otimes \omega^{op}}_{S \times S^{op}}$-module category. Thus, it is of the form $\mathcal{M}(L, \mu)$, where $L < S \times S^{op}$ and $\mu \in C^2(L, \mathbb{C}^\times)$ satisfies $\partial^2 \mu = (\omega \otimes \omega^{op})|_{L \times L \times L}$. Its associativity isomorphisms are defined by a 2-cochain $\tilde{\mu} \in C^2(S \times S^{op}, C)$ induced from μ.

Since S acts transitively on $(S \times S^{op})/L$ on both sides, we have

$$(S \times \{e\})L = (\{e\} \times S^{op})L = S \times S^{op}.$$

Let A_1 be the subgroup of S such that $L \cap (S \times \{e\}) = A_1 \times \{e\}$ and A_2 be the subgroup of S^{op} such that $L \cap (\{e\} \times S^{op}) = \{e\} \times A_2$, and let us denote $\mu^l = \mu|_{(A_1, e) \times (A_1, e)}, \mu^r = \mu|_{(e, A_2) \times (e, A_2)}$. Then $\mathcal{M}(L, \mu)$ viewed as a left (resp., right) $c(0)$-module category is equivalent to $\mathcal{M}(A_1, \mu^l)$ (resp., to $\mathcal{M}(A_2, \mu^r)$).

LEMMA 2.4. *The maps $f_1 : (S \times S^{op})/L \to S/A_1$ and $f_2 : (S \times S^{op})/L \to S^{op}/A_2$ defined by $M \mapsto p_1(M \cap (S \times \{e\}))$ and $M \mapsto p_2(M \cap (\{e\} \times S^{op}))$, respectively, where $p_1 : (s, t^{op}) \mapsto s$, $p_2 : (s, t^{op}) \mapsto t^{op}$, for all $s, t \in S$, are well defined bijections between $Irr(\mathcal{M})$ and S/A_1 (resp., S^{op}/A_2). Let us define $Inv : S^{op}/A_2 \to A_2 \setminus S$ by $s^{op}A_2 \mapsto A_2 s^{-1}$, then the composition $f := Inv \circ f_2 \circ f_1^{-1}$ is a natural bijection:*

$$f : S/A_1 \to A_2 \setminus S$$

such that:

$$L = \{(s, t^{op}) \in S \times S^{op}/f(sA_1) = A_2 t\}. \tag{8}$$

We will denote L in (8) by $L(A_1, A_2, f)$ and, if $A_1 = A_2 = A$, by $L(A, f)$.

The left action of S (resp., S^{op}) on $Irr(\mathcal{M})$ identified with S/A_1 (resp., S^{op}/A_2), is given by the multiplication of left classes. The left action of S^{op} on $Irr(\mathcal{M})$ identified with S/A_1, is the right action of S on S/A_1 given by:

$$(xA_1) \cdot s = xf^{-1}(A_2 s^{-1}).$$

Proof. As $(S \times \{e\})L = S \times S^{op}$, then for any $s, t \in S$ there exists $\sigma \in S$ such that $(\sigma, e) \in (s, t^{op})L$, so $p_1(M \cap (S \times \{e\}))$ is a nonempty class in S/A_1, for any $M \in Irr(\mathcal{M})$. Thus, the map f_1 above is a bijection of $Irr(\mathcal{M})$ and S/A_1, and similarly for f_2. One can see that the natural left action of S (resp., S^{op}) on $(S \times S^{op})/L$ becomes the natural left action of S (resp., S^{op}) on S/A_1 (resp., S^{op}/A_2). Moreover, for all $s, t \in S, f(sA_1) = A_2 t$ means $(s, e)L = (e, (t^{op})^{-1})L$ which is equivalent to $(s, t^{op}) \in L$. So, $L = \{(s, t^{op}) \in$

$S \times S^{op} | f(sA_1) = A_2 t\}$. For any $M \in Irr(\mathcal{M})$, there exists $x \in S$ such that $M = (x, e)L$ and, after identification with S/A_1, $M = xA_1$. One also has, for any $s \in S$: $(e, s^{op})M = (x, s^{op})L$; if $y \in S$ is such that $(x, s^{op})L = (y, e)L$, this means $(x^{-1}y, (s^{op})^{-1}) \in L$, hence $yA_1 = xf^{-1}(A_2 s^{-1})$. ∎

Let us describe $\mathcal{M}(L, \mu)$ in other terms; see [10], 2.1.1. Using notations $\tilde{\mu}^l(x, y, M) := \tilde{\mu}((x, e), (y, e), M)$, $\tilde{\mu}^r(M, x, y) := \tilde{\mu}((e, y^{op}), (e, x^{op}), M)$, and finally also let us define $\tilde{\chi}(x, M, y) := \tilde{\mu}((x, e), (e, y^{op}), M)$ and the decomposition

$$\tilde{\mu}((x_1, x_2^{op}), (y_1, y_2^{op}), M) = \tilde{\chi}(x_1, y_1 \cdot M, y_2)\tilde{\mu}^l(x_1, y_1, M)\tilde{\mu}^r((x_1 y_1) \cdot M, y_2, x_2), \qquad (9)$$

(see [11], p. 27), one can check that $\underline{\partial}^2 \tilde{\mu}^l = \omega, \partial^2 \tilde{\mu}^r = \omega$ and that the following compatibility conditions hold:

$$\tilde{\mu}^l(x, y, M \cdot z)\tilde{\chi}(xy, M, z) = \tilde{\chi}(y, M, z)\tilde{\chi}(x, y \cdot M, z)\tilde{\mu}^l(x, y, M), \qquad (10)$$

$$\tilde{\mu}^r(M, y, z)\tilde{\chi}(x, M, yz) = \tilde{\chi}(x, M \cdot y, z)\tilde{\chi}(x, M, y)\tilde{\mu}^r(x \cdot M, y, z). \qquad (11)$$

Here $\tilde{\mu}^l$ and $\tilde{\mu}^r$ define left and right Vec_S^ω-module category structures on $\mathcal{M}(L, \mu)$, respectively, $x, x_1, x_2, y, y_1, y_2, z \in S$, $M \in Irr(\mathcal{M}(L, \mu))$.

LEMMA 2.5. *If $\mathcal{M}(L, \mu)$ is an invertible $c(0)$-bimodule category, then:*

(i) *A_1 and A_2 are normal abelian subgroups equipped with S-invariant ω-2-cocycles μ^l and μ^r, respectively.*
(ii) *The map $f : S/A_1 \xrightarrow{\sim} S/A_2$ in Lemma 2.4 is a group anti-isomorphism.*

Proof. [5], Definition 4.1 and Proposition 3.5 imply that the dual category of $c(0)$ with respect to $\mathcal{M}(L, \mu)$ viewed as a right $c(0)$-module category (and so equivalent to $\mathcal{M}(A_2, \mu^r)$), is equivalent to $c(0)$ itself which is pointed. But due to [16], Theorem 3.4, this is possible if and only if the pair (A_2, μ^r) satisfies conditions (i). Similarly for (A_1, μ^l).

(ii) Follows from (i) and from Lemma 2.4. ∎

In what follows we denote by $\chi(\cdot, \cdot)$ the bicharacter $\tilde{\chi}(\cdot, \mathbf{1}, \cdot)|_{A_1 \times A_2}$.

2.2.3. The additive endofunctor $L(s)$ of $\mathcal{M}(L, \mu)$ defined by left multiplication by (s, e), where $s \in S$, is isomorphic to the identity if and only if $s \in A_1$. Due to (11), its right $c(0)$-module functor structure can be defined by

$$\tilde{\chi}^{-1}(s, M, x)id_{s \cdot M \cdot x} : L(s)(M \cdot x) \xrightarrow{\sim} (L(s)(M)) \cdot x, \qquad s, x \in S.$$

Similarly, the additive endofunctor $R(s)$ of $\mathcal{M}(L, \mu)$ defined by left multiplication by (e, s^{op}), has a structure of a left $c(0)$-module functor:

$$\tilde{\chi}(x, M, s)id_{x \cdot M \cdot s} : R(s)(x \cdot M) \xrightarrow{\sim} x \cdot (R(s)(M)), \qquad x, s \in S.$$

Let us summarize these observations:

LEMMA 2.6. *$L(s)$ is equivalent to the identity as a right $c(0)$-module autoequivalence of $\mathcal{M}(L, \mu)$ if and only if $s \in A_1$ and $a_s(\cdot) = 1$ on A_2, where the group homomorphism $a : A_1 \to \widehat{A_2}$ is defined by*

$$a : s \mapsto a_s(x) := \chi^{-1}(s, x), x \in A_2. \qquad (12)$$

A similar statement is valid for $R(s)$.

PROPOSITION 2.7. $\mathcal{M}(L, \mu)$ *is invertible if and only if:*

(i) *The conditions of Lemma 2.5 hold.*
(ii) *There is μ' cohomologous to μ such that the well defined S-invariant bicharacter*
$\chi := \mu'|_{(A_1, e) \times (e, A_2)} : A_1 \times A_2 \to \mathbb{C}^\times$ *is non-degenerate.*

Proof. The only thing to prove is (ii). [5], Prop. 4.2 claims that $\mathcal{M} = \mathcal{M}(L, \mu)$ is invertible if and only if the functors $c(0) \to Fun_{c(0)}(\mathcal{M}, \mathcal{M}) : s \mapsto L(s)$ (respectively, $c(0) \to Fun(\mathcal{M}, \mathcal{M})_{c(0)} : s \mapsto R(s)$) are equivalences. Since those functors are tensor, the latter condition is equivalent to $L(s) \ncong id_\mathcal{M}$ as a right $c(0)$-module functor (respectively, $R(s) \ncong id_\mathcal{M}$ as a left $c(0)$-module functor), for all $s \neq e$. Due to Lemma 2.6, these conditions hold if and only if the group homomorphisms (12) and

$$A_2 \to \widehat{A_1} : x \mapsto a'_x, \text{where } a'_x(s) := \chi(s, x), \ s \in A_1, \tag{13}$$

are injective. This is equivalent to χ being non-degenerate on $A_1 \times A_2$. ∎

3. Z/2Z-extensions of $c(0) = Vec_S^\omega$

3.1. Group homomorphisms $c : \mathbb{Z}/2\mathbb{Z} \to BrPic(Vec_S^\omega)$ and fusion rings

3.1.1. Homomorphisms $c : \mathbb{Z}/2\mathbb{Z} \to BrPic(Vec_S^\omega)$. Such a group homomorphism c is defined by an invertible $c(0)$-bimodule category $\mathcal{M}(L, \mu)$ which is equivalent to its opposite. The last one, $\mathcal{M}^{op}(L, \mu)$, has the same simple objects as $\mathcal{M}(L, \mu)$ on which $S \times S^{op}$ acts as $(s, t^{op}) \cdot_{op} M = (s, t^{op})^\vee M$, where $\vee : (s, t^{op}) \mapsto (t^{-1}, (s^{-1})^{op})$ is canonical involutive automorphism of $S \times S^{op}$ (see [5]). The corresponding subgroup of $S \times S^{op}$ is $L^{op} = \vee(L)$ and $\mu_{op} = \mu(\vee \times \vee)^{-1}$. Passing to the quotient we have a canonical bijection $\overline{\vee} : (S \times S^{op})/L \mapsto (S \times S^{op})/L^{op}$ which transforms the opposite action of $S \times S^{op}$ on $(S \times S^{op})/L$ into the left natural action $\cdot_\vee$ of $S \times S^{op}$ on $(S \times S^{op})/L^{op}$, and one can check that $\mathcal{M}^{op}(L, \mu) \cong \mathcal{M}^{op}(L^{op}, \mu_{op})$. In particular, $\mu_{op}^l = (\mu^r \circ \vee)^{-1}, \mu_{op}^r = (\mu^l \circ \vee)^{-1}, \chi_{op} = \chi \circ \vee$.

In terms of Lemma 2.5, $L(A_1, A_2, f)^{op} = L(A_2, A_1, f^{-1})$, so $\mathcal{M}(L, \mu) \cong \mathcal{M}(L, \mu)^{op}$ if and only if subgroups $L(A_1, A_2, f)$ and $L(A_2, A_1, f^{-1})$ are conjugate and μ_{op} is cohomologous to the conjugate of μ. In particular, this implies $A_1 = A_2 = A$, so f is an anti-isomorphism of S/A.

LEMMA 3.1. *If f and f' are anti-isomorphisms of S/A, then $L = L(A, f)$ and $L' = L(A, f')$ are conjugate subgroups of $S \times S^{op}$ if and only if $f = Ad(xA) \circ f'$ for some $x \in S$. For such an x, we have $L' = Ad(1, x)L$ in $S \times S^{op}$. In particular, L and L^{op} are conjugate if and only if*

$$f^2 = Ad(xA) \qquad \text{for some} \quad x \in S. \tag{14}$$

For such an x, we have $L^{op} = Ad(x, 1)L = Ad(1, x)L$ in $S \times S^{op}$.

Proof. The sufficiency is clear. Let $(h, k) \in S \times S^{op}$ be such that $L' = Ad(h, k)L$. If $(s', t') \in L'$, then $f'(s'A) = t'A$, but $Ad(h, k)^{-1}(s', t') \in L$, so $f((hA)(s'A)(hA)^{-1}) = (kA)(t'A)(k^{-1}A)$, from where $f = Ad(xA) \circ f'$ for $xA = f(hA)^{-1}(kA)$.

For any $x \in S$ such that $f = Ad(xA) \circ f'$ and any $(s', t') \in L'$, one has in $S \times S^{op}$: $Ad(1, x^{-1})(s', t') = (s', xt'x^{-1})$, but $(xA)(t'A)(x^{-1}A) = Ad(xA) \circ f'(s'A) = f(s'A)$, so $Ad(1, x^{-1})(s', t') \in L$. The proof of the remaining statements is similar. ∎

REMARK 3.2. If $\mathcal{M}(L_1, \mu_1)$ and $\mathcal{M}(L_0, \mu_0)$ are equivalent invertible $c(0)$-bimodule categories, then $L_1(A_1, f_1)$ and $L_0(A_0, f_0)$ are conjugate and μ_1 is cohomologous to a conjugate of μ_0, so $A_1 = A_0 = A$ and $f_1 = Ad(xA) \circ f_0$ for some $x \in S$. In terms of the triples $(\tilde{\mu}_1^l, \tilde{\mu}_1^r, \tilde{\chi}_1)$ and $(\tilde{\mu}_0^l, \tilde{\mu}_0^r, \tilde{\chi}_0)$, this equivalence is a bijection F of S/A equipped with left and right $\mathcal{D}$-module functor structures $f^l(s, M) : F(sM) \mapsto sF(M), f^r(M, s) : F(Ms) \mapsto F(M)s$ such that, for all $s, t \in S, M \in Irr(\mathcal{M}(L_1, \mu_1))$:

$$\tilde{\mu}_1^l(s, t, F(M)) f^l(st, M) = f^l(t, M) f^l(s, t \cdot M) \tilde{\mu}_0^l(s, t, M),$$

$$\tilde{\mu}_1^r(F(M), s, t) f^r(M, st) = f^r(M, s) f^r(M \cdot s, t) \tilde{\mu}_0^r(M, s, t),$$

$$\tilde{\chi}_1(s, F(M), t) f^l(s, M) f^r(s \cdot M, t) = f^r(M, t) f^l(s, M \cdot t) \tilde{\chi}_0(s, M, t)$$

(see [10], Remark 2.14). This implies that μ_1^l and μ_1^r are cohomologous, respectively, to μ_0^l and μ_0^r, and that $\chi_1 = \chi_0$.

As $S \cong A \underset{\rho}{\rtimes} T, \ T = S/A$ (see [3], IV, 3), we have the following

LEMMA 3.3. *Identifying f with an anti-automorphism ϕ of T such that ϕ^2 is inner, we have $L(A, f) \cong (A \times A) \underset{\overline{\rho}}{\rtimes} T$ with the action of T on $A \times A$ given by ${}^{\overline{t}}(a, a') = ({}^t a, {}^{\phi(t^{-1})} a')$ and the 2-cocycle $\overline{\rho}(t, t') = (\rho(t, t'), {}^{(t't)^{-1}} \rho(t', t))$.*

The map $\tilde{\phi}$: $((x, t), (y, s)) \mapsto s^{-1} \phi(t)$ factors through L to a bijection $(S \times S^{op})/L \to T$. If T is abelian, $L(A, f) \triangleleft (S \times S^{op})$ and $Irr(\mathcal{M}(L, \mu))$ is a group isomorphic to T via the canonical decomposition of $\tilde{\phi}$.

Proof. Obviously,

$$L(A, f) = \{((a, t), (b, \phi(t)) / a, b \in A, t \in T\},$$

and the map $((a, t), (b, \phi(t)) \mapsto (a, \phi(t)^{-1} b, t)$ is an isomorphism $L(A, f) \to (A \times A) \underset{\overline{\rho}}{\rtimes} T$ with the above action of T on $A \times A$ and 2-cocycle $\overline{\rho}$. Let $x = ((a, t), (b, s))$ and $y = ((a', t'), (b', s'))$, then $x^{-1}y \in L(A, f)$, means that $\phi(t^{-1}t') = s's^{-1}$ or $s'^{-1}\phi(t') = s^{-1}\phi(t)$, the remaining statements follow. ∎

COROLLARY 3.4. *1) Given a triple (μ^l, μ^r, χ) as above, let us construct the $\omega \otimes \omega^{op}$-2-cocycle on $A \times A$:*

$$\mu_0((a, b), (a', b')) := \mu^l(a, a') \mu^r(b, b') \chi(a, b').$$

In fact, $\mu_0 = \mu|_{A \times A}$. Let $h \in C^2(L, \mathbb{C}^\times)$ be such that $(\omega \otimes \omega^{op})|_{L \times L \times L} = \partial^2 h$. Then there exists an $\omega \otimes \omega^{op}$-2-cocycle μ on L such that $\mu^l = \mu|_{(A,e) \times (A,e)}, \mu^r = \mu|_{(e,A) \times (e,A)}, \chi = \mu|_{(A,e) \times (e,A)}$ if and only if the 2-cocycle $\dfrac{\mu_0}{h_{|A \times A}}$ on $A \times A$ satisfies the conditions of Proposition 2.1 with respect to the action ${}^{\overline{t}}(a, b) = ({}^t a, {}^{\phi(t^{-1})} b)$ of S/A on $A \times A$, the 2-cocycle $\overline{\rho} = (\rho, \rho^{op}) \in Z^2(S/A, A \times A)$, and some $\psi \in Z^1(S/A, Fun(A \times A, \mathbb{C}^\times))$, $\nu \in C^2(T, \mathbb{C}^\times)$.

2) The condition $(\mu \circ (\vee \times \vee))^{-1} \cong \mu^{(e,x)}$, for some $x \in S^{op}$ such that $f^2 = Ad(x)$, implies the conditions $\mu^l(a,b) \cong (\mu^r(b^{-1},a^{-1}))^{-1}, \chi(a,b) = \chi(b^{-1},a^{-1})$; in particular, the bicharacter χ is symmetric.

Proof. 1) Lemma 3.3 and relation $(\omega \otimes \omega^{op})|_{L \times L \times L} = \partial^2 h$ allow to use Remark 2.2, 1) which gives the needed result. Note that the relation between μ and extensions of μ^l, μ^r and χ (in fact, of μ^l_0, μ^r_0 and χ_0) to L is:

$$\mu((a,b,t),(a',b',t')) = \mu^l((a,b,t),(a',b',t'))\mu^r((a',b',t),(a,b,t'))$$
$$\times \chi((a,b,t),(a',b',t')), \text{ for all } a,a',b,b' \in A, t,t' \in T. \quad (15)$$

2) Direct computation. ∎

Let us summarize the above considerations. We will denote $\psi := \mu_{T,A \times A}$, $\nu := \mu_{T,T}$ (see Preliminaries), $\varepsilon(M) := f^{-1}(M^{-1})$, for all $M \in S/A$, and $(BrPic(Vec^\omega_S))_{(A,\varepsilon)}$ the subset of $BrPic(Vec^\omega_S)$ composed by order two elements attached to $L < S \times S^{op}$, i.e., to the couple (A, ε).

LEMMA 3.5. *Homomorphisms $c : \mathbb{Z}/2\mathbb{Z} \to \pi_1$ are indexed by collections $(A, \mu^r, \chi, \varepsilon, \psi, \nu)$, where $A \triangleleft S$ is abelian and equipped with a symmetric non-degenerate bicharacter χ, $\mu^r \in (\Omega_{A,\omega})^S$, $\varepsilon \in Aut(S/A)$ such that ε^2 is inner, $\psi \in Z^1(S/A, Fun(A \times A, \mathbb{C}^\times))$, $\nu \in C^2(S/A, \mathbb{C}^\times)$ such that:*

(i) $\frac{\mu_0}{{}^t\mu_0} = \partial^1_{A \times A} \psi(t, \cdot)$, *for any $t \in S/A$, where the $\omega \otimes \omega^{op}$-2-cocycle μ_0 is defined above;*

(ii) $\partial^2 \nu(t,t',t'') = \psi(t, \overline{\rho}(t',t'')) \frac{\mu_0({}^t\overline{\rho}(t',t''), \overline{\rho}(t,t't''))}{\mu_0(\overline{\rho}(t,t'), \overline{\rho}(tt',t''))}, \forall t,t',t'' \in S/A;$

(iii) *There exist $k \in C^1(A \times A, \mathbb{C}^\times), q \in C^1(S/A, \mathbb{C}^\times)$ such that:*

$$\mu_0[\mu_0 \circ (\vee \times \vee)] = \partial^1 k,$$
$$\psi[\psi \circ (\varepsilon \times \vee)] = \partial^0 \tilde{k},$$
$$\nu[\nu \circ (\varepsilon \times \varepsilon)] = \frac{\partial^1 q}{k \circ \overline{\rho}},$$

where $\tilde{k}$ is the constant map $t \mapsto k$ on S/A;

(iv) $(BrPic(Vec^\omega_S))_{(A,\varepsilon)}$ *is not empty.*

Two collections, $(A, \mu^r, \chi, \varepsilon, \psi, \nu)$ and $(A', \mu'^r, \chi', \varepsilon', \psi', \nu')$, define the same homomorphism if and only if: $A = A', \varepsilon = \varepsilon'$ in $Out(S/A)$, $\chi = \chi', \mu^r \cong \mu'^r$ and there exist $\eta \in C^1(A \times A, \mathbb{C}^\times), \phi \in C^1(S/A, \mathbb{C}^\times)$ such that $\frac{\mu'_0}{\mu_0} = \partial^1 \eta, \frac{\psi'}{\psi} = \partial^0 \tilde{\eta}$, where $\tilde{\eta}$ is constant map $t \mapsto \eta$, and $\frac{\nu'}{\nu} = \frac{\partial^1 \phi}{\eta \circ \overline{\rho}}$.

Proof. 1) Any homomorphism c is defined by an equivalence class of invertible $c(0)$-bimodule categories $c(1)$ such that $c(1) \cong c(1)^{op}$, i.e., of the form $\mathcal{M}(L, \mu)$ (see Proposition 2.7), where $\mu \cong (\mu \circ (\vee \otimes \vee))^{-1}$. The subgroup L of $S \times S^{op}$ is of the form $L(A, f)$ - see Lemmas 2.5 and 3.1, this gives $A, \varepsilon(t) := f^{-1}(t^{-1})$. Now we can use Corollary 3.4 which describes the relation between μ and triples (μ^l, μ^r, χ) in terms of ψ and ν and gives the relations $\mu^l(a,b) \cong (\mu^r(b^{-1},a^{-1}))^{-1}, \chi(a,b) = \chi(b^{-1},a^{-1})$. To do this, due to

condition (iv), we can use as h a particular fixed $\overline{\mu}$ corresponding to one of invertible $c(0)$-bimodule categories attached to the couple (A, ε) and remark that $\overline{\mu}/\mu \in Z^2(L, \mathbb{C}^\times)$ can be chosen normal changing, if necessary, a representative μ in the same class. Finally, one can check, using Remark 2.2, 2), that conditions (iii) are equivalent to the relation $\mu \cong (\mu \circ (\vee \otimes \vee))^{-1}$.

2) Two $c(0)$-bimodule categories, $\mathcal{M}(L(A, f), \mu)$ and $\mathcal{M}(L(A', f'), \mu')$, are equivalent if and only if $A = A'$ and there exists $x \in S$ such that $L' = Ad(1, x)L$ and $Ad(x) \circ f = f'$ for some $x \in S$ (this gives relations between (A, ε) and (A', ε'); see Lemma 3.1 and Remark 3.2) and $\mu' \cong \mu^{(e,x)}$. In order to show that this last condition is equivalent to the above conditions, one can apply Remark 2.2, 2) to the coboundary $\dfrac{\mu'}{\mu^{(e,x)}}$ and its restrictions $\dfrac{\mu_0'}{\mu_0}, \dfrac{\psi'}{\psi}, \dfrac{\nu'}{\nu}$. ∎

REMARK 3.6. When $\omega = 1$, condition (iv) follows from other conditions because we can take in the proof $h = 1$.

3.1.2. Fusion ring structures for $\mathcal{C} = c(0) \oplus c(1)$. In order to equip the category $\mathcal{C}$ with a tensor product and a duality, we have to define an involutive $c(0)$-bimodule equivalence $\gamma : c(1) \to c(1)^{op} \cong c(1)$ by $\gamma : M \mapsto M^*$, for any $M \in Irr(\mathcal{M}(L, \mu))$, such that $(s \cdot M \cdot t)^* = t^{-1} \cdot M \cdot s^{-1}$. This implies some restrictions on the structure of the fusion ring of $\mathcal{C}$ (see also [14], Lemma 2.3):

PROPOSITION 3.7. *1) Given an invertible $c(0)$-bimodule category $c(1) = \mathcal{M}(L(A, f), \mu)$ equivalent to $c(1)^{op}$, let us suppose that $c(0) \oplus c(1)$ is a fusion category with some tensor product $\otimes$. Then all possible $\mathbb{Z}/2\mathbb{Z}$-graded fusion rings for $c(0) \oplus c(1)$ are characterized by elements $\delta \in S/A$ such that $f^2 = Ad(\delta)$ and $f(\delta) = \delta^{-1}$. Namely, for all $s, s' \in S, M, N \in Irr(\mathcal{M}(L(A, f), \mu))$:*

$$s^* = s^{-1}, \quad M^* = f(M)\delta,$$

$$s \otimes s' = ss', \quad s \otimes M = p(s)M, \quad M \otimes s = Mf^{-1}(p(s)^{-1}), M \otimes N^* = \underset{x \in MN^{-1}}{\oplus} x.$$

2) Let (f, δ) and (f', δ') be as above, then $c(0)$-bimodule categories $c(1) = \mathcal{M}(L(A, f), \mu))$ and $c(1)' = \mathcal{M}(L(A, f'), \mu'))$ are equivalent if and only if there is $x \in S$, such that

$$f' = Ad(xA) \circ f, \ \mu' = \mu \circ Ad(1, x) \text{ in } H^2(L(A, f'), \mathbb{C}^\times).$$

3) $\mathbb{Z}/2\mathbb{Z}$-graded fusion rings for $c(0) \oplus c(1)$ and $c(0) \oplus c(1)'$ are isomorphic if and only if there is $F \in Aut(S)$ such that:

$$F(A) = A, \ F(\delta) = \delta', \ F \circ f \circ F^{-1} = f'.$$

Proof. 1) Relations $s^* = s^{-1}, s \otimes s' = ss'$ are obvious, relations $s \otimes M = p(s)M, \ M \otimes s = Mf^{-1}(p(s)^{-1})$ follow from Lemmas 2.4 and 2.5. The map $\gamma : M \mapsto M^*$ is involutive and, for any $N \in S/A, (M \cdot (au(N)))^* = (au(N))^{-1} \cdot M^*$, so $\gamma(Mf^{-1}(N^{-1})) = N^{-1}\gamma(M)$. If $M = \mathbf{1}$ and $N = f(N')^{-1}$, this gives $\gamma(N') = f(N')\gamma(\mathbf{1})$, and if $N' = \delta := \gamma(\mathbf{1})$, we have $f(\delta) = \delta^{-1}$.

The $\mathbb{Z}/2\mathbb{Z}$-grading implies that $M \otimes N = \oplus_{x \in S} n_x x$, where $n_x \in \mathbb{N}$, and the properties of duality, that $n_e = 1$ if and only if $M = N^*$ and $n_e = 0$ if not. This gives $n_x = 1$ if

and only if $M = xN^*$ (the set of such x is an A-coset) and $n_x = 0$ if not. In the above mentioned terms we have

$$M \otimes N = \bigoplus_{\{x \in S/M^* . x = N\}} x = \bigoplus_{\{x \in S/f(M)\delta f^{-1}(p(x)^{-1}) = N\}} x = \bigoplus_{x \in M\delta^{-1}f(N^{-1})} x$$

which gives $M \otimes N^* = \oplus_{x \in MN^{-1}} x$, so $N = \mathbf{1} = A$ is the only simple object of $c(1)$ such that, for any other simple object M of $c(1)$, one has: $M \otimes N^* = \oplus_{x \in M} x$, hence $\mathbf{1}$ is an invariant of fusion rings.

2) Follows from Lemma 3.1.

3) Let (f, δ) and (f', δ') be two pairs as above, then any isomorphism φ of $\mathbb{Z}/2\mathbb{Z}$-graded fusion rings for $c(0) \oplus c(1)$ and $c(0) \oplus c(1)'$ is given by a pair (F, ϕ) where $F \in Aut(S)$ and $\phi : S/A \to S/A$ are such that, for any $s \in S$, $M, N \in S/A$, one has:

$$\phi(s \otimes M) = F(s) \otimes \phi(M), \phi(\mathbf{1}) = \mathbf{1}, \phi(M \otimes s) = \phi(M) \otimes F(s),$$
$$\phi(M^*) = \phi(M)^*, \varphi(M \otimes N^*) = \phi(M) \otimes \phi(N)^*.$$

The first equality gives $\phi(p(s)M) = p(F(s))\phi(M)$, so $F(A) \subset A$ and F factors through A. Let us write the last one as $F(M)F(N)^{-1} = \phi(M)\phi(N)^{-1}$; together with $\phi(\mathbf{1}) = \mathbf{1}$, this gives $\phi = F$. The remaining equalities can be written as $\phi(Mf^{-1}(p(s)^{-1})) = \phi(M)f'^{-1}(p(F(s^{-1})))$ and $\phi(f(M)\delta) = f'(\phi(M))\delta'$, respectively, which can be summarized by:

$$f' = F \circ f \circ F^{-1}, \ \delta' = F(\delta). \tag{16}$$

The converse also holds: given (f, δ) and $F \in Aut(S)$ such that $F(A) \subset A$, let us define $\phi := F$ and f', δ' by (16); then routine computations show that f' is an anti-isomorphism of S/A such that $f'^2 = Ad(\delta')$ and $f'(\delta') = \delta'^{-1}$. If, moreover, there is $x \in S$ such that $f' = ad(xA) \circ f$ and if one defines $\mu' := \mu \circ Ad(1, x)$, then $\mathcal{M}(L(A, f'), \mu')$ is an invertible $c(0)$-bimodule category equivalent to its opposite and such that ϕ associated with (F, ϕ) gives an isomorphism of the corresponding $\mathbb{Z}/2\mathbb{Z}$-graded fusion rings. ∎

3.2. Vec_S^ω-bimodule equivalences $M_{g,h}$. Given a group homomorphism $c : \mathbb{Z}/2\mathbb{Z} \to BrPic(c(0)) : 0 \mapsto \overline{c(0)}, 1 \mapsto \overline{c(1)} := \overline{\mathcal{M}(L, \mu)}$, there exist (by definition of an invertible $c(0)$-bimodule category) $c(0)$-bimodule equivalences $M_{g,h} : c(g) \boxtimes_{c(0)} c(h) \to c(gh)$ $(g, h \in \{0, 1\})$. They are defined by linear functors respecting the fusion rule of Proposition 3.7: $M_{0,0} : t \boxtimes s \mapsto ts$, $M_{0,1} : t \boxtimes M \mapsto t \cdot M$, $M_{1,0} : M \boxtimes t \mapsto M \cdot t$, $M_{1,1} : M \boxtimes N \mapsto \oplus_{M = xN^*} x$ equipped with natural isomorphisms $M_{g,h}^l$ and $M_{g,h}^r$.

The $c(0)$-bimodule category structure on $c(0)$ is defined by $\mu^l(s, t, x) = \omega(s, t, x)$, $\mu^r(x, t, s) = \omega^{-1}(x, t, s), \chi(t, x, s) = \omega(t, x, s)$, so the definition of a $c(0)$-bimodule functor and relations (6), (7), (10), (11) give, for all $t, x, y \in S$, $M \in S/A$:

$$M_{0,0}^l(t, x \boxtimes y) = \omega(t, x, y)id_{txy}, \ M_{0,0}^r(x \boxtimes y, t) = \omega^{-1}(x, y, t)id_{xyt},$$
$$M_{1,0}^l(t, M \boxtimes x) = \tilde{\chi}(t, M, x)id_{tMx}, \ M_{1,0}^r(M \boxtimes x, t) = \tilde{\mu}^r(M, x, t)id_{Mxt},$$
$$M_{0,1}^l(t, x \boxtimes M) = \tilde{\mu}^l(t, x, M)id_{txM}, \ M_{0,1}^r(x \boxtimes M, t) = \tilde{\chi}^{-1}(x, M, t)id_{xMt}.$$

LEMMA 3.8. *Given $c(1) := \mathcal{M}(L, \mu)$, there exists an equivalent $c(0)$-bimodule category for which*

$$\tilde{\mu}^l(s, t, M) = \tilde{\mu}^r((stM)^*, s, t). \tag{17}$$

In this equivalent category we have, up to a functor isomorphism:

$$M^l_{1,1}(t, M \boxtimes_{c(0)} N) := \bigoplus_{M=x \cdot N^*} [\tilde{\mu}^r(M^* t^{-1}, t, x)]^{-1} id_{tx}, \qquad (18)$$

$$M^r_{1,1}(M \boxtimes_{c(0)} N, t) := \bigoplus_{M=x \cdot N^*} [\tilde{\mu}^r(M^*, x, t)]^{-1} id_{xt}. \qquad (19)$$

Proof. $M^l_{1,1}$ and $M^r_{1,1}$ must be defined by the equalities

$$M^l_{1,1}(t, M \boxtimes_{c(0)} N) = \bigoplus_{M=x \cdot N^*} \gamma^l(t, M, N, x) id_{tx},$$

$$M^r_{1,1}(M \boxtimes_{c(0)} N, t) = \bigoplus_{M=x \cdot N^*} \gamma^r(x, M, N, t) id_{xt},$$

where the functions $\gamma^l(t, M, N, x)$ and $\gamma^r(x, M, N, t)$ satisfy, for all $s, t \in S$, $M, N \in Irr(\mathcal{M})$, the conditions

$$\omega(s, t, x)\gamma^l(st, M, N, x) = \gamma^l(t, M, N, x)\gamma^l(s, tM, N, tx)\tilde{\mu}^l(s, t, M),$$

$$\omega^{-1}(x, s, t)\gamma^r(x, M, N, st) = \gamma^r(x, M, N, s)\gamma^r(xs, M, Ns, t)\tilde{\mu}^r(N, s, t),$$

$$\omega(s, x, t)\gamma^l(s, M, N, x)\gamma^r(sx, sM, N, t) = \gamma^r(x, M, N, t)\gamma^l(s, M, Nt, xt).$$

The first of them with $M = N^*$, $x = e$ gives, denoting $g^l(t, M) := \gamma^l(t, M, M^*, e)$:

$$\gamma^l(t, M, N, x) = \frac{g^l(tx, N^*)}{g^l(x, N^*)}[\tilde{\mu}^l(t, x, N^*)]^{-1},$$

and, similarly, the second condition gives, denoting $g^r(t, M) := \gamma^r(e, M, M^*, t)$:

$$\gamma^r(x, M, N, t) = \frac{g^r(xt, M)}{g^r(x, M)}[\tilde{\mu}^r(M^*, x, t)]^{-1}.$$

Now the last condition becomes equivalent to the relation

$$\tilde{\mu}^l(s, t, M) = \tilde{\mu}^r((stM)^*, s, t)\frac{\gamma(t, M)\gamma(s, t \cdot M)}{\gamma(st, M)},$$

where $\gamma(t, M) := \dfrac{g^r(t, t \cdot M)}{g^l(t, M)}$, and one can check that $M^l_{1,1}$ and $M^r_{1,1}$ as above define $M_{1,1}$ as a surjective $c(0)$-bimodule functor.

Next, let us show that, passing to an isomorphic $c(0)$-bimodule functor, we can choose $g^l(t, M) \equiv 1$ and $g^r(t, M) = \gamma(t, t^{-1} \cdot M)$. Indeed, the definition of a $c(0)$-bimodule functor isomorphism implies that two equivalences, $M_{1,1}$ and $M'_{1,1}$, corresponding to two couples, (g^l, g^r) and (g'^l, g'^r), respectively, are isomorphic if and only if there is a natural transformation

$$\alpha(M \boxtimes_{c(0)} N) = \bigoplus_{M=xN^*} \alpha(M, x, N) id_x,$$

where $\alpha(M, x, N)$ satisfies the following system of two equations:

$$\alpha(M, x, N)\frac{g^l(tx, N^*)}{g^l(x, N^*)} = \frac{g'^l(tx, N^*)}{g'^l(x, N^*)}\alpha(t \cdot M, tx, N),$$

$$\alpha(M, x, N)\frac{g^r(xt, M)}{g^r(x, M)} = \frac{g'^r(xt, M)}{g'^r(x, M)}\alpha(M, xt, N \cdot t).$$

Putting $M = N^*$, $x = e$, and denoting $\delta(t, N^*) := \alpha(t \cdot N^*, t, N)$, we have:

$$g^l(t, N^*) = \delta(t, N^*)g'^l(t, N^*), \quad g^r(t, N^*) = \delta(t, (N \cdot t)^*)g'^r(t, N^*).$$

As $\delta(t, N^*)$ can be arbitrary, we can choose $\delta(t, N^*) = g^l(t, N^*)$ and $g'^l(t, N^*) \equiv 1$ which gives $g'^r(t, N^*) = \gamma(t, (N \cdot t)^*)$.

Finally, replacing $\tilde{\mu}^l(s, t, M)$ by $\tilde{\mu}^l(s, t, M)\dfrac{\gamma(t, M)\gamma(s, t \cdot M)}{\gamma(st, M)}$ and also $\tilde{\chi}(s, M, t)$ by

$\tilde{\chi}(s, M, t)\dfrac{\gamma(s, M \cdot t)}{\gamma(s, M)}$ and taking the same $\tilde{\mu}^r(M, s, t)$, we pass to the equivalent $c(0)$-bimodule category, where the relations (17), (18) and (19) are true. ∎

Let us deduce from (17) some properties of $\tilde{\chi}(s, M, t)$ $(s, t \in S, M \in S/A)$.

REMARK 3.9. Theorem 3.4(iii) from [16] and relation (11) imply the existence of $\tilde{\eta}_{p(x)} \in C^1(S, C)$ (defined up to an element of $Z^1(S, C)$) such that map $\tilde{\chi}(x, M, y)\tilde{\eta}_{p(x)}(M, y) \in Z^1(S, C), \tilde{\eta}_{\mathbf{1}} \equiv 1$. Similarly, (10) and (17) imply that, for any fixed $y \in S$, one must have $\tilde{\chi}(x, M, y)\tilde{\eta}_{p(y^{-1})}^{-1}((xM)^*, x) \in \underline{Z}^1(S, C)$. Then Shapiro's lemma shows that $\tilde{\chi}(x, M, y)$ is completely defined, up to a 1-coboundary, by one of its restrictions, $\tilde{\chi}(x, \mathbf{1}, b)$ or $\tilde{\chi}(a, \mathbf{1}, y)$, where $a, b \in A$.

LEMMA 3.10. *For any $x \in S$, there is $\eta_{p(x)} \in C^1(A, \mathbb{C}^\times)$ such that*

$$\tilde{\chi}(x, \mathbf{1}, b) = \chi(\kappa_{\mathbf{1},x}, b)\eta_{p(x)}^{-1}(b).$$

Proof. Fixing $\tilde{\eta}_{p(x)}$ and applying the restriction map to (10), we have

$$\frac{\eta'_{p(x)}(a)\eta'_{p(x)}(b)}{\eta'_{p(x)}(ab)} = \frac{\tilde{\chi}(x, \mathbf{1}, ab)}{\tilde{\chi}(x, \mathbf{1}, a)\tilde{\chi}(x, \mathbf{1}, b)},$$

where $\eta'_{p(x)}(b) := \tilde{\eta}_{p(x)}(b, \mathbf{1})$. So, for any $x \in S, b \in A$, $\tilde{\chi}(x, \mathbf{1}, b)\eta'_{p(x)}(b) = r_x(b)$ is a character on A; this can be written as $\tilde{\chi}(u(p(x)), \mathbf{1}, b)\eta'_{p(x)}(b) = \chi(\kappa_{\mathbf{1},x}, b)^{-1}r_x(b)$, where the character in the right hand side depends only on $p(x)$. Since $\eta'_{p(x)}$ is defined modulo $\hat{A}$, we have the needed equality, where $\eta_{p(x)}(b) := \eta'_{p(x)}(b)\chi(\kappa_{\mathbf{1},x}, b)r_x^{-1}(b)$. ∎

COROLLARY 3.11. *(i) As $\tilde{\chi}(x, N, y)$ satisfies (9), then, putting there $N = \mathbf{1}, M = p(y)$, and $z = b$, we get*

$$\tilde{\chi}(x, M, b) = \frac{\chi(\kappa_{M,x}, b)\eta_M(b)}{\eta_{x \cdot M}(b)}.$$

Similar formula holds for $\tilde{\chi}(a, M, y)$ $(a \in A, y \in S, M \in S/A)$. In particular, $\tilde{\chi}(a, M, b) = \chi(a, b)$, for all $a, b \in A$.

(ii) The following equalities hold, up to a product of two 1-coboundaries:

$$\tilde{\chi}(a, M, x) = \tilde{\chi}(x, x^{-1}M^*, a), \ \tilde{\chi}(x, M, a) = \tilde{\chi}(a, M^*x^{-1}, x) \text{ for all } a \in A, x \in S.$$

Proof. (i) follows from Lemma 3.10, and (ii) from Lemma 3.10 and the symmetry of χ. ∎

Rewriting (10) and (11) with the usage of (17), then comparing again with these relations and, finally, taking into account Corollary 3.11 (ii), we get:

COROLLARY 3.12. *If $\tilde{\mu}^l$ and $\tilde{\mu}^r$ satisfy (17), then the 2-cochain*

$$\beta(z, N, x) := \frac{\tilde{\chi}(x, x^{-1}N^*z^{-1}, z)}{\tilde{\chi}(z, N, x)} \tag{20}$$

is a product of two 1-coboundaries.

Vice versa, let $\tilde{\mu}^r$ satisfy (7) and $\tilde{\chi}$ satisfy the above mentioned properties. Then $\tilde{\mu}^l$ defined by (17) satisfies (6) and (10).

3.3. Vanishing of $O_3(c)$. Associativity isomorphisms. Main result

3.3.1. Quasi-tensor product on $\mathcal{C} = c(0) \oplus c(1)$. Now, according to [5], 8.4, we have to equip $\mathcal{C}$ with *a quasi-tensor product*, i.e., with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that $\otimes \circ (\otimes \times Id_{\mathcal{C}}) \cong \otimes \circ (Id_{\mathcal{C}} \times \otimes)$. If we know a group homomorphism $c : \mathbb{Z}/2\mathbb{Z} \to \pi_1$ equipped with $c(0)$-bimodule equivalences $M_{g,h} : c(g) \boxtimes_{c(0)} c(h) \cong c(gh)$, then [5], Theorem 8.4 says that such a bifunctor exists if and only if the cohomological obstruction $O_3(c)$ vanishes or, in other terms, if the following $c(0)$-bimodule autoequivalences are isomorphic to Id, for all $f, g, h \in \mathbb{Z}/2\mathbb{Z}$:

$$T_{f,g,h} : M_{fg,h}(M_{f,g} \boxtimes_{c(0)} Id_{c(h)})(Id_{c(f)} \boxtimes_{c(0)} M_{g,h}^{-1})M_{f,gh}^{-1} : c(fgh) \to c(fgh). \tag{21}$$

Equivalently, we have to find conditions on parameters of c such that the $c(0)$-bimodule functors $F_{f,g,h} := M_{f,gh} \circ [id \boxtimes M_{g,h}] : \mathcal{C}_f \boxtimes_{c(0)} \mathcal{C}_g \boxtimes_{c(0)} \mathcal{C}_h \to \mathcal{C}_{fgh}$ and $G_{f,g,h} := M_{fg,h} \circ [M_{f,g} \boxtimes id] : \mathcal{C}_f \boxtimes_{c(0)} \mathcal{C}_g \boxtimes_{c(0)} \mathcal{C}_h \to \mathcal{C}_{fgh}$ are isomorphic. In this case, they define the corresponding associativity isomorphisms:

$$\alpha_{f,g,h} : M_{f,gh}(Id_{c(f)} \boxtimes_{c(0)} M_{g,h}) \cong M_{fg,h}(M_{f,g} \boxtimes_{c(0)} Id_{c(h)}). \tag{22}$$

First, we compute $F_{f,g,h}^l, F_{f,g,h}^r, G_{f,g,h}^l$ and $G_{f,g,h}^r$ using the formulas for left and right bimodule structures of a composition of two bimodule functors:

$$\begin{aligned}
(F_2 \circ F_1)^l(t, M) &= F_2^l(t, F_1(M)) \circ F_2(F_1^l(t, M)), \\
(F_2 \circ F_1)^r(M, t) &= F_2^r(F_1(M), t) \circ F_2(F_1^r(M, t)).
\end{aligned} \tag{23}$$

This and the fact that $[id \boxtimes M_{g,h}]^l$ and $[M_{f,g} \boxtimes id]^r$ are identities, give:

$$F_{f,g,h}^l(t, X \boxtimes Y \boxtimes Z) = M_{f,gh}^l(t, X \boxtimes (Y \cdot Z)),$$

$$F_{f,g,h}^r(X \boxtimes Y \boxtimes Z, t) = M_{f,gh}^r(X \boxtimes (Y \cdot Z), t)M_{f,gh}[id_X \boxtimes M_{g,h}^r(Y \boxtimes Z, t)],$$

$$G_{f,g,h}^r(X \boxtimes Y \boxtimes Z, t) = M_{fg,h}^r((X \cdot Y) \boxtimes Z, t),$$

$$G_{f,g,h}^l(t, X \boxtimes Y \boxtimes Z) = M_{fg,h}^l(t, (X \cdot Y) \boxtimes Z)M_{fg,h}[M_{f,g}^l(t, X \boxtimes Y) \boxtimes id_Z].$$

3.3.2. Associativity isomorphisms for $\mathcal{C} = c(0) \oplus c(1)$. The associativity isomorphisms $\alpha_{f,g,h}(X, Y, Z)$, if they exist, must satisfy the system of relations:

$$\alpha_{f,g,h}(t \cdot X, Y, Z)F_{f,g,h}^l(t, X \boxtimes Y \boxtimes Z) = \alpha_{f,g,h}(X, Y, Z)G_{f,g,h}^l(t, X \boxtimes Y \boxtimes Z),$$

$$\alpha_{f,g,h}(X, Y, Z \cdot t)F_{f,g,h}^r(X \boxtimes Y \boxtimes Z, t) = \alpha_{f,g,h}(X, Y, Z)G_{f,g,h}^r(X \boxtimes Y \boxtimes Z, t).$$

At the same time, if this system has a solution, this means that the $c(0)$-bimodule functors $F_{f,g,h}$ and $G_{f,g,h}$ are isomorphic. We compute step by step, using (17) and (20):

1) $f = g = h = 0$. $F_{0,0,0}^l(t, x \boxtimes y \boxtimes z) = \omega(t, x, yz)id_{txyz}$, $F_{0,0,0}^r(x \boxtimes y \boxtimes z, t) = \omega(x, yz, t)\omega(y, z, t)id_{xyzt}$, $G_{0,0,0}^r(x \boxtimes y \boxtimes z, t) = \omega(xy, z, t)id_{xyzt}$, $G_{0,0,0}^l(t, x \boxtimes y \boxtimes z) = \omega(t, xy, z)\omega(t, x, y)id_{txyz}$,

$$\alpha_{0,0,0}(x, y, z) = \omega(x, y, z)id_{xyz}.$$

2) $f = 1, g = h = 0$. $F^l_{1,0,0}(t, M \boxtimes x \boxtimes y) = \tilde{\chi}(t, M, xy)id_{tMxy}$, $F^r_{1,0,0}(M \boxtimes x \boxtimes y, t) = \tilde{\mu}^r(M, xy, t)\omega^{-1}(x, y, t)id_{Mxyt}$, $G^r_{1,0,0}(M \boxtimes x \boxtimes y, t) = \tilde{\mu}^r(Mx, y, t)id_{Mxyt}$, $G^l_{1,0,0}(t, M \boxtimes x \boxtimes y) = \tilde{\chi}(t, M \cdot x, y)\tilde{\chi}(t, M, x)id_{tMxy}$,

$$\alpha_{1,0,0}(M, x, y) = [\tilde{\mu}^r(M, x, y)]^{-1}id_{Mxy}.$$

3) $f = g = 0, h = 1$. $F^l_{0,0,1}(t, x \boxtimes y \boxtimes M) = \tilde{\mu}^l(t, x, y \cdot M)id_{txyM}$, $F^r_{0,0,1}(x \boxtimes y \boxtimes M, t) = \tilde{\chi}^{-1}(x, y \cdot M, t)\tilde{\chi}^{-1}(y, M, t)id_{xyMt}$, $G^r_{0,0,1}(x \boxtimes y \boxtimes M, t) = \tilde{\chi}^{-1}(xy, M, t)id_{xyMt}$, $G^l_{0,0,1}(t, x \boxtimes y \boxtimes M) = \tilde{\mu}^l(t, xy, M)\omega(t, x, y)id_{txyM}$,

$$\alpha_{0,0,1}(x, y, M) = \tilde{\mu}^l(x, y, M)id_{xyM}.$$

4) $f = h = 0, g = 1$. $F^l_{0,1,0}(t, x \boxtimes M \boxtimes y) = \tilde{\mu}^l(t, x, M \cdot y)id_{txMy}$, $F^r_{0,1,0}(x \boxtimes M \boxtimes y, t) = \tilde{\chi}^{-1}(x, M \cdot y, t)\tilde{\mu}^r(M, y, t)id_{xMyt}$, $G^r_{0,1,0}(x \boxtimes M \boxtimes y, t) = \tilde{\mu}^r(x \cdot M, y, t)id_{xMyt}$, $G^l_{0,1,0}(t, x \boxtimes M \boxtimes y) = \tilde{\mu}^l(t, x, M)\tilde{\chi}(t, x \cdot M, y)id_{txMy}$,

$$\alpha_{0,1,0}(x, M, y) = \tilde{\chi}(x, M, y)id_{xMy}.$$

5) $f = g = 1, h = 0$. $F^l_{1,1,0}(t, M \boxtimes N \boxtimes s) = \oplus_{M=xN^*}[\tilde{\mu}^l(t, xs, (N \cdot s)^*)]^{-1}id_{txs}$, $F^r_{1,1,0}(M \boxtimes N \boxtimes s, t) = \oplus_{M=xN^*}[\tilde{\mu}^r(M^*, xs, t)]^{-1}\tilde{\mu}^r(N, s, t)id_{xst}$, also one gets $G^r_{1,1,0}(M \boxtimes N \boxtimes s, t) = \oplus_{M=xN^*}\omega^{-1}(x, s, t)id_{xst}$, $G^l_{1,1,0}(t, M \boxtimes N \boxtimes s) = \oplus_{M=xN^*}\omega(t, x, s)[\tilde{\mu}^l(t, x, N^*)]^{-1}id_{txs}$

$$\alpha_{1,1,0}(M, N, s) = \oplus_{M=xN^*}\tilde{\mu}^r(M^*, x, s)id_{xs}.$$

6) $f = 0, g = h = 1$. $F^l_{0,1,1}(t, s \boxtimes M \boxtimes N) = \oplus_{M=xN^*}\omega(t, s, x)id_{tsx}$, $F^r_{0,1,1}(s \boxtimes M \boxtimes N, t) = \oplus_{M=xN^*}\omega^{-1}(s, x, t)[\tilde{\mu}^r(M^*, x, t)]^{-1}id_{sxt}$, $G^r_{0,1,1}(s \boxtimes M \boxtimes N, t) = \oplus_{M=xN^*}[\tilde{\mu}^r((s \cdot M)^*, sx, t)]^{-1}id_{sxt}$, $G^l_{0,1,1}(t, s \boxtimes M \boxtimes N) = \oplus_{M=xN^*}[\tilde{\mu}^l(t, sx, N^*)]^{-1} \times \tilde{\mu}^l(t, s, M)id_{tsx}$,

$$\alpha_{0,1,1}(s, M, N) = \oplus_{M=xN^*}[\tilde{\mu}^l(s, x, N^*)]^{-1}id_{sx}.$$

7) $f = h = 1, g = 0$. $F^l_{1,0,1}(t, M \boxtimes s \boxtimes N) = \oplus_{Ms=xN^*}[\tilde{\mu}^l(t, x, (s \cdot N)^*)]^{-1}id_{tx}$, $F^r_{1,0,1}(M \boxtimes s \boxtimes N, t) = \tilde{\chi}^{-1}(s, N, t) \oplus_{Ms=xN^*} [\tilde{\mu}^r(M^*, x, t)]^{-1}id_{xt}$, and also $G^r_{1,0,1}(M \boxtimes s \boxtimes N, t) = \oplus_{Ms=xN^*}[\tilde{\mu}^r((M \cdot s)^*, x, t)]^{-1}id_{xt}$, and finally one gets $G^l_{1,0,1}(t, M \boxtimes s \boxtimes N) = \tilde{\chi}(t, M, s) \times \oplus_{Ms=xN^*}[\tilde{\mu}^l(t, x, N^*)]^{-1}id_{tx}$.

Then

$$\alpha_{1,0,1}(M, s, N) = \oplus_{Ms=xN^*}\alpha(M, s, N, x)id_x,$$

where the function $\alpha(M, s, N, x)$ satisfies the following two equations:

$$\alpha(t \cdot M, s, N, tx)[\tilde{\mu}^l(t, x, (s \cdot N)^*)]^{-1} = \alpha(M, s, N, x)[\tilde{\mu}^l(t, x, N^*)]^{-1}\tilde{\chi}(t, M, s).$$

$$\alpha(M, s, N \cdot t, xt)[\tilde{\mu}^r(M^*, x, t)]^{-1}\tilde{\chi}^{-1}(s, N, t) = \alpha(M, s, N, x)[\tilde{\mu}^r((M \cdot s)^*, x, t)]^{-1}.$$

Putting $M = (s \cdot N)^*$, $x = e$ and denoting $\alpha(s, N) := \alpha((s \cdot N)^*, s, N, e)$, one can deduce, for arbitrary M, N and x such that $M \cdot s = x \cdot N^*$:

$$\alpha(M, s, N, x) = \alpha(s, N)\tilde{\chi}(x, x^{-1} \cdot M, s),$$

$$\alpha(M, s, N, x) = \alpha(s, N \cdot x^{-1})\tilde{\chi}(s, N \cdot x^{-1}, x).$$

Comparing these equalities and using (20), where one can write

$$\beta(z, N, s) = \frac{\alpha(z, N)}{\alpha(z, N \cdot s)} \quad \text{with } \alpha(z, N) \in C^1(S, C),$$

one has the following

LEMMA 3.13. *The $c(0)$-bimodule autoequivalence $T_{1,0,1}$ is isomorphic to the identity, and we have:*

$$\alpha_{1,0,1}(M, s, N) = \alpha(s, N) \oplus_{Ms=xN^*} \tilde{\chi}(x, x^{-1} \cdot M, s) id_x. \tag{24}$$

8) $f = g = h = 1$. $F_{1,1,1}^l(t, M \boxtimes N \boxtimes K) = \oplus_{N=yK^*} \tilde{\chi}(t, M, y) id_{tMy}$, $F_{1,1,1}^r(M \boxtimes N \boxtimes K, t) = \oplus_{N=yK^*} \frac{\tilde{\mu}^r(M,y,t)}{\tilde{\mu}^r(N^*,y,t)} id_{Myt}$, $G_{1,1,1}^r(M \boxtimes N \boxtimes K, t) = \oplus_{M=xN^*} \tilde{\chi}(x, K, t)^{-1} id_{xKt}$, $G_{1,1,1}^l(t, M \boxtimes N \boxtimes K) = \oplus_{M=xN^*} \frac{\tilde{\mu}^l(t,x,K)}{\tilde{\mu}^l(t,x,N^*)} id_{txK}$.

As $\alpha_{1,1,1}(M, N, K) : \oplus_{M=xN^*} x \cdot K \mapsto \oplus_{N=yK^*} M \cdot y$ (let us note that $x \cdot K = M \cdot y$), this isomorphism is defined by an $|A| \times |A|$-matrix

$$(\alpha(M, N, K, x, y) id_{x \cdot K})_{M=xN^*; N=yK^*},$$

whose coefficients satisfy the system of the following equations

$$\alpha(t \cdot M, N, K, tx, y) \tilde{\chi}(t, M, y) = \alpha(M, N, K, x, y) \frac{\tilde{\mu}^l(t, x, K)}{\tilde{\mu}^l(t, x, N^*)},$$

$$\alpha(M, N, K \cdot t, x, yt) \frac{\tilde{\mu}^r(M, y, t)}{\tilde{\mu}^r(N^*, y, t)} = \alpha(M, N, K, x, y) \tilde{\chi}(x, K, t)^{-1}.$$

Putting here $M = N^* = K, x = y = e$, we have that $\alpha(t \cdot N^*, N, N^*, t, e) = \alpha(N^*, N, N^*, e, e)$ which we denote by $\tau(N)$, $\alpha(N^*, N, N^* \cdot t, e, t) = \tau(N)$. Inserting these expressions again into the above equations, we have

$$\alpha(M, N, K, x, y) = \tau(N) \tilde{\chi}^{-1}(x, N^*, y).$$

Thus, we have proved

LEMMA 3.14. *The associativity isomorphism $\alpha_{1,1,1}(M, N, K)$ is defined by the matrix*

$$(\tau(N) \tilde{\chi}^{-1}(x, N^*, y) id_{x \cdot K})_{M=xN^*; N=yK^*}. \tag{25}$$

REMARK 3.15. The above results show that the obstruction $0_3(c)$ vanishes.

3.3.3. Vanishing of the obstruction $O_4(c, M)$. As $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = \{0\}$, the obstruction $O_4(c, M)$ (see [5], 8.6) vanishes automatically, so there is a choice of $\alpha_{f,g,h}$ satisfying the pentagon equations (see [5], (51)) which can be given a form similar to that in [23]:

$$M_{f,gh,k}(id_f \boxtimes_{c(0)} \alpha_{g,h,k}) \circ \alpha_{f,gh,k}(Id_{\mathcal{C}_f} \boxtimes_{c(0)} M_{g,h} \boxtimes_{c(0)} Id_{\mathcal{C}_k}) \circ M_{fgh,k}(\alpha_{f,g,h} \boxtimes_{c(0)} id_k)$$
$$= \alpha_{f,g,hk}(Id_{\mathcal{C}_f} \boxtimes_{c(0)} Id_{\mathcal{C}_g} \boxtimes_{c(0)} M_{h,k}) \circ \alpha_{fg,h,k}(M_{f,g} \boxtimes_{c(0)} Id_{\mathcal{C}_h} \boxtimes_{c(0)} Id_{\mathcal{C}_k}). \tag{26}$$

Moreover, given two systems of associativity isomorphisms, $\alpha_{f,g,h}$ and $\alpha'_{f,g,h}$, [5], Theorem 8.9 shows that $\beta(f, g, h) = \alpha_{f,g,h}(\alpha'_{f,g,h})^{-1}$ can be viewed as an element of $Z^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$, and that they give equivalent $\mathbb{Z}/2\mathbb{Z}$-extensions of $c(0)$ if and only if $\beta \in B^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$. As the only nontrivial 3-cocycle on $\mathbb{Z}/2\mathbb{Z}$ is defined by $\beta(1, 1, 1) = -1$, there are exactly 2 different classes of associativity isomorphisms which differ by a sign of $\tau(N)$. The simplest choice of representatives in these classes corresponds to $\alpha(t, M) \equiv 1$ and $\tau(N) \equiv \tau$. Exactly like in [23], one can find easily from the pentagon equation for $\alpha_{1,1,1}$, using the non-degeneracy of χ, that $\tau = \pm |A|^{-1/2}$.

3.3.4. Main result. Let us recall, for the convenience of the reader, some notations, for other notations see Preliminaries. For a triple (μ^r, μ^l, χ), where $\mu^r \in (\Omega_{A,\omega})^S$, $\mu^l \cong (\mu^r \circ (\vee \times \vee))^{-1}$, $(x,y)^\vee := (y^{-1}, x^{-1})$, $\forall (x,y) \in (S \times S^{op})$, we denote $\mu_0((a,b),(c,d)) := \mu^l(a,c)\mu^r(b,d)\chi(a,d)$ - the $\omega \otimes \omega^{op}$-2-cocycle on $A \times A$; the action of S/A on $A \times A$ is denoted by ${}^{\bar{t}}(a,b) := ({}^t a, {}^{\varepsilon(t)} b)$, and we put ${}^{\bar{t}}\mu_0((a,b),(c,d)) := \mu_0({}^{\bar{t}}(a,b), {}^{\bar{t}}(c,d))$. Finally, we introduce the 3-cochain on S/A:

$$\zeta(t,t',t'') := \psi(t, \bar{\rho}(t',t'')) \frac{\mu_0({}^t\bar{\rho}(t',t''), \bar{\rho}(t,t't''))}{\mu_0(\bar{\rho}(t,t'), \bar{\rho}(tt',t''))},$$

where $\bar{\rho} = (\rho, \rho^{op}) \in Z^2(S/A, A \times A)$ and $\rho(t,t'), \rho^{op}(t,t') = {}^{(t't)^{-1}} \rho(t',t)$ are 2-cocycles giving, together with the action $a \mapsto {}^t a$ of S/A on A, the structure of twisted semidirect product on S and S^{op}, respectively.

THEOREM 3.16. $\mathbb{Z}/2\mathbb{Z}$-extensions $\mathcal{C}$ of $c(0) = Vec_S^\omega$ are parameterized, up to a tensor equivalence, by collections $(A, \chi, \mu^r, \tau, \varepsilon, \delta, \psi, \nu)$, where:

• A is an abelian normal subgroup of S equipped with a symmetric non-degenerate bicharacter χ, μ^r represents an equivalence class $(\Omega_{A,\omega})^S$ modulo restrictions on A of $Z^2(S, \mathbb{C}^\times)$, $\tau = \pm |A|^{-1/2}$;

• $\varepsilon \in Aut(S/A)$, $\delta \in S/A$ such that $\varepsilon^2 = Ad(\delta)$, $\varepsilon(\delta) = \delta$, and the set $(BrPic(Vec_S^\omega))_{(A,\varepsilon)}$ is not empty;

• $\psi \in Z^1(S/A, Fun(A \times A, \mathbb{C}^\times))$ such that $\frac{\mu_0}{{}^{\bar{t}}\mu_0} = \partial^1 \psi$, for any $t \in S/A$, $\nu \in C^2(S/A, \mathbb{C}^\times)$ satisfying $\zeta = \partial^2 \nu$, $\mu_0[\mu_0 \circ (\vee \times \vee)] = \partial^1 k$, $\psi[\psi \circ (\varepsilon \times \vee)] = \partial^0 \tilde{k}$, $\nu[\nu \circ (\varepsilon \times \varepsilon)] = \frac{\partial^1 q}{k \circ \bar{\rho}}$, where $k \in C^1(A \times A, \mathbb{C}^\times)$, $q \in C^1(S/A, \mathbb{C}^\times)$, $\tilde{k}$ is constant map $t \mapsto k$ on S/A.

Namely, $\mathcal{C} = c(0) \oplus c(1)$ with $Irr(c(0)) = S, Irr(c(1)) = S/A$, the fusion rule is: $x^* = x^{-1}, M^* = \varepsilon^{-1}(M^{-1})\delta, x \otimes y = xy, x \otimes M = p(x)M, M \otimes x = M\varepsilon(p(x)), M \otimes N^* = \oplus_{x \in MN^{-1}} x$ (where $x,y \in S$, $M, N \in S/A$).

The associativity isomorphisms are defined, for any $x,y,z \in S$, $K, L, M \in Irr(c(1))$, by 2-cochains $\tilde{\mu}^r$ and $\tilde{\chi}$ induced, respectively, from μ^r and from the extension of χ obtained by ψ and ν:

$$\alpha_{0,0,0}(x,y,z) = \omega(x,y,z)id_{xyz},$$
$$\alpha_{1,0,0}(K,x,y) = [\tilde{\mu}^r(K,x,y)]^{-1}id_{Kxy},$$
$$\alpha_{0,1,0}(x,K,y) = \tilde{\chi}(x,K,y)id_{xKy},$$
$$\alpha_{0,0,1}(x,y,K) = \tilde{\mu}^r((xyK)^*,x,y)id_{xyK},$$
$$\alpha_{0,1,1}(x,K,L) = \oplus_{K=sL^*}[\tilde{\mu}^r(K^*x^{-1},x,s)]^{-1}id_{xs},$$
$$\alpha_{1,1,0}(K,L,x)) = \oplus_{K=sL^*}\tilde{\mu}^r(K^*,s,x)id_{sx},$$
$$\alpha_{1,0,1}(K,x,L) = \oplus_{Kx=sL^*}\tilde{\chi}(s,(xL)^*,x)id_s,$$
$\alpha_{1,1,1}(K,L,M)$ is defined by the matrix $(\tau \tilde{\chi}^{-1}(s,L^*,t)id_{sM})_{K=sL^*; L=tM^*}$.

Two such collections, $(A, \chi, \mu^r, \tau, \varepsilon, \delta, \psi, \nu)$ and $(A', \chi', \mu'^r, \tau', \varepsilon', \delta', \psi', \nu')$, define equivalent $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_S^ω if and only if:

• $A = A', \tau = \tau'$ and there are $F \in Aut(S)$ and $\varphi \in C^2(S^{op}, \mathbb{C}^\times)$ such that $\omega \circ (F \times F \times F)/\omega = \partial^2 \varphi$, $F(A) = A$ (so F factors through A), $F(\delta) = \delta'$ and, modulo inner automorphisms: $\mu'^r \cong \varphi|_{A \times A} \cdot \mu^r \circ (F \times F)$, $\chi' = \chi \circ (F \times F)$, $F \circ \varepsilon = \varepsilon' \circ F$;

• $\dfrac{\mu_0 \circ (F \times F)}{\mu_0'} = \partial^1 \eta$, $\dfrac{\psi \circ F}{\psi'} = \partial^0 \tilde{\eta}$, $\dfrac{\nu \circ F}{\nu'} = \dfrac{\partial^1(\phi \circ F)}{\eta \circ \overline{\rho}}$, where $\eta \in C^1(A \times A, \mathbb{C}^\times)$, $\tilde{\eta} : t \mapsto \eta$ is constant function on S/A, and $\phi \in C^1(S/A, \mathbb{C}^\times)$.

Proof. (i) If $\mathcal{C} = c(0) \oplus c(1)$ is a fusion category, then $c(1) \cong c(1)^{op}$ is an invertible $c(0)$-bimodule category, so it corresponds to a collection $(A, \mu^r, \chi, \varepsilon, \psi, \nu)$ described in Lemma 3.5. The existence of δ and its properties follow from Proposition 3.7, the existence and the value of τ, as explained above, from [5], Theorem 8.9 and from the pentagon equation for $\alpha_{1,1,1}$. Thus, we have associated with $\mathcal{C}$ the collection $(A, \mu^r, \chi, \tau, \varepsilon, \delta, \psi, \nu)$.

(ii) Vice versa, given a collection $(A, \chi, \mu^r, \tau, \varepsilon, \delta, \psi, \nu)$ as above, Lemma 3.5 allows to construct an invertible $c(0)$-bimodule category $c(1)$ such that $c(1) \cong c(1)^{op}$, and Proposition 3.7 a fusion ring. Now, Lemmas 3.13, 3.14 and the triviality of $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$ show that there is a tensor product on $\mathcal{C} = c(0) \oplus c(1)$ giving a structure of a fusion category, with two choices of τ.

(iii) Let $(A, \chi, \mu^r, \tau, \varepsilon, \delta, \psi, \nu)$ and $(A', \chi', \mu'^r, \tau', \varepsilon', \delta', \psi', \nu')$ be two collections corresponding to equivalent $\mathbb{Z}/2\mathbb{Z}$-extensions, $\mathcal{C}$ and $\mathcal{C}'$, respectively. By definition, a tensor equivalence of $\mathcal{C}$ and $\mathcal{C}'$ contains:

1) A tensor autoequivalence of $c(0)$ defined by a couple (F, φ), where $F \in Aut(S)$ and $\varphi \in C^2(S, \mathbb{C}^\times)$ are such that $\omega \circ (F \times F \times F)/\omega = \partial^2 \varphi$ (it is more convenient for us to pass to $c(0)^{op}$ and so to deal with $\varphi \in C^2(S^{op}, \mathbb{C}^\times)$).

2) An equivalence of $c(0)$-bimodule categories $c(1) = \mathcal{M}(L(A, f), \mu)$ and $c(1)' = \mathcal{M}(L(A', f'), \mu')$ which implies the following equalities with some $f^l, f^r \in C^1(S, C)$ (see Remark 3.2):

$$\frac{\tilde{\chi}'^s}{\tilde{\chi} \circ (F \times F)} = \frac{\partial^0 f^l}{\partial^0 f^r}, \quad \frac{(\tilde{\mu}'^r)^s}{\tilde{\mu}^r \circ (F \times F)} = \varphi \cdot \partial^1 f^r$$

whose restriction to A gives $(\chi')^s = \chi \circ (F \times F)$ and $\varphi|_{A \times A} \cdot \mu^r \circ (F \times F) \cong (\mu'^r)^s$ (these equalities become simpler when considered modulo inner automorphisms: $\mu'^r \cong \varphi|_{A \times A} \cdot \mu^r \circ (F \times F)$, $\chi' = \chi \circ (F \times F)$); Lemma 3.5 gives the remaining list of relations between the components of the above collections.

3) An isomorphism of their fusion rings, so Proposition 3.7 implies that $F(A) = A$, $F(\delta) = \delta'$, and $F \circ \varepsilon = \varepsilon' \circ F$ in $Out(S/A)$.

Finally, the explicit formulas for the associativity isomorphisms for two given $\mathbb{Z}/2\mathbb{Z}$-extensions of $c(0)$ and the above relations between the two corresponding collections allow to construct an equivalence of these categories. ∎

COROLLARY 3.17. *If $S \cong A \rtimes (S/A)$ is a usual semidirect product, i.e., $\rho = 1$, one can omit μ^r in the above parameterization and the conditions on ψ and ν are simpler: $\nu \in Z^2(S/A, \mathbb{C}^\times)$, $\psi' \circ (\varepsilon \otimes \vee) = (\psi \circ F)^{-1}$, $\nu' \circ (\varepsilon \otimes \varepsilon) \cong (\nu \circ F)^{-1}$.*

Proof. Let us fix a $\mathbb{Z}/2\mathbb{Z}$-extension $c(0) \oplus c(1)$ of $c(0)$ and the ω-2-cocycle μ_1^r giving a structure of right $c(0)$-bimodule category on $c(1)$. Let $\mathcal{C}$ be any other $\mathbb{Z}/2\mathbb{Z}$-extension with

its μ_2^r. We want to show, using tensor equivalences of the form (Id, φ), $\varphi \in Z^2(S^{op}, \mathbb{C}^\times)$, that there is a $\mathbb{Z}/2\mathbb{Z}$-extension $\mathcal{C}'$ equivalent to $\mathcal{C}$ for which $\mu_2'^r = \mu_1^r$. We have, on the one hand, $\mu_2'^r = \varphi|_{A \times A}\mu_2^r$, and on the other hand, $\mu_2^r = Z \cdot \mu_1^r$, where $Z \in Z^2(A, \mathbb{C}^\times)$. So, it suffices to choose as φ any extension of Z^{-1}, if it exists.

But the conditions imposed on (μ_1^r, χ_1) and (μ_2^r, χ_2) imply that $Z = \dfrac{\mu_2^r}{\mu_1^r}$ satisfies all the conditions of [12], Lemma 2.2.4, so its extension exists.

Thus, if $S \cong A \rtimes (S/A)$, one can omit μ^r in the above parameterization of $\mathbb{Z}/2\mathbb{Z}$-extensions of $c(0)$. Also, as $\mu_2'^r = \mu_1^r$, we have $\eta = 1$ in the conditions of Theorem 3.16, so they take the above mentioned simpler form. ∎

REMARK 3.18. a) If $\omega = 1$, the condition that $(BrPic(Vec_S^\omega))_{(A, \varepsilon)}$ is not empty follows from other conditions (see Remark 3.6); if also $S \cong A \rtimes (S/A)$, one can choose $\mu^r = 1$.

b) If $A = S$, Theorem 3.16 gives the result of Tambara and Yamagami [23]. This result was also obtained by the methods of graded fusion categories in [5], Proposition 9.3, Example 9.4.

c) The case $A = \{e\}$ was treated in much more generality in [5].

4. Examples

4.1. Non-isomorphic fusion rings if $S/A \cong \mathbb{Z}/p\mathbb{Z}$, p is prime

LEMMA 4.1. *If $S/A \cong \mathbb{Z}/p\mathbb{Z}$, p is prime, and let n be the number of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded fusion rings with basis $X = S \cup S/A$, then:*

(i) if $p = 2$, one has $n = 2$;

(ii) if $p \geq 3$, $\rho = 1$, i.e. $S \cong A \underset{\alpha}{\rtimes} \mathbb{Z}/p\mathbb{Z}$ is a usual semidirect product, where α is an action of $\mathbb{Z}/p\mathbb{Z}$ on A, then $n = 3$ if S is abelian and $n = p + 1$ otherwise.

Proof. (i) If $p = 2$, then Proposition 3.7 gives $f = Id$ and $\delta = 0$ or $\delta = 1$ give two non-isomorphic fusion rings corresponding to the single subgroup $L = L^{op}$ of $S \times S^{op}$.

(ii) If $p > 2$, $S/A \cong \mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p - 1\}$, then $f_1 = id$ and $f_{-1} : x \mapsto -x$ are the only involutive elements of $Aut(S/A)$ which give two subgroups, $L_1 = L_1^{op}$ and $L_{-1} = L_{-1}^{op}$ - see Proposition 3.7.

Now, only $\delta = 0$ satisfies equation $f_1(\delta) = -\delta$; on the contrary, any $\delta \in S/A$ satisfies equation $f_{-1}(\delta) = -\delta$. As $F \circ id \circ F^{-1} = id$, for any $F \in Aut(S/A)$, the fusion ring corresponding to the couple $(id, 0)$ is not isomorphic to any other. Similarly, as $F(0) = 0$, for any $F \in Aut(S/A)$, the fusion ring corresponding to the couple $(f_1, 0)$ is not isomorphic to any other. Let us also remark that $F \circ f_{-1} \circ F^{-1} = f_{-1}$, for any $F \in Aut(S/A)$.

Next, Proposition 3.7 shows that the only condition under which two couples, (f_{-1}, δ) and (f_{-1}, δ') $(\delta, \delta' \neq 0)$, generate isomorphic fusion rings, is the existence of $F \in Aut(S/A)$ such that $F(\delta) = \delta'$ and F can be extended to $G \in Aut(S)$ such that $G(A) = A$. If $S = A \underset{\alpha}{\rtimes} \mathbb{Z}/p\mathbb{Z}$, this is equivalent to

$$\alpha_{\delta'\delta^{-1}x}G(a) = G(\alpha_x(a)), \text{ for all } a \in A, x \in \mathbb{Z}/n\mathbb{Z}.$$

Putting here $x = 1$, we get $\alpha_\delta = \alpha_{\delta'}$, i.e, $\delta'\delta^{-1} \in Ker(\alpha)$, where $\alpha : \delta \to \alpha_\delta$ is a homomorphism from S/A to $Aut(A)$. Conversely, if $\delta'\delta^{-1} \in Ker(\alpha)$, then one can put $G(a) = a$, for any $a \in A$, in order to extend F. As p is prime, $Ker(\alpha)$ is S when S is abelian or $\{e\}$ otherwise, and the result follows. ∎

4.2. Number of non-equivalent $\mathbb{Z}/2\mathbb{Z}$-extensions in examples

DEFINITION 4.2. If S is a finite group, $\omega \in H^3(S, \mathbb{C}^\times)$, $A \lhd S$ is abelian and such that $\omega|_{A \times A \times A} = 1$ in $H^3(A, \mathbb{C}^\times)$, we denote by $n(S, \omega, A)$ the number of $\mathbb{Z}/2\mathbb{Z}$-graded fusion categories (up to equivalence) associated with it.

PROPOSITION 4.3. *If S is abelian, $|S| = 2p$ (p is prime) and A is a nontrivial subgroup of S, then $n(S, 1, A)$ equals to 6 when $S \cong \mathbb{Z}/2p\mathbb{Z}$, $A \cong \mathbb{Z}/2\mathbb{Z}$, $p > 2$, to 8 when $S \cong \mathbb{Z}/4\mathbb{Z}$ or $S \cong \mathbb{Z}/2p\mathbb{Z}$, $A \cong \mathbb{Z}/p\mathbb{Z}$, $p > 2$, and to 16 when $S \cong (\mathbb{Z}/2\mathbb{Z})^2$, for any of 3 its nontrivial subgroups.*

Proof. If $S \neq \mathbb{Z}/4\mathbb{Z}$, all the subgroups L are direct products of $A \times A$ and S/A. If $|A| = 2$, there is only one symmetric non-degenerate bicharacter on A; if $A \cong \mathbb{Z}/p\mathbb{Z}$, $p > 2$, [26] tells us that there are 2 orbits in the set of non-degenerate symmetric bicharacters on A with respect to $Aut(A)$. As S is abelian, $Inn(A) = \{Id\}$, as S/A is cyclic, $H^2(S/A, \mathbb{C}^\times) = 0$.

We have $H^1(S/A, A \times A) = 0$ when $S \cong \mathbb{Z}/2p\mathbb{Z}$, $p > 2$ because $|S/A|$ and $|A|$ are relatively prime; see [12], Proposition 1.3.1. According to Lemma 4.1, the number of non-isomorphic fusion rings equals to 2 if $|S/A| = 2$ and to 3 otherwise. As in all cases there are 2 choices for τ, we already have the needed result for $S \cong \mathbb{Z}/2p\mathbb{Z}$, $p > 2$.

To complete the proof if $S \cong (\mathbb{Z}/2\mathbb{Z})^2$, it suffices to remark that $H^1(\mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2)$, with trivial action, is $(\mathbb{Z}/2\mathbb{Z})^2$.

If $S \cong \mathbb{Z}/4\mathbb{Z}$, $A \cong \mathbb{Z}/2\mathbb{Z}$, then $S \cong A \underset{\rho}{\rtimes} S/A$ with trivial action of S/A on A and the 2-cocycle $\rho : S/A \to A$ given by $\rho(1,1) = 1$, and $\rho = 0$ otherwise. The subgroup $L < S \times S^{op}$ is isomorphic to $(A \times A) \underset{\bar\rho}{\rtimes} S/A$ with trivial action and the 2-cocycle $\bar\rho : S/A \to A \times A$ coming from ρ. As above, $H^1(S/A, A \times A) = (\mathbb{Z}/2\mathbb{Z})^2$; in order to check condition 3) of Proposition 2.1, remark that the function $\gamma(s, t, u) := \alpha_{(0,S/A),(A \times A,0)}(s, \bar\rho(t, u))$ equals identically 1 for two of them and equals the nontrival 3-cocycle on $\mathbb{Z}/2\mathbb{Z}$ for two others.

Now, the equation $\partial^2\alpha = \gamma$ has no solutions when γ is a nontrival 3-cocycle because the left-hand side is clearly a 3-coboundary. For two elements of $H^1(S/A, A \times A)$ for which the solution of the above equation exists, it is unique because $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = 0$. Thus, taking into account 2 choices for τ, we see that the number of non-equivalent $\mathbb{Z}/2\mathbb{Z}$-extensions in this example is 4 for any of the two non-isomorphic fusion rings. ∎

REMARK 4.4. It was shown earlier in [2] (see also [14]) that the number of fusion categories coming from $S \cong \mathbb{Z}/4\mathbb{Z}$, $\omega = 1$, $A \cong \mathbb{Z}/2\mathbb{Z}$, $\delta = 1$ is 4.

PROPOSITION 4.5. $n(D_p, 1, \mathbb{Z}/p\mathbb{Z}) = 8$, *where D_p is dihedral group, $p > 2$ is prime, $A = \mathbb{Z}/p\mathbb{Z}$.*

Proof. $D_p \cong A \rtimes \mathbb{Z}/2\mathbb{Z}$ where $A = \mathbb{Z}/p\mathbb{Z}$, the subgroup $L = L^{op} < D_p \times D_p^{op}$ is unique and isomorphic to $(A \times A) \rtimes \mathbb{Z}/2\mathbb{Z}$ with the action $(a, b) \mapsto (-a, -b)$, so any bicharacter

on $A \times A$ is S/A-invariant. Since $[S : A] = 2$, there are 2 non-isomorphic fusion rings; see Lemma 4.1.

As above, there are two orbits in the set of non-degenerate symmetric bicharacters on A with respect to automorphisms of S satisfying the conditions of Theorem 3.16.

Finally, $H^2(S/A, \mathbb{C}^\times) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = 1$ and $H^1(S/A, A \times A) = 1$ because $|S/A|$ and $|A \times A|$ are relatively prime; see [12], Proposition 1.3.1.

Thus, non-equivalent $\mathbb{Z}/2\mathbb{Z}$-graded extensions of $Vec^1_{D_p}$ correspond to 2 choices of fusion rings, to 2 choices of χ, and to 2 choices of τ. ∎

PROPOSITION 4.6. $n(A_4, 1, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = 8$.

Proof. $A_4 \cong A \rtimes \mathbb{Z}/3\mathbb{Z}$, where $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and both subgroups, L_1 and L_{-1}, are isomorphic to usual semidirect products of type $A^2 \rtimes \mathbb{Z}/3\mathbb{Z}$. Since $[S : A] = 3$, there are 4 non-isomorphic fusion rings (see Lemma 4.1), so we have to show that, for any fixed ε, δ, there are exactly 2 fusion categories.

There are 4 symmetric non-degenerate bicharacters on A : $\chi_K((a_0, a_1), (b_0, b_1)) = (-1)^{\Sigma^1_{i,j=0} k_{ij} a_i b_j}$, where $a_i, b_j, k_{ij} \in \{0, 1\}$, k_{ij} are matrix coefficients of a symmetric invertible 2×2-matrix K.

Due to Theorem 3.16, in order to find non-equivalent $\mathbb{Z}/2\mathbb{Z}$-graded extensions of $Vec^1_{A_4}$, we have to look at the automorphisms of A_4 under which A is stable. It is straightforward to show that inner automorphisms of A_4 generated by $Z/3Z$ transform one into another the three bicharacters χ_K with $K \neq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and that the fourth one is stable under $Aut(A)$. A simple computation gives that in the case of L_1 the three bicharacters in the same orbit are S/A-cohomologically invariant but not the fourth one; conversely, in the case of L_{-1} only the fourth one is S/A-cohomologically invariant.

Finally, $H^2(S/A, \mathbb{C}^\times) = H^2(\mathbb{Z}/3\mathbb{Z}, \mathbb{C}^\times) = 1$ and $H^1(S/A, A \times A) = 1$ because $|S/A|$ and $|A \times A|$ are relatively prime - see [12], Proposition 1.3.1.

Thus, non-equivalent $\mathbb{Z}/2\mathbb{Z}$-graded extensions of $Vec^1_{A_4}$ correspond to 4 choices of fusion rings, to 1 choice of χ, and to 2 choices of τ. ∎

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