

## ON THE PROBLEM OF CLASSIFYING SIMPLE COMPACT QUANTUM GROUPS

SHUZHOU WANG

*Department of Mathematics, University of Georgia*

*Athens, GA 30602, USA*

*E-mail: szwang@math.uga.edu*

*Dedicated to Professor S. L. Woronowicz on the occasion of his 70th birthday*

**Abstract.** We review the notion of simple compact quantum groups and examples, and discuss the problem of construction and classification of simple compact quantum groups.

**1. Introduction.** There are two main areas in the operator algebraic approach to quantum groups: compact quantum groups and locally compact quantum groups. The former has a satisfactory axiomatic theory due to Woronowicz [83, 85]. While the latter has witnessed great progress due to concerted efforts of several generations of mathematicians (cf. an incomplete list including [47, 48, 41, 42, 1, 50, 54]), it still does not have an axiomatic framework that contains the non-compact Drinfeld-Jimbo quantum groups [40] as examples except for special cases such as  $SL_q(2, \mathbb{C})$  (cf. e.g. [55]), nor has the existence of Haar weight been established in general. This is in stark contrast with the fact that compact real forms of the Drinfeld-Jimbo quantum groups are special examples in the theory of compact quantum groups (see [59, 60, 62, 61, 51]), and Haar measure, Peter-Weyl theorem and Tannaka-Krein duality can be established from a very simple set of axioms [83, 84, 85]. Within this well established framework of compact quantum groups, recent work on compact quantum groups has been primarily on the construction, classification, structure and other (operator) algebraic properties of specific classes of compact quantum groups.

The modern impetus in the theory of quantum groups came as a result of the discovery of new examples of Hopf algebras by Drinfeld-Jimbo on the algebraic side [40]

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and Woronowicz on the analytic side [82]. These are deformation quantizations of the classical Lie algebras and Lie groups, and much of the literature on quantum groups had been devoted to this approach to quantum groups.

Starting in his thesis [70], the author took a different direction than the traditional deformation quantization method by viewing quantum groups as intrinsic objects and found in a series of papers (including [69] in collaboration with Van Daele) several classes of universal compact quantum groups that *cannot be obtained* as deformations of Lie groups or Lie algebras, the most important of these are the universal compact quantum groups of Kac type  $A_u(n)$  and their orthogonal counterpart  $A_o(n)$  [71], the more general universal compact quantum groups  $A_u(Q)$  and their self-conjugate counterpart  $B_u(Q)$  [69, 74], where  $Q \in GL(n, \mathbb{C})$ , and the quantum automorphism groups  $A_{aut}(B, tr)$  [75] of finite dimensional  $C^*$ -algebras  $B$  endowed with a tracial functional  $tr$ , including the quantum permutation groups  $A_{aut}(X_n)$  on the space  $X_n$  of  $n$  points. These objects have been an international focus of study in the subject of compact quantum groups and interest in them continues unabated (cf. [2]-[35], [44], [49], [63]-[68]).

The quantum groups  $A_u(Q)$  have the remarkable universal property that can be used to give following alternative and *concrete definition* of compact matrix quantum groups that was originally defined by Woronowicz [83] more abstractly: a compact matrix quantum group is a quotient  $A_u(Q)/I$ , where  $I$  is a Woronowicz  $C^*$ -ideal in  $A_u(Q)$ . This means in geometric language that every compact matrix quantum group (including compact Lie group) is a quantum subgroup of  $A_u(Q)$  for an appropriate choice of  $Q$ . In contrast, Drinfeld-Jimbo quantum groups and other deformations of Lie groups do not enjoy this property, but are quantum subgroups of  $A_u(Q)$ . Similarly, the quantum groups  $B_u(Q)$  have the universal property that every compact matrix quantum group with self-conjugate fundamental representation is of the form  $B_u(Q)/I$ , where  $I$  is a Woronowicz  $C^*$ -ideal in  $B_u(Q)$ .

Without universal compact quantum groups, the Drinfeld-Jimbo quantum groups and other quantum groups obtained by deformation would be the end of the story. However, the outpouring of papers on the universal quantum groups in the last few years (e.g. [2]-[35], [44], [49], [63]-[68]) demonstrates depth of the subject: despite much work achieved so far, we have only seen the tip of the iceberg and the story is far from the end.

Although compact quantum groups have a satisfactory axiomatic framework and Drinfeld-Jimbo quantum groups are their special examples, after the discovery of these new classes of universal compact quantum groups, it is a natural program to classify simple compact quantum groups. This program was initiated in [79], where it was shown that all compact quantum groups mentioned above are simple in generic cases.

The *main goals* of the program on simple compact quantum groups are: (1) construct and classify simple compact quantum groups and their irreducible representations, (2) understand the structure of simple compact quantum groups and structure of compact quantum groups in terms of the simple ones, and (3) develop new applications of simple compact quantum groups in other areas of mathematics and physics, such as quantum symmetries in noncommutative geometry and algebraic quantum field theory. For goal (1), one would like to develop a theory of simple compact quantum groups that parallels

the Killing-Cartan theory and the Cartan-Weyl theory for simple compact Lie groups. For this purpose, one must first construct all simple compact quantum groups. Though the work so far provides several infinite classes of examples of these, it should be pointed out that the construction of simple compact quantum groups is only at the beginning stage for this task at the moment, as all the simple compact quantum groups known so far are almost classical in the sense that their representation rings are isomorphic to those of ordinary compact groups and in particular are commutative. The first examples of simple compact quantum groups that are not almost classical should be directly related to the universal quantum groups  $A_u(Q)$  (see the footnote after Problem 4.1), where  $Q \in GL(n, \mathbb{C})$  are positive,  $n \geq 2$ , though these quantum groups are not simple themselves. The representation ring of  $A_u(Q)$  is highly noncommutative, being roughly the free product of two copies of the ring of integers, according to Banica [3]. To construct other simple compact quantum groups, the most natural idea is to study quantum automorphism groups of appropriate quantum spaces, such as those in the author's papers [75, 76], the papers of Banica, Bichon, Goswami and their collaborators [5]-[11], [32, 33], [44], [24]-[31]. In retrospect, both simple Lie groups and finite simple groups are automorphism groups. This suggests viability of this approach to the program. Natural mathematical and physical structures that have compact quantum automorphic group symmetries are compact commutative and noncommutative Riemannian manifolds in the sense of Connes [36, 38]. Such symmetries should be investigated first.

The following heuristic may indicate the depth of the problem on classification problem of simple compact quantum groups. The finite dimensional factors are classified by the *discrete set* of natural numbers while the classification of infinite dimensional von Neumann factors involves *continuous parameters*. Similarly, simple compact Lie groups are classified by a *discrete set* of Cartan matrices, but the classification of simple compact quantum groups involves *continuous parameters*. However, since the algebraic structures of compact quantum groups are richer and more rigid than those of von Neumann algebras, the classification of simple compact quantum groups might be more accessible than the classification of infinite dimensional factors. Even if a classification of simple compact quantum groups up to isomorphism is unattainable, just as von Neumann factors are far from being classified up to isomorphism, experience has demonstrated that the study of universal quantum groups and quantum automorphism groups is fruitful (cf. [71, 69, 74, 76, 78, 79], [2]-[35], [44], [49], [63]-[68] and references therein), and other types of classification theories may also be considered, such as the classification of easy quantum groups [21, 18, 19, 81] and the classification of restricted classes of quantum automorphism groups [8]-[12] by Banica et al.

An outline of the paper is as follows. In §2, we review the notion of compact quantum groups and simple compact quantum groups. In §3, we review examples of simple compact quantum groups constructed so far in [79]. In §4, we discuss problem of the fine structure of  $A_u(Q)$  and simple quantum quotient groups from  $A_u(Q)$ , as well as quantum subgroups from free products of compact quantum groups. In §5, we give a list of problems related to almost classical compact quantum groups and compact quantum groups with property  $F$ . In §6 and §7, we discuss the problem of constructing simple compact quantum groups from

quantum automorphisms of finite graphs and other quantum subgroups of the quantum permutation groups, easy quantum groups and quantum isometry groups. In §6, several new quantum groups constructed by Banica, Curran and Speicher [17] since [79] are shown to be simple using results in Raum [56] and Weber [81] along with results in [79].

**2. Compact quantum groups and simple compact quantum groups.** We first recall the definition of compact (matrix) quantum groups and then the notion of simple compact quantum groups.

There are several equivalent definitions of compact (matrix) quantum groups, each has its own advantages over the others. We briefly describe below another equivalent definition, which has the advantage of letting the reader “visualize” all compact quantum groups more concretely. This equivalent definition is essentially in the literature, but not in the explicit form we describe below.

**Notation:** For elements  $u_{ij}$  ( $i, j = 1, \dots, n$ ) of a  $C^*$ -algebra  $A$ , we define the following elements in the  $n \times n$  matrix algebra  $M_n(A)$  over  $A$ :  $u := (u_{ij})_{i,j=1}^n$ ,  $\bar{u} := (u_{ij}^*)_{i,j=1}^n$ ,  $u^t := (u_{ji})_{i,j=1}^n$  and  $u^* := \bar{u}^t$ , i.e.  $u^* = (u_{ji}^*)_{i,j=1}^n$ .

**DEFINITION 2.1** (cf. [70, 71, 69, 74, 78]). The *universal compact matrix quantum groups* are defined to be the family of pairs  $(A_u(Q), \Delta_u)$ , where  $Q \in GL(n, \mathbb{C})$ ,  $Q > 0$ , and  $u_{ij}$  ( $i, j = 1, \dots, n$ ) are generators of the universal  $C^*$ -algebra  $A_u(Q)$  that satisfies the following sets of relations:

$$u^*u = I_n = uu^*, \quad u^t Q \bar{u} Q^{-1} = I_n = Q \bar{u} Q^{-1} u^t,$$

and  $\Delta_u : A_u(Q) \rightarrow A_u(Q) \otimes A_u(Q)$  is the uniquely defined morphism such that

$$\Delta_u(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

Note that instead of restricting  $Q$  to positive matrices, one can still define  $(A_u(Q), \Delta_u)$  for any invertible  $Q$ . For such  $Q$ , one has the following free product decomposition [78],

$$A_u(Q) \cong A_u(P_1) * A_u(P_2) * \cdots * A_u(P_k)$$

for appropriate positive matrices  $P_1, P_2, \dots, P_k$  with compatible coproducts as in [71].

**DEFINITION 2.2** (cf. [83]). A *compact matrix quantum group* is a triple  $(A, \Delta, \pi)$ , where  $\pi : A_u(Q) \rightarrow A$  and  $\Delta : A \rightarrow A \otimes A$  are  $C^*$ -morphisms such that

- (1)  $\pi$  is surjective, and
- (2)  $\Delta\pi = (\pi \otimes \pi)\Delta_u$ .

Note that using [71], conditions (1) and (2) above are equivalent to (1) plus the following condition:

$$(2)' \Delta_u(\ker(\pi)) \subset \ker(\pi \otimes \pi).$$

Hence a compact matrix quantum group can be defined even more simply as a pair  $(A, \pi)$  satisfying (1) and (2)'.

The  $C^*$ -algebra  $A$  in the definition of a compact matrix quantum group  $(A, \Delta, \pi)$  is called a *finitely generated Woronowicz  $C^*$ -algebra*. The morphism  $\Delta$  is called the *coproduct* of  $A$ .

It can be shown that there is a Hopf  $*$ -algebra structure  $(\mathcal{A}_u(Q), \Delta, \varepsilon, S)$  on the dense  $*$ -subalgebra  $\mathcal{A}_u(Q)$  of  $A_u(Q)$  such that

$$S(u_{ij}) = u_{ji}^*, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad i, j = 1, 2, \dots, n.$$

This  $*$ -Hopf algebra structure induces Hopf  $*$ -algebra structure on the dense  $*$ -subalgebra  $\mathcal{A}$  of  $A$  in the above definition of compact matrix quantum group, and on the dense  $*$ -subalgebra  $\mathcal{A}$  of  $A$  in the definition of the compact quantum group below.

As in 2.3 of [71], one can define morphisms between compact matrix quantum groups as opposite of morphisms of finitely generated Woronowicz  $C^*$ -algebras. Under these morphisms, compact matrix quantum groups form a category, though this category is not closed under inverse limits, which leads to the following notion of compact quantum groups using 3.1 of [71]:

**DEFINITION 2.3** (cf. [1, 85]). A *compact quantum group*  $(A, \Delta)$  is an inverse limit of compact matrix quantum groups  $(A_\lambda, \Delta_\lambda, \pi_{\lambda\lambda'})$ .

We will see in the next few paragraphs that the above notion of compact quantum groups is equivalent to the elegant and abstract one in [85].

As an inductive limit (instead of inverse limit) of finitely generated Woronowicz  $C^*$ -algebras  $A_\lambda$ , the  $C^*$ -algebra  $A$  in the definition above is called a *Woronowicz  $C^*$ -algebra*. Kernels of morphisms between Woronowicz  $C^*$ -algebras are called *Woronowicz  $C^*$ -ideals*, which can also be intrinsically defined as in 2.3 of [71].

**REMARK.** Intuitively, we think of  $A = C(G)$ , where  $G$  is a compact quantum group, even there might be no points in “ $G$ ” other than the identity. We also use the notations  $A_G = C(G)$  and  $C(G_A) = A$ . As in the literature, by abuse of terminology,  $A$  is also called a compact (matrix) quantum group besides being called a (finitely generated) Woronowicz  $C^*$ -algebra.

As usual, one defines the Haar state/measure. One then establishes the existence and uniqueness of the Haar state/measure on any compact matrix quantum group just as in [83]. Using this and 3.3 of [71], one establishes the existence and uniqueness of the Haar state/measure on any compact quantum group  $G$ . In the following, the Haar state on  $A = C(G)$  is denoted by  $h_G$  or simply  $h$  if no confusion arises.

Using the Haar state/measure  $h$ , the Peter-Weyl theory for all compact quantum groups can be developed as in [85] or [53]. As a result, we see that the above definition of compact quantum groups is equivalent to the one in [85].

We use  $\mathcal{A}_G$  to denote the dense  $*$ -subalgebra of  $A_G$  consisting of coefficients of finite dimensional representations of  $G$ . As a consequence of the Peter-Weyl theory for compact quantum groups,  $\mathcal{A}_G$  is a Hopf  $*$ -algebra.

We need to recall the notion of normal quantum subgroups [71, 80] to define simple compact quantum groups.

Let  $(N, \pi)$  be a *quantum subgroup* of a compact quantum group  $G$ , which, as defined in [70, 71], means that  $\pi : C(G) \rightarrow C(N)$  is a surjection of  $C^*$ -algebras such that  $(\pi \otimes \pi)\Delta_G = \Delta_N \pi$ , where  $\Delta_G, \Delta_N$  are coproducts of  $C(G)$  and  $C(N)$ , respectively.

Define

$$C(G/N) := \{a \in C(G) \mid (id \otimes \pi)\Delta(a) = a \otimes 1_N\},$$

$$C(N \setminus G) := \{a \in C(G) \mid (\pi \otimes id)\Delta(a) = 1_N \otimes a\},$$

$\Delta$  being coproduct on  $C(G)$ ,  $1_N$  the unit of  $C(N)$ .

DEFINITION 2.4 (cf. [71, 79]). We say  $N$  is *normal* in  $G$  if it satisfies one of the equivalent conditions in the proposition below.

PROPOSITION 2.5 (cf. [79]). *Let  $N$  be a quantum subgroup of a compact quantum group  $G$ . The following conditions are equivalent:*

- (1)  $C(N \setminus G)$  is a Woronowicz  $C^*$ -subalgebra of  $C(G)$ .
- (2)  $C(G/N)$  is a Woronowicz  $C^*$ -subalgebra of  $C(G)$ .
- (3)  $C(G/N) = C(N \setminus G)$ .
- (4) For every irreducible representation  $u^\lambda$  of  $G$ , either  $h_N \pi(u^\lambda) = I_{d_\lambda}$  or  $h_N \pi(u^\lambda) = 0$ , where  $h_N$  is the Haar measure on  $N$ ,  $d_\lambda = \dim(u^\lambda)$  and  $I_{d_\lambda}$  the  $d_\lambda \times d_\lambda$  identity matrix.

Among the above four equivalent formulations of the notion of normal quantum subgroups, condition (4) is the most convenient for our purposes. If  $G$  is a compact group, then the above definition coincides with the usual notion of closed normal subgroups.

To avoid complications with classification of finite quantum groups, we want to restrict the notion of simple quantum groups to quantum groups that are connected.

DEFINITION 2.6 (cf. [79]). A compact quantum group  $G$  is called *connected* if for each non-trivial irreducible representation  $u^\alpha \in \hat{G}$ , the  $C^*$ -algebra  $C^*(u_{ij}^\alpha)$  generated by the coefficients of  $u^\alpha$  is of infinite dimension.

A compact matrix quantum group  $G$  is called *simple* (resp. *absolutely simple*) if it is connected and has no non-trivial connected normal quantum subgroups (resp. non-trivial normal quantum subgroups) and no non-trivial representations of dimension one.

Note that compact Lie groups are (commutative) examples of compact quantum groups. It is easy to show that a compact Lie group is simple (resp. connected) in the usual sense if and only if it is simple (resp. connected) in the sense above.

PROBLEM 2.7. *In the definition of simple compact quantum groups, if one replaces the condition “it has no non-trivial representations of dimension one” with the apparently less stringent condition “ $C(G) \neq C^*(\Gamma)$  where  $\Gamma$  is a discrete group”, do we get an equivalent definition?*

Note according to [82], the condition  $C(G) \neq C^*(\Gamma)$  above means that  $G$  is a non-abelian compact quantum group. For a compact Lie group  $G$ , this condition simply means  $G$  is a nonabelian Lie group, and the answer to the above question is affirmative by Weyl’s dimension formula.

Just as the notion of simple compact Lie groups excludes the torus groups, the definition of simple quantum groups above (including the alternative one formulated in

Problem 2.7) excludes the compact quantum groups coming from group  $C^*$ -algebras  $C^*(\Gamma)$  of discrete groups  $\Gamma$  (i.e., abelian quantum groups). This is important because the classification of discrete groups is out of reach.

**3. Examples of simple compact quantum groups that are almost classical and have property  $F$ .** The simple compact quantum groups that have been constructed so far share many properties common to compact Lie groups. We recall two of these properties [79].

Just as for compact groups, the *representation ring* (also called the *fusion ring*)  $R(G)$  of a compact quantum group  $G$  is a partially ordered algebra over the integers  $\mathbb{Z}$  generated by the irreducible characters of  $G$ . The set of characters of  $G$  is a semi-ring that defines the order of  $R(G)$ .

DEFINITION 3.1 (cf. [79]). A compact quantum group  $G$  is said to have *property  $F$*  if each Woronowicz  $C^*$ -subalgebra of  $C(G)$  is of the form  $C(G/N)$  for some normal quantum subgroup  $N$  of  $G$ .

A compact quantum group is called *almost classical* if its representation ring  $R(G)$  is order isomorphic to the representation ring of a compact group.

In plain language, a compact quantum  $G$  is said to have property  $F$  if its quantum function algebra  $C(G)$  behaves exactly as the function algebras of compact groups with respect to normal subgroups. Note that compact quantum groups  $C^*(\Gamma)$  for the dual  $\Gamma$  of a discrete group  $\Gamma$  do not have this property unless the discrete group  $\Gamma$  is abelian, in which case the group  $C^*$ -algebra  $C^*(\Gamma)$  is a genuine function algebra over the Pontryagin dual  $\hat{\Gamma}$  of the discrete abelian group  $\Gamma$ .

The notion dual to property  $F$  is given by following definition, which captures the property that compact quantum group  $C^*(\Gamma)$  has with respect to normal quantum subgroups:

DEFINITION 3.2 (cf. [79]). A compact quantum  $G$  is said to have *property  $FD$*  if each of quantum subgroup of  $G$  is normal.

Proofs of assertions in [79] concerning properties  $F$  and  $FD$ , along with other related properties of compact quantum groups, can be found in [80].

**Quantum groups  $B_u(Q)$**  (cf. [70, 71, 69, 74, 78]). Keeping the notation for definition of  $A_u(Q)$  in §2. Let  $Q \in GL(n, \mathbb{C})$  be such that  $Q\bar{Q} = \pm I_n$ ,  $n \geq 2$ .  $B_u(Q)$  is defined to be the universal  $C^*$ -algebra with generators  $u_{ij}$  ( $i, j = 1, 2, \dots, n$ ) that satisfy that following sets of relations:

$$u^*u = I_n = uu^*, \quad u^t Qu Q^{-1} = I_n = Qu Q^{-1} u^t.$$

It can be shown that there is a well-defined morphism

$$\Delta : B_u(Q) \rightarrow B_u(Q) \otimes B_u(Q)$$

such that

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

and that  $(B_u(Q), \Delta)$  is a compact matrix quantum group. The quantum groups  $B_u(Q)$  have the universal property that every compact matrix quantum group with self-conjugate fundamental representation is of the form  $B_u(Q)/I$ , where  $I$  is a Woronowicz  $C^*$ -ideal in  $B_u(Q)$ .

**Note:**  $B_u(Q)$  is also denoted by  $A_o(Q^*)$  by Banica et al.

When  $Q = I_n$ ,  $B_u(Q)$  is just  $A_o(n)$ , the universal orthogonal quantum group of Kac type introduced in 4.5 of [71]. Banica et al. also invented the notation  $O^+(n)$  to signify  $A_o(n) = C(O^+(n))$ .

In addition, when  $Q = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ ,  $B_u(Q)$  is a quantum symplectic group. This is one of the reasons that we use the notation  $B_u(Q)$  instead of  $A_o(Q)$ , as the latter only captures the special case  $Q = I_n$ . Following the notation  $O^+(n)$  of Banica et al., we denote the above universal symplectic quantum group by  $Sp^+(n)$ . The quantum symplectic group has also appeared in recent work of Bhowmick, D'Andrea, Das and Dąbrowski on quantum gauge symmetries (see [26]).

For any invertible  $Q \in GL(n, \mathbb{C})$  that does not satisfy  $Q\bar{Q} = \pm I_n$ ,  $B_u(Q)$  can be defined by the same relations as above, but

$$B_u(Q) \cong A_u(P_1) * A_u(P_2) * \cdots * A_u(P_k) * \\ * B_u(Q_1) * B_u(Q_2) * \cdots * B_u(Q_l).$$

for certain  $P_i > 0$  and  $Q_j$  such that  $Q_j\bar{Q}_j$ 's are scalars (cf. [78]).

We note that both  $A_u(Q)$  and  $B_u(Q)$  can be alternatively described as quantum automorphism groups of appropriate spaces (cf. [76, 26]).

**Quantum automorphism groups**  $A_{aut}(B, \tau)$  (cf. [75]). Let  $B$  be a finite dimensional  $C^*$ -algebra and  $\phi$  a functional on  $B$ . In general the quantum automorphism group  $A_{aut}(B, \phi)$  that preserves the system  $(B, \phi)$  exists, which is the universal object in the category of compact quantum groups acting on the system  $(B, \phi)$ . However, as shown in Banica [4], only when  $\phi$  is the *canonical trace*  $\tau$  does the quantum group  $A_{aut}(B, \tau)$  have a representation theory that is relatively easy to describe, where the trace  $\tau$  on  $B$  is called *canonical* if it coincides with the restriction to  $B$  of the unique tracial state on the algebra  $L(B)$  of operators with  $B$  acting by the GNS representation associated with the trace  $\tau$ . If one identifies  $B$  with  $\bigoplus_{k=1}^m M_{n_k}(\mathbb{C})$ , then

$$\tau\left(\sum_{k=1}^m b_k\right) = \sum_{k=1}^m \frac{n_k^2}{n} Tr(b_k),$$

where  $b_k \in M_{n_k}$ ,  $n$  is the dimension of  $B$ , and  $Tr$  is ordinary trace on  $M_{n_k}$ , i.e.  $Tr(b_k)$  is the sum of the diagonal entries of the matrix  $b_k$ .

In either case, explicit description of  $A_{aut}(B, \phi)$  or  $A_{aut}(B, \tau)$  in terms of generators and relation is complicated for a general finite dimensional  $C^*$ -algebra  $B$ . However, when  $B = C(X_n)$  is the commutative  $C^*$ -algebra of functions on the space where  $X_n$  is the space of  $n$  points, the *quantum permutation group*  $A_{aut}(X_n) := A_{aut}(C(X_n))$  has a surprisingly simple description in terms of generators and relations: The  $C^*$ -algebra



$A_{aut}(X_n)$  is generated by self-adjoint projections  $a_{ij}$  such that each row and column of the matrix  $(a_{ij})_{i,j=1}^n$  adds up to 1, i.e.,

$$\begin{aligned} a_{ij}^2 &= a_{ij} = a_{ij}^*, \quad i, j = 1, \dots, n, \\ \sum_{j=1}^n a_{ij} &= 1, \quad i = 1, \dots, n, \\ \sum_{i=1}^n a_{ij} &= 1, \quad j = 1, \dots, n. \end{aligned}$$

Banica et al. invented the very convenient geometric notation  $S_n^+$  so that  $A_{aut}(X_n) = C(S_n^+)$ . In [34], Bichon generalizes the quantum permutation groups to purely algebraic context and establishes the universal property of the quantum permutation groups in complete generality.

**THEOREM 3.3** (cf. 4.1 and 4.7 in [79]). (a) *For  $Q \in GL(n, \mathbb{C})$  such that  $Q\bar{Q} = \pm I_n$  and  $n \geq 2$ ,  $B_u(Q)$  is an almost classical simple compact quantum group with property  $F$ .*

(b) *Let  $B$  be a finite dimensional  $C^*$ -algebra endowed with its canonical trace  $\tau$  and  $\dim(B) \geq 4$ . Then the quantum group  $A_{aut}(B, \tau)$  is an almost classical absolutely simple compact quantum groups with property  $F$ .*

As special cases of  $B_u(Q)$  and  $A_{aut}(B, \tau)$ , we have

**COROLLARY 3.4.** (a) *The universal orthogonal quantum groups  $O^+(n)$  (for  $n \geq 2$ ) and the universal symplectic quantum groups  $Sp^+(n)$  (for  $n \geq 1$ ) are simple.*

(b) *The quantum permutation groups  $S_n^+$  are simple for  $n \geq 4$ .*

The main ideas used in the proof of Theorem 3.3 include

- (1) Banica's fundamental work of on the structure of fusion rings of these quantum groups (cf. Théorème 1 and Theorem 4.1 in [2, 4] respectively);
- (2) Correspondence between Hopf  $*$ -ideals and Woronowicz  $C^*$ -ideals (cf. 4.2-4.3 in [79]);
- (3) Reconstruction of a normal quantum group from the identity in the quotient quantum group (cf. 4.4 in [79]).

In addition to the fusion rings of compact quantum groups such as those considered in [2, 4], the matters in (2) and (3) above are of interest in their own right and worth further investigation. The correspondence between Hopf  $*$ -ideals and Woronowicz  $C^*$ -ideals and related matters in (2) relate algebraic and analytical aspects of compact quantum groups. In purely Hopf algebras context, reconstruction of a normal quantum group from the quotient quantum group in (3) has been an issue since 1970's and is related to several other important and old open questions [80].

**Quantum groups  $K_q$ ,  $K_q^u$  and  $K_J$**  (cf. [62, 61, 52, 58, 73] and [59, 60]). A unified study of the compact quantum groups  $K_q$ ,  $K_q^u$  is due to Soibelman & Vaksman, Leventodskii [62, 61, 52]. The  $*$ -Hopf algebras  $\mathcal{A}_{K_q}$  are algebras of “representative functions” of Drinfeld-Jimbo quantum groups  $U_q(\mathfrak{g})$  and define in a sense their “compact real form”. The quantum group  $K_q$  is a deformation of the Poisson Lie group  $K(1, 0)$  (cf. [52]). The

\*-Hopf algebras  $\mathcal{A}_{K_q^u}$  are twisting of the \*-Hopf algebras  $\mathcal{A}_{K_q}$  by an element  $u \in \wedge^2 \mathfrak{h}_{\mathbb{R}}$ . The quantum group  $K_q^u$  is a deformation of the Poisson Lie group  $K(1, u)$  (cf. [52]). As shown in [73],  $K_q^u$  is an example of Rieffel's deformation from action of finite dimensional vector space as conjectured by Rieffel (cf. [57, 58] for the background).

Rieffel's quantum group deformation  $K_J$  [58] depends on  $J = S \oplus (-S)$ , where  $S$  is a skew symmetric operator on the Lie algebra (viewed as  $\mathbb{R}^n$ ) of a torus subgroup of the compact Lie group  $K$ . For appropriate choice of  $S$ ,  $K_J$  is a deformation of Poisson Lie group  $K(0, u)$  [52, 58]. An action of  $\mathbb{R}^d := \mathbb{R}^n \times \mathbb{R}^n$  on  $A = C(K)$  can be constructed and Rieffel's theory of deformation for action of  $\mathbb{R}^d$  [57] can be applied to obtain  $A_J$  [58], also denoted  $C(K_J)$ .

A precise description of  $K_q$ ,  $K_q^u$  and  $K_J$  would require more space than appropriate in this paper. For our purposes, these quantum groups can be roughly described as follows:

- (1) The associated dense Hopf \*-algebras  $\mathcal{A}_{K_q}$ ,  $\mathcal{A}_{K_q^u}$  and  $\mathcal{A}_{K_J}$  are the same vector space as the un-deformed/un-twisted ones  $\mathcal{A}_K$ ,  $\mathcal{A}_{K_q}$  and  $\mathcal{A}_K$  respectively, but the algebras  $\mathcal{A}_{K_q}$ ,  $\mathcal{A}_{K_q^u}$  and  $\mathcal{A}_{K_J}$  have deformed products;
- (2) The Hopf \*-algebras  $\mathcal{A}_{K_q}$ ,  $\mathcal{A}_{K_q^u}$  and  $\mathcal{A}_{K_J}$  have the same coproduct as the un-deformed/un-twisted ones  $\mathcal{A}_K$ ,  $\mathcal{A}_{K_q}$  and  $\mathcal{A}_K$  respectively;
- (3) Representation theories of the deformed/twisted quantum groups  $K_q$ ,  $K_q^u$  and  $K_J$  are the same as the un-deformed/un-twisted ones  $K$ ,  $K_q$  and  $K$  respectively.

**THEOREM 3.5** (cf. (5.1, 5.4 and 5.6 in [79])). *If  $K$  is a simple compact Lie group, then  $K_q$ ,  $K_q^u$ ,  $K_J$  are almost classical simple compact quantum groups with property  $F$ .*

The main ideas used in the proof of Theorem 3.5 include

- (1) Representation theory of these quantum groups;
- (2) Correspondence between Hopf \*-ideals and Woronowicz  $C^*$ -ideals, as in the proof of Theorem 3.3;
- (3) Reconstruction of a normal quantum group from the identity in the quotient quantum group, as in the proof of Theorem 3.3;
- (4) The normal subgroups of the undeformed Lie group remain to be normal subgroups of the deformed quantum groups and explicit identification of normal quantum subgroups of the deformed quantum groups.

Other deformations of compact Lie groups, though constructed not as systematic as the ones considered by Drinfeld-Jimbo, Soibelman et al. and Rieffel, are scattered in the literature. We believe the general ideas used in the proof of Theorem 3.5 can also be applied to such deformations.

**4. Quotient quantum groups from  $A_u(Q)$  and free products.** In [78], the quantum groups  $A_u(Q)$  are classified up to isomorphism for positive matrices  $Q > 0$  and the quantum groups  $B_u(Q)$  are classified up to isomorphism for matrices  $Q$  with  $Q\bar{Q} = \pm I_n$ . It is shown that the corresponding  $A_u(Q)$  (resp.  $B_u(Q)$ ) is not a free product, or a tensor product, or a crossed product. However, for general non-singular matrices  $Q$ , we have the decomposition theorem expressing  $A_u(Q)$  and  $B_u(Q)$  in terms of free product of the

forgoing quantum groups (cf. Theorem 3.1 and 3.3 in [78]). In the light of these results, the following problem seems to be fundamental:

**PROBLEM 4.1.** *Study further the fine structure of  $A_u(Q)$  for positive matrices  $Q \in GL(n, \mathbb{C})$  and  $n \geq 2$ ; Determine their simple quotient quantum groups.*

A solution of this problem will also provide the first examples<sup>1</sup> of simple compact quantum groups that are not almost classical (see §3). Note that the  $A_u(Q)$ 's have the 1 dimensional diagonal torus  $\mathbb{T}$  as their (connected) normal quantum subgroup, as observed by Bichon (private communication, cf. 4.5 in [79]), so they are not simple. However, they are very close to being simple. For example, they have no non-trivial irreducible representations of dimension one [3, 78].

It is worth noting that in Problem 4.1, simple quotient quantum groups of  $A_u(Q)$  should be easier to determine than simple quantum subgroups of  $A_u(Q)$ , since the latter is tantamount to finding all simple quantum groups due to the universal property of  $A_u(Q)$ , which include all simple compact quantum groups in [79] as reviewed in §3, simple compact Lie groups, as well as all the other unknown simple compact quantum groups.

By investigating the fine structure of concrete quantum groups such as  $A_u(Q)$ , one can expect to gain insights into the structure of general compact quantum groups and simple quantum groups. Sections 6 and 7 below contain more directions of research on this approach to quantum groups.

Suitable modifications of the method for the proofs of the main results in section 4 of the paper [79] should yield a solution to Problem 4.1. The extra work needed for this problem that does not appear in [79] is that there are more Woronowicz subalgebras in  $A_u(Q)$  to consider than therein. Some preliminary computations of these subalgebras give optimism to a positive solution of the problem. One of the main ingredients in this calculation is Banica's fundamental result on fusion rules of the irreducible representations of the quantum group  $A_u(Q)$  [3].

The general  $A_u(Q)$  (resp.  $B_u(Q)$ ) for arbitrary  $Q \in GL(n, \mathbb{C})$  is not simple if  $C^*(\mathbb{Z})$  appears in its free product decompositions as described in [78]. This is because of the fact that  $A_u(Q') = C^*(\mathbb{Z}) = C(\mathbb{T})$  (resp.  $B_u(Q') = C^*(\mathbb{Z}/2\mathbb{Z})$ ) for  $Q' \in GL(1, \mathbb{C})$  and the following result ([80]):

**PROPOSITION 4.2.** *Let  $G_1, G_2$  be compact quantum groups. Let  $G = G_1 \hat{*} G_2$  be the free product compact quantum group [71] underlying  $A_{G_1} * A_{G_2}$ . Let  $\pi_1$  be the natural embed-*

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<sup>1</sup>After this paper was accepted for publication, Alexandru Chirvăsitu informed the author that in the preprint "Free unitary quantum groups are (almost) simple", he showed that the quantum group generated by  $u_{ij}u_{kl}^*$  and  $u_{kl}^*u_{ij}$  is a simple compact quantum group with noncommutative representation ring and without property  $F$ , where  $u_{ij}$  are the generators of  $A_u(Q)$  with  $Q > 0$ . In the same preprint, he also showed that the quotient quantum group of  $A_u(Q)$  by its central subgroup  $\mathbb{T}$  in the proof of 4.5 in [79] has no normal quantum subgroups but is not finitely generated, and all normal subgroups are subgroups of  $\mathbb{T}$ , giving a complete classification of simple quotient groups of  $A_u(Q)$ . See also footnote before Problem 5.4 below.

ding of  $G_1$  into  $G$  defined by the surjection

$$\pi_1 : A_{G_1} * A_{G_2} \rightarrow A_{G_1}, \quad \pi_1 = id_1 * \epsilon_2.$$

If  $G_1$  has at least one irreducible representation of dimension greater than one, then  $(G_1, \pi_1)$  is not a normal quantum subgroup of  $G_1 \hat{*} G_2$ . Otherwise,  $(G_1, \pi_1)$  is normal in  $G_1 \hat{*} G_2$ .

The hat in the symbol  $\hat{*}$  above signifies the “Fourier transform” of the free product  $*$  reminiscent of the classical case in which  $G_k = \hat{\Gamma}_k$ , where  $\Gamma_k$  are discrete abelian groups.

Note that a compact Lie group being simple means roughly that it is not a direct product of proper connected subgroups. A similar result also holds for quantum groups: If  $G_A$  is a simple compact quantum group, then  $A_G$  is not a tensor product (i.e.  $G$  is not a direct product of its non-trivial quantum subgroups) [72, 80].

However, the proposition above says that the evident quantum subgroups  $(G_1, \pi_1)$  and  $(G_1, \pi_2)$  of  $G_1 \hat{*} G_2$  are not normal in  $G_1 \hat{*} G_2$  when  $G_1$  and  $G_2$  have no non-trivial representations of dimension one. Along with other results of the author, this may suggest that the following problem on the structure of simple quantum groups has a positive solution.

**PROBLEM 4.3.** *Let  $G_1$  and  $G_2$  be simple compact quantum groups. Is  $G_1 \hat{*} G_2$  also simple?*

The results in the author’s paper [71] should be useful for a solution of this problem. In particular, Theorem 1.1 there should play a role, as it did in [2, 3, 33, 78]. Note that the formula for the Haar measure on  $G_1 \hat{*} G_2$  and the classification its irreducible representations are given in explicit formulas in Theorem 1.1 in [71]. According to the postulates in the definition of a normal quantum subgroup (Definition 2.4), these are important ingredients in determining whether  $G_1 \hat{*} G_2$  has normal quantum subgroups. The results and methods of the paper [79] (especially section 4 therein) should also be useful for this problem.

A positive solution to Problem 4.3 would have the following implication:  $G_1 \hat{*} G_2$  would be a *simple compact quantum group* when both  $G_1$  and  $G_2$  are merely *simple compact Lie groups*.

The following easier variation of Problem 4.3 is also of interest and is related to the problem of determining whether several families of easy quantum groups are simple (cf. §6 below).

**PROBLEM 4.4.** *Let  $G_1$  be a simple compact quantum group and  $G_2$  a finite quantum group. Is  $G_1 \hat{*} G_2$  also simple?*

Note that in view of the problems in §6 below, it would be interesting to solve Problem 4.5 for  $G_2$  a finite group. Also related problems can be formulated for Bichon’s free wreath product of compact quantum groups [33] (cf. §6 below):

**PROBLEM 4.5.** *Let  $G_1$  be a simple compact quantum group and  $G_2$  a finite quantum group. Is  $G_1 \hat{*} G_2$  also simple?*

**5. Problems related to almost classical compact quantum groups and property  $F$ .** As reviewed in §3, the simple compact quantum groups known so far are almost

classical and have property  $F$ . Such quantum groups seem to be most accessible at the moment. The following problem evidently is less difficult than the general problem of classifying simple compact quantum groups and should be attempted first:

**PROBLEM 5.1.** *Classify simple compact quantum groups that are almost classical and have property  $F$ .*

The following closely related problems should be considered also:

**PROBLEM 5.2.** (a) *Classify almost classical simple compact quantum groups.*

(b) *Classify simple compact quantum groups with property  $F$ .*

**PROBLEM 5.3.** *Does simple compact quantum groups with property  $FD$  exist? If so, construct and classify them.*

In Problem 5.1 and Problem 5.2, it would be interesting enough to restrict consideration to almost classical simple compact quantum groups that have the same representation rings as simple compact Lie groups.

In another direction, for the apparently more difficult problem of classifying simple compact quantum groups that are not almost classical or without property  $F$ , we do not have a single example of them. Therefore the following is a basic problem<sup>2</sup>:

**PROBLEM 5.4.** (a) *Construct an example of simple compact quantum group that is not almost classical.*

(b) *Construct an example of simple compact quantum group that does not have property  $F$ .*

A more concrete problem than Problem 5.4 is the following

**PROBLEM 5.5.** *Construct simple compact quantum groups with noncommutative representation ring.*

For the problems in this section, results on general structure of compact quantum groups such as those in the previous sections should also be useful. In this direction, we have the following result (cf. [80]).

**THEOREM 5.6.** *Let  $G$  be a compact quantum group with property  $F$ . Then its quantum subgroups and quotient groups  $G/N$  by normal quantum subgroups  $N$  also have property  $F$ .*

It would be of interest to develop other general results on the structure of compact quantum groups.

## 6. Quantum automorphism groups of finite graphs and easy quantum groups.

How and where do we find quantum groups satisfying the properties in the problems in §5 above? The most natural approach, in our opinion, is by considering quantum automorphism groups of appropriate commutative and noncommutative spaces and their quantum subgroups. Much of the recent work on compact quantum groups falls into

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<sup>2</sup>Alexandru Chirvăsitu informed the author that he has since solved Problem 5.4 and Problem 5.5—see footnote after Problem 4.1.

this category. We would like to mention three classes of these quantum groups: quantum automorphism groups of finite graphs and other quantum subgroups of the quantum permutation groups, easy quantum groups and quantum isometry groups. The last of these three is discussed in the next section. We briefly look at the first two in this section.

In part to understand quantum subgroups of the quantum permutation groups [75], Bichon [32] constructed the *quantum automorphism groups of finite graphs*, which are quantum subgroups of the former preserving the edges of the graphs. To further the understanding of this new class of quantum groups, Bichon [33] also defined free wreath product of compact quantum groups using the quantum permutation groups and proved the beautiful formula stating that the free wreath product of the quantum automorphism group of a graph by the quantum permutation group  $A_{\text{aut}}(X_n)$  is the quantum automorphism group of  $n$  disjoint copies of the graph. Banica also independently studied quantum subgroups of the quantum permutation groups [5, 6]. These works lead to a great deal of further studies of quantum automorphism of finite graphs and quantum subgroups of the quantum permutation groups, cf. [5]–[12] and references therein.

It is instructive to see an immediate application of the quantum automorphism groups of finite graphs to related problems in the last section. Clearly, a quantum quotient group  $G/N$  of an almost classical quantum group  $G$  is also almost classical. However a *crucial* observation is that a quantum subgroup of an almost classical quantum group needs not be almost classical. For example, the quantum permutation groups  $A_{\text{aut}}(X_n)$  are almost classical (cf. [4, 75, 79] and §3), but according to of Bichon [33], their quantum subgroups  $A_2(\mathbb{Z}/m\mathbb{Z})$ , as the quantum automorphism groups of certain graphs, are not almost classical if  $m \geq 3$  (see Corollary 2.7 and the paragraph following Corollary 4.3 of [33]). Though the quantum groups  $A_2(\mathbb{Z}/m\mathbb{Z})$  are not simple (see proposition 2.6 of [33]), and noting that the quantum automorphism group of the trivial graph (i.e. the quantum permutation group) is simple, it is natural to expect that it is possible to obtain simple compact quantum groups that are not almost classical by considering quantum automorphism groups of other appropriate finite graphs, including other free wreath products, thus solving Problem 5.4. Note that since the free wreath is constructed from the free product, such problems are related to see Problem 4.1 and the discussions following it.

In another important and related new direction that has origins in works on free wreath product and quantum subgroups of the quantum permutation group discussed above, Banica and Speicher [21] initiated the study of easy quantum groups and found several interesting families of new compact quantum groups. A compact matrix quantum group  $G$  with fundamental representation  $u$  is called *easy* if

- (1)  $G$  lies between  $S_n$  and  $O^+(n)$  (cf. §3 on  $B_u(Q)$  with  $Q = I_n$ ); and
- (2) For any  $k, l \geq 0$ ,  $\text{Hom}(u^{\otimes k}, u^{\otimes l})$  is linearly generated by operators  $T_p$  canonically associated with partitions  $p$  of  $k + l$ .

See [21] or [17] or [81] for a description of  $T_p$ .

In addition to the six families of free easy quantum groups (also called orthogonal quantum groups) in [21, 17], Weber also found another new family  $B_n'^+$  of free easy quantum groups in [81], where we follow his different notation from [21]. In [56], Raum

computed the fusion rings of several easy quantum groups in [21] using free products. In works of tour de force, Banica and Vergnioux computed the fusion rings of the quantum reflection group  $H_n^s$  in [22] and the half-librated orthogonal quantum group  $O^*(n)$  in [23]. It would be interesting to see if any of these quantum groups provide solutions to some of the problems in this section. As the fusion rings of  $H_n^s$  and  $O^*(n)$  are non-commutative, the following problems seem to be most appealing:

PROBLEM 6.1. (a) *Is the quantum reflection group  $H_n^s$  simple?*

(b) *Is the half-librated orthogonal quantum group  $O^*(n)$  simple?*

A positive answer to either (a) or (b) would provide the first simple compact quantum group that is not almost classical and has noncommutative representation ring.

In the light of Theorem 4.1 in Raum [56] and 3.1 and 3.2 in Weber [81] whose notation we follow, the relevant quantum groups considered in that theorem are either not simple or rely on solution of problems in §4. For instance, using their results above along with our Theorem 3.3, we see that

- (1)  $B_n^+$  is simple for  $n \geq 3$ ;
- (2)  $B_n'^+$  ( $n \geq 3$ ) and  $S_n'^+$  ( $n \geq 4$ ) are simple modulo two components (i.e. disconnected);
- (3)  $B_n^{\#++}$  is not simple because of Proposition 4.2, but it seems not far from being simple (a concept to be made precise) because of 4.1.(3) in [56] and 3.2.(a) in [81].

It is not clear if the quantum groups  $H_n^+$ ,  $H_n^*$ ,  $H_n^{(s)}$  and  $H_n^{[s]}$  are simple. Note that as a free wreath product, the quantum group  $H_n^+$  is related to the general Problem 4.5.

**7. Quantum isometry groups of commutative and noncommutative Riemannian spaces.** After Banica's initial investigation of quantum isometry groups of finite spaces [5, 6], a recent conceptual breakthrough in compact quantum groups is Debashish Goswami's [44] theory of quantum isometry groups of spectral triples à la Connes [36], where the universal quantum groups  $A_u(Q)$  plays an essential rôle in the proof of existence. Using this notion he computed with Bhowmick [27, 24] several examples of quantum isometry groups and found that they are either isomorphic to classical isometry groups or are among the examples studied earlier by Rieffel and the author [58, 73]. Using the same circle of ideas they subsequently developed [28] an improved notion of quantum isometry group without relying on existence of a good Laplacian as required in [44].

The quantum isometry groups computed so far are either classical groups, or known quantum groups, or combinations of both based on free product or tensor product. Because of this, Goswami made earlier a rigidity conjecture to the effect that connected spaces do not admit non-trivial quantum symmetries. On the other hand, Huichi Huang [46] has shown that connected non-smooth metric spaces admit faithful action even by quantum permutation groups, disproving this earlier conjecture. Most striking of all is that Goswami, along with his collaborators, have recently shown in a series of papers [45, 39] a modified rigidity result stating that a connected and oriented Riemannian manifold does not have quantum symmetries other than the classical ones.

As a fundamental new concept, one would naturally wonder if fundamentally new compact quantum groups can be constructed using quantum isometry groups. In the light of results of Huang and Goswami et al. above, one should look at quantum isometry groups of non-smooth metric spaces or disconnected Riemannian manifolds that are non-classical (i.e. not compact groups). It would be interesting to find if any such quantum groups are simple:

**PROBLEM 7.1.** *Construct examples of simple quantum isometry groups of non-smooth metric spaces or disconnected Riemannian manifolds.*

It is conceivable that quantum isometry groups of disconnected Riemannian manifolds will be related to quantum subgroups of the quantum permutation group as considered in §6, since quantum permutation group can permute the connected components in quantum manner just as it does on the finite space.

Another direction of research in the theory of quantum isometry group is the following. In the newly developed notion of quantum isometry group in [28], the quantum groups in the categories  $\mathbf{Q}'_R$  and  $\mathbf{Q}'$  that are used to define the quantum isometry group there does not carry a  $C^*$ -algebraic action on the spectral triple, and as a result, the universal object (i.e. the quantum isometry group) does not carry a  $C^*$ -algebraic action in general. See [30] for an example of this situation. (We refer the reader to the above cited papers for detailed description of  $\mathbf{Q}'_R$  and  $\mathbf{Q}'$  due to space limitation.)

To address this problem, we believe the categories  $\mathbf{Q}'_R$  and  $\mathbf{Q}'$  are too large, and propose the following alternative for the notion of quantum isometry groups that will always carry  $C^*$ -algebraic action.

First, by a *compact quantum transformation group*  $(A, \alpha, u)$  of a compact type spectral triple  $(\mathcal{B}, \mathcal{H}, D)$  we mean a compact quantum transformation group (cf. [75])  $(A, \alpha)$  of  $B$  that satisfies

(QT1) There is a unitary representation  $u \in L(\mathcal{H} \otimes A_G)$  of  $G_A$  on  $\mathcal{H}$  such that  $(\rho, u)$  is  $\alpha$ -covariant:  $(\rho \otimes 1)\alpha(b) = u(\rho(b) \otimes 1)u^*$ ,  $b \in B$ ;

(QT2)  $D$  is an intertwiner of  $u$  with itself:  $(D \otimes 1)u = u(D \otimes 1)$ .

A *morphism* from another quantum transformation group  $(\tilde{A}, \tilde{\alpha}, \tilde{u})$  to  $(A, \alpha, u)$  is defined to be a morphism  $\pi$  from  $(\tilde{A}, \tilde{\alpha})$  to  $(A, \alpha)$  (cf. [75]) that satisfies  $\tilde{u} = (id_{\mathcal{H}} \otimes \pi)u$ .

The above defines *category  $\mathcal{C}$  of compact quantum transformation groups* of  $(\mathcal{B}, \mathcal{H}, D)$  with objects  $\{(A, \alpha, u)\}$  and morphisms  $\{\pi\}$ . Similar to [75], the *quantum isometry group* of  $(\mathcal{B}, \mathcal{H}, D)$  is defined to be a universal object of category  $\mathcal{C}$  if it exists.

As in [75], universal object does not always exist in  $\mathcal{C}$  in general. Therefore, as in [75] and [28], consider *measured spectral triple*  $(\mathcal{B}, \mathcal{H}, D, \phi)$  with a (usually positive) functional  $\phi$  on  $\mathcal{B}$  and consider the category  $\mathcal{C}_\phi$  of compact quantum transformation groups that satisfies (QT1)-(QT2) and

(QT3)  $(\phi \otimes id)(\alpha(b)) = \phi(b)1_A$  for  $b \in \mathcal{B}$

The *quantum isometry group* of  $(\mathcal{B}, \mathcal{H}, D, \phi)$  is defined to be the universal object of category  $\mathcal{C}_\phi$  if it exists. The following can be proved and justifies in part the above notion of quantum isometry group.



PROPOSITION 7.2. (1) *Let  $(\mathcal{B}, \mathcal{H}, D)$  be the spectral triple associated with a compact Riemannian manifold  $M$ . Then the universal object in the category of compact transformation groups of  $(\mathcal{B}, \mathcal{H}, D)$  in the sense above is the isometry group of  $M$ .*

(2) *For an arbitrary spectral triple  $(\mathcal{B}, \mathcal{H}, D)$ , the universal object in the category of compact transformation groups of  $(\mathcal{B}, \mathcal{H}, D)$  in the sense above is the compact group of isometries of  $(\mathcal{B}, \mathcal{H}, D)$  in the sense of Connes (see p6200 of [37]).*

Since composition of two continuous maps is continuous, the quantum isometry group defined above is evidently contained in the quantum isometry groups defined by Goswami [44] and Bhowmick and Goswami [28]. Because of this, and using ideas in [75] and [28], the following seems to be quite plausible:

PROBLEM 7.3. *For a measured spectral triple  $(\mathcal{B}, \mathcal{H}, D, \phi)$ , show that universal object exists in the category  $\mathcal{C}_\phi$ .*

Many other problems naturally arise:

(i) Calculate the quantum isometry groups in the sense above for the classical spaces such as the flat spheres, tori, and other Riemannian spaces;

(ii) Calculate the quantum isometry groups in the sense above for the noncommutative tori;

(iii) Study the properties of quantum isometry groups in general and apply them to study other properties of noncommutative spaces;

(iv) Construct simple compact quantum groups through the study of quantum isometry groups.

As in the definitions of quantum isometry groups by Goswami and Bhowmick, and because a quantum isometry group in our sense is contained in theirs, examples of simple quantum isometry groups in our sense might need to be constructed out of non-smooth metric spaces or disconnected Riemannian manifolds, as in Problem 7.1.

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