GAME THEORY AND
MATHEMATICAL ECONOMICS
BANACH CENTER PUBLICATIONS, VOLUME 71
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2006

THE EXISTENCE OF GLOBALLY STABLE PRICE MECHANISMS FOR PURE EXCHANGE MODELS WITH UPPER SEMICONTINUOUS MULTIVALUED EXCESS DEMAND

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1. Introduction. The aim of the article is to present sufficient conditions for an upper semicontinuous multivalued excess demand, guaranteeing the existence of some globally stable price mechanism. We consider two different price mechanisms: sign-compatible and angle-compatible with the excess demand. Our conditions depend on vectors from the excess demand sets and corresponding price systems, with respect to the equilibrium price system. We show that there exist adequate price mechanisms in Scarf's example (when the excess demand is single-valued) and in examples with upper semicontinuous multivalued excess demand.

2. Model. Consider a model of pure exchange with a multivalued excess demand $E : \mathbb{R}^n_+ \rightsquigarrow \mathbb{R}^n$ (where $\mathbb{R}_+ = (0, +\infty)$) satisfying the following natural hypothesis:

(a0) E has nonempty, closed and convex values;
(a1) E is upper semicontinuous;
(a2) E satisfies Walras' Law: $\langle u, p\rangle = 0$ for all $u \in E(p)$ (where $\langle \cdot, \cdot\rangle$ denotes the inner product);
(a3) E is positive homogeneous of degree zero, i.e. $E(tp) = E(p)$ for $t > 0$;
(a4) E satisfies boundary condition: if $p^k \overset{k\to\infty}{\longrightarrow} p$, where p is such that $p_i = 0$ for some $i = 1, \dots, n$, then $d(E(p^k), \mathbf{0}) \overset{k\to\infty}{\longrightarrow} \infty$, where $d(A, \mathbf{0}) = \sup\{|a| : a \in A\}$.

The excess demand sets consist of differences between the total demand and the total supply of commodities, which are exchanged on the market. We assume that this map depends only on commodity bundle's price vector. The hypotheses (a0)–(a4) guarantee

2000 *Mathematics Subject Classification*: Primary 91B50.
The paper is in final form and no version of it will be published elsewhere.

the existence of at least one Walrasian equilibrium, i.e. a point p^*, such that $\mathbf{0} \in E(p^*)$ (compare [De]). Following Samuelson ([Sa]), we assume that the path of prices, which starts at fixed p^0, is a solution of the differential equation

$$\frac{dp}{dt} = g(p), \qquad p(0) = p^0. \tag{1}$$

The continuous function $g : \mathbb{R}^n_+ \to \mathbb{R}^n$ on the right hand side of Eq. (1) is called a *price mechanism* if it satisfies (when substituted for E) (a2)–(a4) and the condition: $g(p^*) = \mathbf{0}$ if and only if $p^* \in \mathcal{P}_E = \{p \in \mathbb{R}^n_+ : \ \mathbf{0} \in E(p)\}$. We say that a price mechanism is *globally asymptotically stable* if any price trajectory $p(t)$, which is a solution of Eq. (1) for any initial point p^0, converges to some $p^* \in \mathcal{P}_E$, when t tends to infinity and for any $\varepsilon > 0$ there exist $t_0 \geq 0$ and $\delta > 0$ such that for every solution $p(t)$ of Eq. (1) if $|p(t_0) - p^*| < \delta$ then $|p(t) - p^*| < \varepsilon$ for all $t > t_0$.

Let us recall that every price trajectory for the price mechanism g is located on the nonnegative part of the sphere $S_+(|p^0|) = \{p \in \mathbb{R}^n_+ : \ |p| = |p^0|\}$ (because of (a2) and (a4)). Since g satisfies (a3) we can regard such price adjustment process as a continuous tangent vector field on $S_+ = \{p \in \mathbb{R}^n_+ : \ |p| = 1\}$. This is the reason why we can restrict a domain of price mechanisms to S_+.

3. Problem. One can consider different kinds of price mechanisms. Let F be a given multivalued map from S_+ into $2^{\mathbb{R}^n} \setminus \{\emptyset\}$. We say that a price mechanism g is specified by F if $g(p) \in F(p)$ for all $p \in S_+$. Since g has to have zeros at equilibrium points p^* we impose on F the following condition: $F(p^*) = \{\mathbf{0}\}$ if and only if $\mathbf{0} \in E(p^*)$. We are going to give sufficient conditions for the excess demand, guaranteeing the existence of some globally asymptotically stable price mechanism g specified by F, which in turn is derived from E by sign- or angle-compatibility rule.

4. Stability. First, we ask when there exists the continuous selection g of the multivalued map F, such that any trajectory of an autonomous equation $x'(t) = g(x(t))$, $x(0) = x^0 \in Q$, is convergent to some equilibrium point $x^* \in Q$.

Let

$$T_Q(x) = \bigcap_{\varepsilon>0} \bigcap_{\eta>0} \bigcup_{0<h<\eta} \left(\frac{1}{h}(Q - x) + \varepsilon B(\mathbf{0}, 1) \right)$$

denote the contingent cone to Q at x.

THEOREM 1 (Nagumo). *Let Q denote a compact subset of $\mathbb{R}^n$. Let $g : Q \to \mathbb{R}^n$ be the continuous function such that*

$$g(x) \in T_Q(x), \ \textit{for all } x \in Q. \tag{2}$$

Then for any $x^0 \in Q$ there exists a solution $x : [0, \infty) \to \mathbb{R}^n$ of the equation $x'(t) = g(x(t))$ with the initial condition $x(0) = x^0$ such that $x(t) \in Q$ for all $t \geq 0$.

Let $H_0^-(y) = \{u \in \mathbb{R}^n : \langle u, y \rangle < 0\}$ and $H_0^+(y) = \{u \in \mathbb{R}^n : \langle u, y \rangle > 0\}$ for $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

THEOREM 2. *Let Q denote an open subset of $\mathbb{R}^n$. Let $g : Q \to \mathbb{R}^n$ be a function with $x^* \in Q$ being the only point such that $g(x^*) = \mathbf{0}$. If there exists a continuously differentiable function $V : Q \to [0, \infty)$ such that $V(x) = 0$ if and only if $x = x^*$ and*

$$g(x) \in H_0^-(\nabla V(x)) \text{ for all } x \in Q \setminus \{x^*\} \tag{3}$$

then for any $\varepsilon > 0$ there exist $t_0 \geq 0$ and $\delta > 0$ such that for every solution $x(t)$ of the equation $x'(t) = g(x(t))$ if $|x(t_0) - x^*| < \delta$ then $|x(t) - x^*| < \varepsilon$ for all $t > t_0$ and $\lim_{t\to\infty} x(t) = x^*$.

Proof. According to Lyapunov Theorem (see for instance [Ha, Theorem 8.2]) we have to show that $\langle g(x), \nabla V(x)\rangle < 0$ for all $x \in Q \setminus \{x^*\}$. Indeed, by definition of $H_0^-(y)$ we have $\langle g(x), \nabla V(x)\rangle < 0$ if and only if $g(x) \in H_0^-(\nabla V(x))$. ∎

REMARK 1. If the function g satisfies condition (3) for all $x \neq x^*$ then $g(x) \in T_{\mathcal{L}_V(x)}(x)$ for all $x \neq x^*$, where $\mathcal{L}_V(x) = \{y : V(y) \leq V(x)\}$.

Let $B = \{u \in \mathbb{R}^n : |u| < 1\}$.

THEOREM 3. *Let Q denote a compact subset of $\mathbb{R}^n$. Let $F : Q \to \mathbb{R}^n$ be the lower semicontinuous map with only one point $x^* \in Q$ such that $F(x^*) = \{\mathbf{0}\}$ and let $F(x)$ be a closed, convex cone for every $x \in Q$. If there exists a continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$ such that $V(x) = 0$ if and only if $x = x^*$ and*

$$F(x) \cap H_0^-(\nabla V(x)) \neq \emptyset \text{ for all } x \in Q \setminus \{x^*\} \tag{4}$$

then the multivalued map

$$x \mapsto \begin{cases} F(x) \cap H_0^-(\nabla V(x)), & \text{if } x \in Q \setminus \{x^*\}, \\ \{\mathbf{0}\}, & \text{if } x = x^* \end{cases} \tag{5}$$

has a continuous and bounded in Q selection.

Proof. Since all assumptions of Corollary [1.11.1, AuCe] are satisfied (indeed $x \mapsto F(x)$ is lower semicontinuous with closed, convex values and $x \mapsto H_0^-(\nabla V(x))$ has open graph) then there exists a continuous selection f of the multivalued map $x \mapsto F(x) \cap H_0^-(\nabla V(x))$ defined on $Q \setminus \{x^*\}$ with values in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Let $\delta(x) = \frac{|x-x^*|}{1+|x-x^*|}$. Since the sets $F(x)$ and $H_0^-(\nabla V(x))$ are cones and $\delta(x) \in (0, 1)$ then $\delta(x)f(x) \in F(x) \cap H_0^-(\nabla V(x))$ for all $x \in Q \setminus \{x^*\}$. Then the function

$$x \mapsto \begin{cases} \delta(x)\frac{f(x)}{|f(x)|}, & \text{if } x \in Q \setminus \{x^*\}, \\ \mathbf{0}, & \text{if } x = x^* \end{cases}$$

is a continuous selection of (5), bounded in Q.

Theorem 2 concerns dynamic systems in $\mathbb{R}^n$. Since we are going to use it in analysis of dynamic systems on the nonconvex set S_+ we project conformally trajectories of price dynamics characterized by some price mechanism g on a hyperplane orthogonal to the vector $p^* \in S_+$: $H_0(p^*) = \{u \in \mathbb{R}^n : \langle u, p^*\rangle = 0\}$. Let us recall the definition of conformal (stereographic) projection.

DEFINITION 1. A one-to-one smooth mapping ω of $S \setminus \{-p^*\}$ onto $H_0(p^*)$ defined by

$$\omega(p) = \varphi(p)(p + p^*) - 2p^*, \text{ where } \varphi(p) = \frac{2}{1 + \langle p^*, p\rangle}, \ p \in S \setminus \{-p^*\} \tag{6}$$

we call the *conformal projection.*

The matrix of derivatives of the conformal projection is given by:

$$D_p(\omega(p)) = -\frac{1}{2}\varphi(p)^2 (p^*)^{\mathrm{T}}(p + p^*) + \varphi(p)\mathbf{1},$$

where p^{T} denotes the transpose of the vector p, and $\mathbf{1}$ denotes the identity matrix.

The inverse map to ω, i.e. the function $\omega^{-1} : H_0(p^*) \to S \setminus \{-p^*\}$ is defined by

$$\omega^{-1}(x) = \psi(x)(x + 2p^*) - p^*, \quad \text{where } \psi(x) = \frac{4}{4 + |x|^2}, \ x \in H_0(p^*).$$

It is easy to check that $\psi(\omega(p)) \cdot \varphi(p) = 1$. The matrix of derivatives of the inverse map to the conformal projection is given by:

$$D_x(\omega^{-1}(x)) = -\frac{1}{2}\psi(x)^2 (x)^{\mathrm{T}}(x + 2p^*) + \psi(x)\mathbf{1}.$$

For all $p \in S_+$ let

$$\Phi_p(u) = u \cdot D_p(\omega(p)) = \left[-\frac{1}{2}\varphi(p)^2\langle p^*, u\rangle\right](p + p^*) + \varphi(p)u.$$

The map Φ_p is a linear operator defined on $\mathbb{R}^n$ for all $p \in S_+$. Let us observe that $\ker \Phi_p = \{\lambda(p + p^*) : \lambda \in \mathbb{R}\}$. Thus Φ_p is a one-to-one linear operator in $\mathbb{R}^n \setminus (\mathbb{R}^n_+ \cup \mathbb{R}^n_-)$, for all $p \in S_+$.

For all $x \in \omega(S_+)$ let

$$\Psi_x(\hat{u}) = \hat{u} \cdot D_x(\omega^{-1}(x)) = \left[-\frac{1}{2}\psi(x)^2\langle x, \hat{u}\rangle\right](x + 2p^*) + \psi(x)\hat{u}.$$

The map Ψ_x is a linear operator defined on $\mathbb{R}^n$ for all $x \in \omega(S_+)$.

Easy computations show that the maps Φ_p i Ψ_x have the following properties.

LEMMA 4. *Let $u \in H_0(p)$ and $\hat{u} \in H_0(p^*)$ for $p \in S_+$.*

(i) $\Phi_p(u) \in H_0(p^*)$.
(ii) $\langle \omega(p), \Phi_p(u)\rangle = -\varphi(p)^2\langle p^*, u\rangle$.
(iii) $\Psi_{\omega(p)}(\Phi_p(u)) = u$.
(iv) $\langle \Psi_{\omega(p)}(\hat{u}), \Phi_p(u)\rangle = \langle \hat{u}, u\rangle$.

The linear operator $\Psi_{\omega(p)}$ is inverse to the linear operator Φ_p restricted to $H_0(p)$, for all $p \in S_+$, since $H_0(p) \subset (\mathbb{R}^n \setminus (\mathbb{R}^n_+ \cup \mathbb{R}^n_-))$ and by property (iii) of Lemma 4.

It is clear that for any continuously differentiable on $[0, \infty)$ trajectory $p(t) \in S_+$ there exists the only trajectory $x(t) = \omega(p(t))$ in $\omega(S_+)$ (for $t \geq 0$). It is easy to check that

$$\frac{dx(t)}{dt} = \frac{d\omega(p(t))}{dt} = \frac{dp(t)}{dt} \cdot D_p(\omega(p(t))) = \Phi_{p(t)}\left(\frac{dp(t)}{dt}\right).$$

In particular, if $p(t)$ is a solution of the differential equation $\dot{p} = e(p)$, then $x(t)$ is a solution of the differential equation $\dot{x} = \hat{e}(x)$, where $\hat{e}(x) = \Phi_{\omega^{-1}(x)}(e(\omega^{-1}(x)))$.

Similarly, for every continuously differentiable on $[0,\infty)$ trajectory $x(t) \in \omega(S_+)$ there exists a trajectory $p(t)$ in S_+, given by: $p(t) = \omega^{-1}(x(t))$ (for all $t \geq 0$). We have then

$$\frac{dp(t)}{dt} = \frac{d\omega^{-1}(x(t))}{dt} = \frac{dx(t)}{dt} \cdot D_x(\omega^{-1}(x(t))) = \Psi_{x(t)}\left(\frac{dx(t)}{dt}\right).$$

Now we can formulate our main result.

THEOREM 5. *Let $F : S_+ \to \mathbb{R}^n$ be a lower semicontinuous multifunction with nonempty, closed and convex values. Let $F(p)$ be a cone for all $p \in S_+$ and $F(p^*) = \{\mathbf{0}\}$ at only one point $p^* \in S_+$. Moreover, let $F(p) \subset H_0(p)$ for all $p \in S_+$. If for all $p \in S_+ \setminus \{p^*\}$*

$$F(p) \cap H_0^+(\xi(p)) \neq \emptyset, \quad \textit{where } \xi(p) = \left(\frac{(p_1^*)^2}{p_1}, \frac{(p_2^*)^2}{p_2}, \ldots, \frac{(p_n^*)^2}{p_n}\right),$$

then there exists a continuous selection g of the map F such that for any $p^0 \in S_+$ there exists a solution $p : [0,\infty) \to S_+$ of the equation $p'(t) = g(p(t))$ with the initial condition $p(0) = p^0$ such that $\lim_{t\to\infty} p(t) = p^*$.

Moreover, for any $\varepsilon > 0$ there exist $t_0 \geq 0$ and $\delta > 0$ such that all solutions of the equation $p'(t) = g(p(t))$ satisfy the condition: if $|p(t_0) - p^*| < \delta$ then $|p(t) - p^*| < \varepsilon$ for all $t > t_0$.

Proof. Let $V : \mathbb{R}^n_+ \to R$ be defined by:

$$V(p) = v - \prod_{i=1}^n p_i^{(p_i^*)^2}, \quad \text{where } v = \prod_{i=1}^n (p_i^*)^{(p_i^*)^2}.$$

Since the function $p \mapsto \prod_{i=1}^n p_i^{(p_i^*)^2}$ (for $p_i^* < 1$) is increasing and concave on $\mathbb{R}^n_+$, the function V is decreasing and convex. Thus the following system of equations is a necessary and sufficient condition for the minimum of the function V in S_+:

$$\nabla\left(V(p) + \lambda(1 - |p|)\right) = \mathbf{0}, \quad 1 - |p| = 0, \quad \lambda \in \mathbb{R}.$$

We have

$$\nabla V(p) = -\left(\prod_{i=1}^n p_i^{(p_i^*)^2}\frac{(p_j^*)^2}{p_j}\right)_{j=1\ldots n} = -\prod_{i=1}^n p_i^{(p_i^*)^2}\xi(p) = (V(p) - v)\xi(p).$$

Hence, we obtain

$$\begin{cases} (V(p) - v)\frac{(p_i^*)^2}{p_i} - \lambda p_i = 0, \text{ for } i = 1, \ldots, n, \\ \sum_{i=1}^n p_i^2 = 1. \end{cases}$$

For all $i = 1, \ldots, n$ and $\lambda \neq 0$ we have $p_i^2 = \frac{(V(p)-v)(p_i^*)^2}{\lambda}$. Substituting this into the last equation we obtain

$$\sum_{i=1}^n \frac{(V(p) - v)(p_i^*)^2}{\lambda} = 1.$$

Hence $\lambda = V(p) - v$. Then $p = p^*$ is a solution and the minimum of the function V equals 0. Moreover, $V(p) > 0$ for all $p \in S_+ \setminus \{p^*\}$.

It is not difficult to check that $V(S_+) = [0, v)$. Let $\varepsilon \in (0, \frac{1}{\sqrt{n}})$ be such that $p^* \in S_+^\varepsilon = \{p \in S_+ : p_i > \varepsilon \text{ for all } i = 1, \ldots, n\}$ and let $\Sigma = \mathcal{L}_{VS}(\varepsilon) = \{p \in S_+ : V(p) \leq v - \varepsilon\}$. It is easy to verify that Σ is a compact set. Let us observe that for all $p \in \overline{S_+^\varepsilon} = \{p \in S_+ : p_i \geq \varepsilon \text{ for all } i = 1, \dots, n\}$ we have

$$V(p) = v - \prod_{i=1}^{n} p_i^{(p_i^*)^2} \leq v - \prod_{i=1}^{n} \varepsilon^{(p_i^*)^2} = v - \varepsilon^{\sum_{i=1}^{n}(p_i^*)^2} = v - \varepsilon.$$

Thus we conclude that $\overline{S_+^\varepsilon} \subset \Sigma \subset S_+$.

Let us observe that the condition $F(p) \cap H_0^-(\nabla V(p)) \neq \emptyset$ for all $p \in \Sigma \setminus \{p^*\}$ holds since $F(p) \cap H_0^+(\xi(p)) \neq \emptyset$ implies that there exists $u \in F(p)$ such that

$$\langle u, \nabla V(p)\rangle = \langle u, (V(p) - v)\xi(p)\rangle = (V(p) - v)\langle u, \xi(p)\rangle < 0.$$

Hence, all assumptions of the Theorem 3 are satisfied. Thus the multivalued map

$$p \mapsto \begin{cases} F(p) \cap H_0^-(\nabla V(p)), & \text{if } p \in \Sigma \setminus \{p^*\}, \\ \{\mathbf{0}\}, & \text{if } p = p^* \end{cases}$$

defined on Σ, has a continuous and bounded selection g. The proof will be completed by showing that diffeomorphic image $\hat{g}(\cdot)$ of the selection g, which is defined by $\hat{g}(x) = \Phi_{\omega^{-1}(x)}(g(\omega^{-1}(x)))$, fulfills the assumptions of Theorems 2 and 1. Let $Q = \omega(\Sigma)$. We will show that the function $\hat{V}(x) = V(\omega^{-1}(x))$ satisfies the assumptions of Theorem 2. We know that $V(\Sigma) \subset [0, v)$. For all $x \in Q$ we have $\omega^{-1}(x) \in \Sigma$. Thus $\hat{V}(x) = V(\omega^{-1}(x)) \in [0, v)$. Moreover $\hat{V}(\mathbf{0}) = V(\omega^{-1}(\mathbf{0})) = V(p^*) = 0$. We also have

$$\nabla\hat{V}(x) = \nabla V(\omega^{-1}(x)) \cdot D_x(\omega^{-1}(x)) = \Psi_x(\nabla V(\omega^{-1}(x))).$$

Using (iv) of Lemma 4 (with $p = \omega^{-1}(x)$) we have

$$\begin{aligned}\langle \nabla\hat{V}(x), \hat{g}(x)\rangle &= \langle \Psi_x(\nabla V(\omega^{-1}(x))), \Phi_{\omega^{-1}(x)}(g(\omega^{-1}(x)))\rangle = \\ &= \langle \nabla V(\omega^{-1}(x)), g(\omega^{-1}(x))\rangle = \langle \nabla V(p), g(p)\rangle < 0.\end{aligned}$$

Hence, $\hat{g}(x) \in H_0^-(\nabla\hat{V}(x))$ for all $x \in Q \setminus \{\mathbf{0}\}$. By Remark 1, $\hat{g}(x) \in T_{\mathcal{L}_{\hat{V}}(x)}(x)$ for all $x \in Q \setminus \{\mathbf{0}\}$. Therefore we have $\hat{g}(x) \in T_Q(x)$ for all $x \in Q \setminus \{\mathbf{0}\}$. Thus, by Theorem 1, for any $x^0 \in Q$ there exists a solution $x : [0, \infty) \to \mathbb{R}^{n-1}$ of the equation $x'(t) = \hat{g}(x(t))$ with the initial condition $x(0) = x^0$ such that $x(t) \in Q$ for all $t \geq 0$. Moreover, by Theorem 2, we conclude that $x(t)$ is convergent to $\mathbf{0}$. Then there exists a solution $p(t) = \omega^{-1}(x(t)) \in \Sigma$ of the equation $p'(t) = g(p(t))$ with the initial condition $p(0) = \omega^{-1}(x^0)$, where $g(p) = \Psi_{\omega(p)}(\hat{g}(\omega(p)))$, and this $p(t)$ is convergent to p^*.

Moreover, by Theorem 2, we know that for any $\varepsilon > 0$ there exist $t_0 \geq 0$ and $\delta > 0$ such that all solutions of the equation $x'(t) = \hat{g}(x(t))$ satisfy a condition: if $|x(t_0)| < \delta$ then $|x(t)| < \varepsilon$ for all $t > t_0$. It is easy to check that $|p(t_0) - p^*| < |x(t_0)|$ and $|p(t) - p^*| < |x(t)|$ and this completes the proof. ∎

5. The sign-compatibility condition. A price mechanism g describes a classical price adjustment process if it is compatible with the excess demand in the following sense. If at some price system p the i-th coordinates of all $u \in E(p)$ are positive then $g_i(p) \geq 0$. If the i-th coordinates of all $u \in E(p)$ are negative then $g_i(p) \leq 0$. If there exists $u \in E(p)$ such that its i-th coordinate is zero then $g_i(p) = 0$. Thus, g is specified by F defined by

$$F(p) = C^{\uparrow}[E(p)] = \bigcap_{v \in E(p)} \{u \in \mathbb{R}^n : \text{if } v_i \neq 0 \text{ then } u_i v_i \geq 0, \text{ if } v_i = 0 \text{ then } u_i = 0\}.$$

We call such a price mechanism g *sign-compatible* with the excess demand. Observe that for any nonempty convex set A the set $C^{\uparrow}[A]$ is nonempty, closed and convex cone.

THEOREM 6. *Let E be an excess demand multifunction of some pure exchange economy, fulfilling (a0)–(a3), with only one equilibrium point $p^* \in S_+$. Let*

$$C^{\uparrow}[E(p)] \cap H_0(p) = \{\mathbf{0}\} \textit{ if and only if } \mathbf{0} \in E(p). \tag{7}$$

If for all $p \in S_+ \setminus \{p^*\}$ we have

$$C^{\uparrow}[E(p)] \cap H_0(p) \cap H_0^+(\xi(p)) \neq \emptyset, \tag{8}$$

where $\xi(p) = (\frac{(p_1^*)^2}{p_1}, \frac{(p_2^*)^2}{p_2}, \ldots, \frac{(p_n^*)^2}{p_n})$, then there exists a globally asymptotically stable price mechanism which is sign-compatible with the excess demand E.

Proof. According to Theorem 5, it is sufficient to show that the multivalued map $p \mapsto C^{\uparrow}[E(p)] \cap H_0(p)$ is lower semicontinuous on S_+, with nonempty, closed and convex values.

The set $C^{\uparrow}[E(p)] \cap H_0(p)$ is a closed, convex cone for all $p \in S_+$ (as an intersection of two closed, convex cones in $\mathbb{R}^n$). This set is nonempty since $\mathbf{0} \in C^{\uparrow}[E(p)] \cap H_0(p)$ for all $p \in S_+$. Fix $p^0 \in S_+$.

Let $y^0 \in C^{\uparrow}[E(p^0)] \cap H_0(p^0)$. Take any sequence $(p^k)_{k=1}^{\infty} \subset S_+$ which is convergent to p^0 and a sequence $(y^k)_{k=1}^{\infty}$ such that $y^k = (\frac{p_1^0}{p_1^k} y_1^0, \frac{p_2^0}{p_2^k} y_2^0, \ldots, \frac{p_n^0}{p_n^k} y_n^0)$. It is easy to check that $y^k \in H_0(p^k)$. Moreover, $y^k \in C^{\uparrow}[E(p^0)]$ since $\operatorname{sign}(\frac{p_i^0}{p_i^k} y_i^0) = \operatorname{sign} y_i^0$ for all $i = 1, \ldots, n$.

We will show now that for every $p^0 \in \mathbb{R}^n$ there exists a neighbourhood V such that for all $p \in V$ we have $C^{\uparrow}[E(p^0)] \subseteq C^{\uparrow}[E(p)]$. If $\mathbf{0} \in E(p^0)$ then $C^{\uparrow}[E(p^0)] = \{\mathbf{0}\} \subset C^{\uparrow}[E(p)]$. Assume that $\mathbf{0} \notin E(p^0)$. An easy verification shows that for any nonempty, convex sets A, B we have

(a) $A \subseteq B \Rightarrow C^{\uparrow}[B] \subseteq C^{\uparrow}[A]$,
(b) there exists an open and convex set U such that $A \subset U$ and $C^{\uparrow}[A] = C^{\uparrow}[U]$.

Let U denote the open and convex set such that $E(p^0) \subset U$ and $C^{\uparrow}[E(p^0)] = C^{\uparrow}[U]$. Since E is upper semicontinuous, there exists a neighbourhood V of p^0 such that for all $p \in V$ we have $E(p) \subset U$. Applying (a) we have $C^{\uparrow}[U] \subseteq C^{\uparrow}[E(p)]$ and we can conclude that $C^{\uparrow}[E(p^0)] \subseteq C^{\uparrow}[E(p)]$.

Then there exists $K > 0$ such that for all $k > K$ we have $C^{\uparrow}[E(p^0)] \subseteq C^{\uparrow}[E(p^k)]$. Hence $y^k \in C^{\uparrow}[E(p^k)]$ for all $k > K$. Let us check that the sequence $(y^k)_{k=1}^{\infty}$ is convergent to y^0. Indeed,

$$\begin{aligned}
|y^k - y^0| &= \sqrt{\sum_{i=1}^{n} \left(\frac{p_i^0}{p_i^k} - 1\right)^2 (y_i^0)^2} = \sqrt{\sum_{i=1}^{n} (p_i^0 - p_i^k)^2 \left(\frac{y_i^0}{p_i^k}\right)^2} \\
&\leq \sqrt{\sum_{i=1}^{n} (p_i^0 - p_i^k)^2 \left(\frac{\max_i y_i^0}{\min_i p_i^k}\right)^2} \leq |p^k - p^0| \frac{|y^0|}{\min_i p_i^k} \xrightarrow{k\to\infty} 0.
\end{aligned}$$

Hence we conclude that the map $p \mapsto C^{\uparrow}[E(p)] \cap H_0(p)$ is lower semicontinuous. ∎

REMARK 2. The condition (7) is a necessary condition of the existence of globally asymptotically stable price mechanism which is sign-compatible with the excess demand.

6. The angle-compatibility condition. When the price mechanism is sign-compatible with the excess demand, we know that a price change vector and all excess demand vectors from $E(p)$ have to be in the same orthant of $\mathbb{R}^n$. In special cases, it permits the situation when these directions are relatively divergent (the angle between these vectors could be nearly $\frac{\pi}{2}$). This motivates considering a price mechanism g where $g(p)$ forms with all excess demand vectors from $E(p)$ an acute angle and not necessarily $g(p)$ has to have the same signs as all excess demands from $E(p)$ (see [Ar]). Let $\alpha \in [0, \frac{\pi}{2}]$. We say that a price mechanism g is *α-compatible* with the excess demand E if $g(p)$ forms with all vectors $u \in E(p)$ an angle less than or equal to α. Thus, g is specified by F defined by

$$F(p) = C^\alpha[E(p)] \quad \text{with } C^\alpha[A] = \bigcap_{y \in A} C^\alpha(y),$$

where $C^\alpha(y) = \{u \in \mathbb{R}^n : \langle u, y\rangle \ge |u||y| \cos\alpha\}$ for $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $C^\alpha(\mathbf{0}) = \{\mathbf{0}\}$.

Below are some properties of $C^\alpha[A]$ which are presented without proof.

LEMMA 7. *Let $\alpha, \beta \in [0, \frac{\pi}{2}]$ and A, B be nonempty, closed and convex sets. Then*

(i) if $u \in C^\alpha(y)$ and $v \in C^\beta(y)$ and $\alpha + \beta < \frac{\pi}{2}$ then $u \in C^{\alpha+\beta}(v)$;
(ii) $C^\alpha[A]$ is nonempty, closed and convex cone;
(iii) if $\alpha < \beta$ then $C^\alpha[A] \subset C^\beta[A]$;
(iv) if $A \subset B$ then $C^\alpha[B] \subset C^\alpha[A]$;
(v) $v \in C^\alpha[A]$ if and only if $A \subset C^\alpha(v)$;
(vi) let A be compact and $\beta(A) = \inf\{\beta \in [0, \frac{\pi}{2}] : C^\beta[A] \ne \{\mathbf{0}\}\}$, if $\mathbf{0} \notin A$ then $C^{\beta(A)}[A] = \{\lambda z : \lambda \ge 0\}$ for some $z \in A$.

THEOREM 8. *Let E be an excess demand multifunction of some pure exchange economy, fulfilling (a0)–(a3), with only one equilibrium point $p^* \in S_+$. Let*

$$\bar{\alpha} = \sup_{p \in S_+ \setminus \{p^*\}} \inf\{\beta \ge 0 : C^\beta[E(p)] \ne \{\mathbf{0}\}\} < \frac{\pi}{2}. \tag{9}$$

If for $\alpha \in \left(\bar{\alpha}, \frac{\pi}{2}\right]$ and for all $p \in S_+ \setminus \{p^*\}$ we have

$$C^\alpha[E(p)] \cap H_0(p) \cap H_0^+(\xi(p)) \ne \emptyset, \tag{10}$$

where $\xi(p) = (\frac{(p_1^*)^2}{p_1}, \frac{(p_2^*)^2}{p_2}, \dots, \frac{(p_n^*)^2}{p_n})$, then there exists a globally asymptotically stable price mechanism which is α-compatible with the excess demand E.

Proof. According to Theorem 5, it is sufficient to show that the multivalued map $p \mapsto C^\alpha[E(p)] \cap H_0(p)$ is lower semicontinuous on S_+, with nonempty, closed and convex values. The set $C^\alpha[E(p)] \cap H_0(p)$ is a closed, convex cone for all $p \in S_+$ (as an intersection of two closed, convex cones in $\mathbb{R}^n$). This set is nonempty since $\mathbf{0} \in C^\alpha[E(p)] \cap H_0(p)$ for all $p \in S_+$.

Fix $p^0 \in S_+ \setminus \{p^*\}$ and $v^0 \in C^\alpha[E(p^0)]$. Hence, by (v) of Lemma 7, $E(p^0) \subset C^\alpha(v^0)$. Let $\beta(E(p^0)) = \inf\{\beta \ge 0 : C^\beta[E(p^0)] \ne \{\mathbf{0}\}\}$. Let z denote the nonzero vector from $E(p^0)$ such that $\{\lambda z : \lambda \ge 0\} = C^{\beta(E(p^0))}[E(p^0)]$. Then $E(p^0) \subset C^{\beta(E(p^0))}(z)$. Let us take any sequence $(p^k)_{k=1}^\infty \subset S_+ \setminus \{p^*\}$ which is convergent to p^0. Since E is upper semicontinuous then there exists $K_1 > 0$ such that for all $k > K_1$ we have $E(p^k) \subset E(p^0) + r_k B(\mathbf{0}, 1) = F_k$, where $r_k \searrow 0$ for $k \to \infty$. Since $\mathbf{0} \notin E(p^0)$, for any $\theta \in (0, \alpha - \beta(E(p^0)))$

there exists $K_2 > 0$ such that for all $k > K_2$ we have $F_k \subset C^{\beta(E(p^0))+\theta}(z)$ and hence

$$z \in C^{\beta(E(p^0))+\theta}[F_k] \subset C^\alpha[F_k] \subset C^\alpha[E(p^0)]. \tag{11}$$

Let $K = \max\{K_1, K_2\}$. Observe that the segment $[v^0, z]$ has a nonempty intersection with $C^\alpha[F_k]$ for all $k > K$. Let

$$t_k = \sup\{t \in [0,1] : tv^0 + (1-t)z \in C^\alpha[F_k]\}.$$

Let $v^k = t_k v^0 + (1-t_k)z \in C^\alpha[F_k] \subset C^\alpha[E(p^0)]$. The sequence $(t_k)_{k=1}^\infty$ is bounded and increasing. Assume that $\lim_{k\to\infty} t_k = t' < 1$. Let $t'' \in (t', 1)$ and $v'' = t''v^0 + (1-t'')z$. Then $v'' \notin C^\alpha[F_k]$ for all $k > K$. This means that for all $k > K$ there exists $y^k \in F_k$ such that $y^k \notin C^\alpha(v'')$. The subsequence $(y^{k_m})_{m=1}^\infty$ is convergent to some $y \in E(p^0)$ (since the map E has compact values). Hence, $\langle y^{k_m}, v''\rangle < |y^{k_m}||v''|\cos\alpha$, and then $\langle y, v''\rangle \le |y||v''|\cos\alpha$. On the other hand $v'' \in C^\alpha[E(p^0)]$, so for all $y \in E(p^0)$ we have $y \in C^\alpha(v'')$. Hence, $\langle y, v''\rangle \ge |y||v''|\cos\alpha$. Then we conclude that the angle between the vectors y and v'' is α. But this contradicts the fact that the segment $[v^0, z] \subset C^\alpha(y)$, because z belongs to the interior of $C^\alpha(y)$ and $v^0 \in C^\alpha(y)$. Thus the sequence $(v^k)_{k=1}^\infty$ converges to v. Hence, the map $p \mapsto C^\alpha[E(p)]$ is lower semicontinuous. Moreover, the map $p \mapsto H_0(p)$ is continuous in Wijsman topology on S_+ and it has nonempty, closed, convex values. We will verify that $H_0(p) \cap \operatorname{int} C^\alpha[E(p)] \neq \emptyset$. According to (vi) of Lemma 7, for every $p \in S_+$ there exists a nonzero vector $z \in C^{\bar\alpha}[E(p)]$ which belongs to $E(p)$. Hence $E(p) \subset C^{\bar\alpha}(z)$. For $\theta \in (0, \alpha - \bar\alpha)$ let us take any $u \in C^\theta(z)$. Then for all $y \in E(p)$ we have $u \in C^{\bar\alpha+\theta}(y)$ (by (i) of Lemma 7). Hence $C^\theta(z) \subset C^{\bar\alpha+\theta}(y) \subset C^\alpha(y)$ for all $y \in E(p)$. Moreover, since $\theta > 0$, there exists $\gamma > 0$ such that $z + \gamma B(\mathbf{0}, 1) \subset C^\theta(z) \subset C^\alpha[E(p)]$. Thus $z \in E(p) \cap \operatorname{int} C^\alpha[E(p)]$. By Walras' Law $E(p) \subset H_0(p)$ for all $p \in S_+$ therefore we have $H_0(p) \cap \operatorname{int} C^\alpha[E(p)] \neq \emptyset$. By Proposition [2.54, HuPa] we conclude that the map $p \mapsto C^\alpha[E(p)] \cap H_0(p)$ is lower semicontinuous. ■

REMARK 3. The condition (9) is a necessary condition of the existence of globally asymptotically stable price mechanism which is angle-compatible with the excess demand. $\bar\alpha$ is the minimal angle such that the sets $C^\alpha[E(p)]$ contain nonzero vectors at all price systems different from equilibrium point.

REMARK 4. Theorem 8 is true for $\alpha \ge \bar\alpha_* = \beta(E(p^*))$ if we have $\beta(E(p)) < \bar\alpha_*$ for all $p \in S_+ \setminus \{p^*\}$.

7. Examples

7.1. *Scarf's example.* This example concerns the pure exchange model with the single-valued excess demand, i.e. $E(p) = \{e(p)\}$ (for details see [Sc]).

If we take the price mechanism $g = e$, the price trajectories are closed curves on the positive part of the unit sphere S_+. The sets $C^\uparrow[e(p)]$ are presented in the plane $H_0(p^*)$ in the following way (Fig. 1): if some coordinate of all vectors from considered set is nonnegative then we have plus. If some coordinate of all vectors from the considered set is nonpositive then we have minus. On the lines AC', B'C, BA' one coordinate of all vectors from the corresponding set is zero. The equilibrium point $p^* = \frac{1}{\sqrt{3}}(1,1,1)$ is denoted by O.

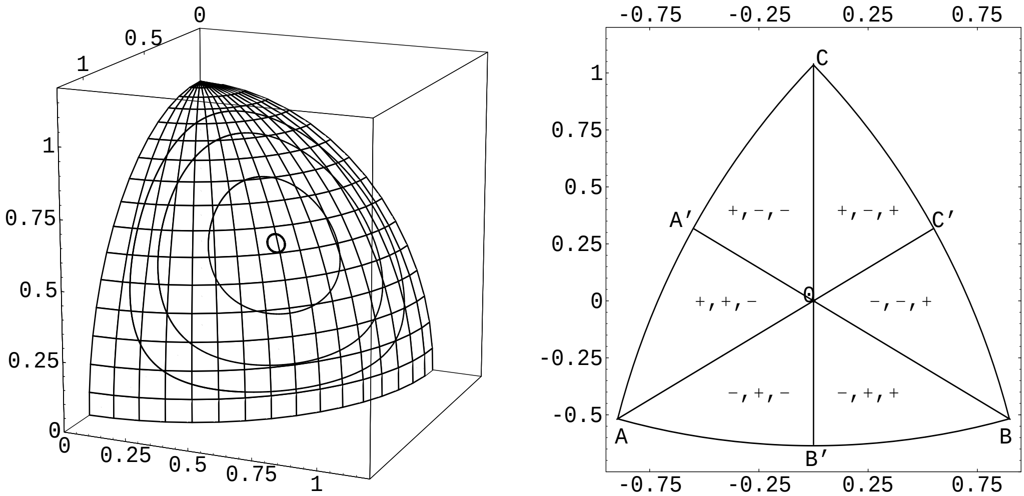

Fig. 1. Scarf's example: price trajectories and the sets $C^{\uparrow}[e(p)]$

The boundary condition does not hold (see Fig. 1). In the set ABO (along the curve AB) the third coordinate should be positive. In the set ACO (along the curve AC) the second coordinate should be positive. In the set BCO (along the curve BC) the first coordinate should be positive.

We cannot use Theorem 6 since the condition (8) does not hold. At $p \in$ BA' $\cup$ AC' $\cup$ CB' we have $C^{\uparrow}[e(p)] = \{\lambda e(p) : \lambda \geq 0\}$ and $e(p) \in H_0(\xi(p))$.

Let $s^k(p)$ be the metric projection of the excess demand vector $e(p)$ on the intersection of the k-th wall of the cone $C^{\uparrow}[e(p)]$ and the set $H_0(p)$:

$$s^k(p) = \Pi(e(p); C^{\uparrow}[e(p)] \cap H_0(p) \cap \{u \in \mathbb{R}^n : u_k = 0\}) \in C^{\uparrow}[e(p)] \cap H_0(p).$$

It is easy to show that $s^1(p) \in H_0^+(\xi(p))$ at $p \in$ AA'O $\cup$ BC'O, $s^2(p) \in H_0^+(\xi(p))$ at $p \in$ BB'O $\cup$ A'CO, $s^3(p) \in H_0^+(\xi(p))$ at $p \in$ AB'O $\cup$ CC'O. The map $g : S_+ \to \mathbb{R}^n$ defined by

$$g(p) = \begin{cases} \delta(p)s(p) + (1-\delta(p))e(p), & \text{if } p \in S_+ \setminus (\mathrm{BA}' \cup \mathrm{AC}' \cup \mathrm{CB}'), \\ e(p), & \text{if } p \in \mathrm{BA}' \cup \mathrm{AC}' \cup \mathrm{CB}', \end{cases}$$

where

$$s(p) = \begin{cases} s^1(p), & \text{if } p \in \mathrm{AA}'\mathrm{O} \cup \mathrm{BC}'\mathrm{O}, \\ s^2(p), & \text{if } p \in \mathrm{BB}'\mathrm{O} \cup \mathrm{A}'\mathrm{CO}, \\ s^3(p), & \text{if } p \in \mathrm{AB}'\mathrm{O} \cup \mathrm{CC}'\mathrm{O}, \end{cases}$$

and

$$\delta(p) = \begin{cases} \frac{6\sqrt{2}(p_2-p_1)^2[(p_2-p_1)^2-(2p_3-p_2-p_1)^2]^2}{(3+\sqrt{3}(p_1+p_2+p_3))[(p_2-p_1)^2+\frac{1}{3}(2p_3-p_2-p_1)^2]^{\frac{5}{2}}}, & \text{if } p \in S_+ \setminus \{p^*\}, \\ 0, & \text{if } p = p^* \end{cases}$$

is a continuously differentiable selection of $C^{\uparrow}[e(p)] \cap H_0(p)$ and $g(p) \in \overline{H_0^+(\xi(p))}$ for $p \neq p^*$.

Moreover, the function V, defined in the proof of Theorem 5, is a strictly decreasing function along any solution of the differential equation $p'(t) = g(p(t))$. Analysis similar to that in the proof of Theorem 5 shows that $\hat{V}$ is a global Lyapunov function for the differential equation $x'(t) = \hat{g}(x(t))$. Thus, by Barbashin–Krasovski Theorem ([Kr], [Gl]), we conclude that there exists a globally asymptotically stable price mechanism in Scarf's example which is sign-compatible with the excess demand.

In the case of angle-compatibility rule we can obtain the existence of globally asymptotically stable price mechanism which is α-compatible with the excess demand for all $\alpha > 0$ by Theorem 8 using only arguments of geometric nature. It is sufficient to notice that $e(p) \in H_0(\xi(p))$ for all $p \in S_+$.

7.2. *Example with multivalued excess demand (1).* The second example concerns the pure exchange model with a multivalued excess demand which is constructed from the following utility functions and initial resources: $u^1(x_1, x_2, x_3) = \min\{x_1 + x_2, x_1 + x_3\}$, $w^1 = (1, 1, 1)$, $u^2(x_1, x_2, x_3) = x_1^{\frac{1}{2}} x_3^{\frac{1}{2}}$, $w^2 = (1, 2, 1)$, $u^3(x_1, x_2, x_3) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, $w^3 = (1, 1, 2)$. The excess demand is defined by $E(p) = \{e(p, t) : t \in [0, 1]\}$ where

– for $p \in \mathrm{I} = \{p \in S_+ : \frac{p_2}{p_1} + \frac{p_3}{p_1} > 1\}$ we have

$$e(p,t) = \left(\frac{5}{2}\frac{p_2}{p_1} + \frac{5}{2}\frac{p_3}{p_1} - 1, \frac{1}{2}\frac{p_1}{p_2} + \frac{p_3}{p_2} - \frac{7}{2}, \frac{1}{2}\frac{p_1}{p_3} + \frac{p_2}{p_3} - \frac{7}{2}\right),$$

– for $p \in \mathrm{II} = \{p \in S_+ : \frac{p_2}{p_1} + \frac{p_3}{p_1} = 1\}$ we have

$$e(p,t) = \left(\frac{3}{2} - 2t, \frac{3}{2}\frac{p_3}{p_2} + 2t - 3, \frac{3}{2}\frac{p_2}{p_3} + 2t - 3\right),$$

– for $p \in \mathrm{III} = \{p \in S_+ : \frac{p_2}{p_1} + \frac{p_3}{p_1} < 1\}$ we have $e(p, t) =$

$$\left(\frac{3}{2}\frac{p_2}{p_1} + \frac{3}{2}\frac{p_3}{p_1} - 2, \frac{1}{2}\frac{p_1}{p_2} + \frac{p_1}{p_2 + p_3} + \frac{p_3}{p_2} - \frac{5}{2}, \frac{1}{2}\frac{p_1}{p_3} + \frac{p_1}{p_2 + p_3} + \frac{p_2}{p_3} - \frac{5}{2}\right).$$

This excess demand is single-valued at such price systems which belong to I or III. These sets are presented in the first diagram (Fig. 2). The excess demand is multivalued at such price systems which belong to II. This set is denoted by the line DI.

In the next diagram (Fig. 2) there are presented the sets $C^{\uparrow}[E(p)]$ at price systems from the sets I and III. It is not difficult to observe that the boundary condition is satisfied. Along the curve AB the third coordinate is positive. Along the curve AC the second coordinate is positive. Along the curve BC the first coordinate is positive. The competitive equilibrium $p^* = \frac{1}{\sqrt{6}}(2, 1, 1)$ (for $\frac{p_3}{p_2} = 1$, $\frac{p_1}{p_3} = 2$, $t = \frac{3}{4}$) is denoted by O.

The sets $C^{\uparrow}[E(p)]$, when the excess demand is multivalued, are presented in the table.

$\frac{p_3}{p_2}$	$(0, \frac{1}{2})$	$[\frac{1}{2}, \frac{2}{3})$	$[\frac{2}{3}, \frac{3}{2}]$	$(\frac{3}{2}, 2]$	$(2, \infty)$
u_1, $(u \in C^{\uparrow}[E(p)])$	0	0	0	0	0
u_2, $(u \in C^{\uparrow}[E(p)])$	-	-	0	0	+
u_3, $(u \in C^{\uparrow}[E(p)])$	+	0	0	-	-
II	HI	GH	FG	EF	DE

Let us observe that $C^{\uparrow}[E(p)] = \{\mathbf{0}\}$ at $p \in$ FG. It means that the excess demand for all goods is both negative and positive at given price system from the line FG. At every price system from the segments EF and GH the intersection of $C^{\uparrow}[E(p)]$ and $H_0(p)$ consists of only zero as well. In this case the excess demand (from $C^{\uparrow}[E(p)]$) is negative for one good. Thus we conclude that condition (7) is not satisfied and there does not exist a globally asymptotically stable price mechanism which is sign-compatible with the excess demand. The necessary condition (7) is satisfied if at all price systems, different from competitive equilibrium, the excess demand (from $C^{\uparrow}[E(p)]$) for at least one good is positive and for at least one good is negative.

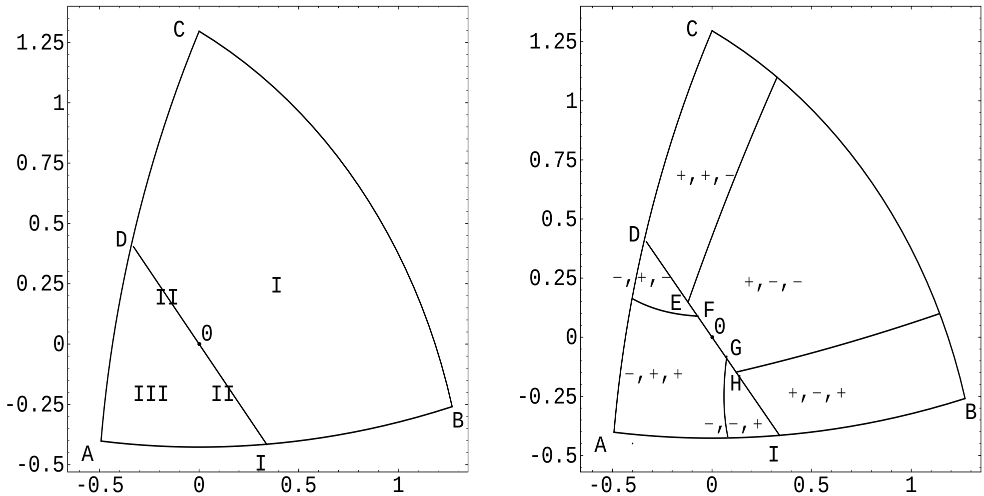

Fig. 2. The sets I, II, III and $C^{\uparrow}[E(p)]$

In the case of angle-compatibility rule $\bar{\alpha}_*$ is equal to $\frac{\pi}{2}$ and the condition (10) is satisfied by the excess demand vectors in I $\cup$ III and by the vector $e(p,0)+e(p,1) \in C^{\frac{\pi}{2}}[E(p)]$ in II. Thus, by Theorem 8, there exists a globally asymptotically stable price mechanism which is $\frac{\pi}{2}$-compatible with the multivalued excess demand E.

7.3. *Example with multivaled excess demand (2).* The third example concerns the pure exchange model with a multivalued excess demand which is constructed from the following utility functions and initial resources: $u^1(x_1,x_2,x_3) = x_2 + x_3$, $w^1 = (2,0,0)$, $u^2(x_1,x_2,x_3) = x_1^{\frac{1}{2}} x_3^{\frac{1}{2}}$, $w^2 = (2,2,0)$, $u^3(x_1,x_2,x_3) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, $w^3 = (0,2,2)$. The excess demand is defined by $E(p) = \{e(p,t) : t \in [0, 2\frac{p_1}{p_2}]\}$ where

– for $p \in \mathrm{I} = \{p \in S_+ : \frac{p_3}{p_2} < 1\}$ we have

$$e(p,t) = \left(2\frac{p_2}{p_1} + \frac{p_3}{p_1} - 3, \frac{p_3}{p_2} - 3, 3\frac{p_1}{p_3} + \frac{p_2}{p_3} - 2\right),$$

– for $p \in \mathrm{II} = \{p \in S_+ : \frac{p_3}{p_2} = 1\}$ we have

$$e(p,t) = \left(3\frac{p_2}{p_1} - 3, t - 2, 3\frac{p_1}{p_2} - t - 1\right),$$

– for $p \in \mathrm{III} = \{p \in S_+ : \frac{p_3}{p_2} > 1\}$ we have

$$e(p,t) = \left(2\frac{p_2}{p_1} + \frac{p_3}{p_1} - 3, 2\frac{p_1}{p_2} + \frac{p_3}{p_2} - 3, \frac{p_1}{p_3} + \frac{p_2}{p_3} - 2\right).$$

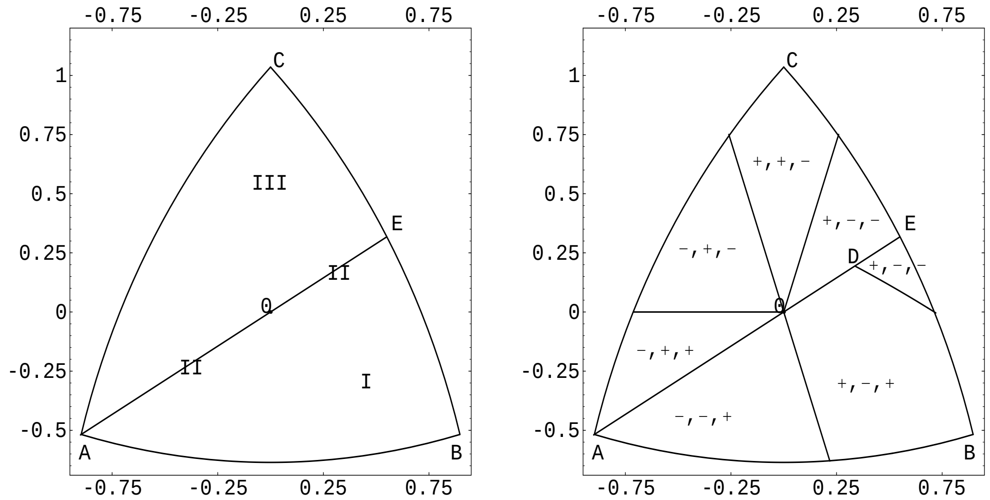

Fig. 3. The sets I, II, III and $C^{\uparrow}[E(p)]$

This excess demand is single-valued at such price systems which belong to I or III. These sets are presented in the first diagram (Fig. 3). The excess demand is multivalued at such price systems which belong to II. This set is denoted by the line AE.

In the next diagram (Fig. 3) there are presented the sets $C^{\uparrow}[E(p)]$ at price systems from the sets I and III. It is not difficult to observe that the boundary condition is satisfied. The competitive equilibrium $p^* = \frac{1}{\sqrt{3}}(1,1,1)$ (for $\frac{p_3}{p_2} = 1$, $\frac{p_2}{p_1} = 1$, $t = 2$) is denoted by O. The sets $C^{\uparrow}[E(p)]$, when the excess demand is multivalued, are presented in the table.

$\frac{p_1}{p_2}$	$(0, \frac{1}{3})$	$\left[\frac{1}{3}, 1\right)$	1	$(1, \infty)$
u_1, $(u \in C^{\uparrow}[E(p)])$	$+$	$+$	0	-
u_2, $(u \in C^{\uparrow}[E(p)])$	-	-	0	0
u_3, $(u \in C^{\uparrow}[E(p)])$	-	0	0	$+$
II	DE	OD	O	AO

The necessary condition (7) is satisfied. $C^{\uparrow}[E(p)] = \{\mathbf{0}\}$ only at competitive equilibrium. The intersection of $C^{\uparrow}[E(p)]$ and $H_0(p)$ consists of nonzero vectors at all p different from p^*. The condition (8) is satisfied by the excess demand vectors in $\mathrm{I} \cup \mathrm{III} \cup \mathrm{DE}$ and by any vector from the set $C^{\uparrow}[E(p)]$ in $\mathrm{AO} \cup \mathrm{OD}$. Thus, by Theorem 6, there exists a globally asymptotically stable price mechanism which is sign-compatible with the multivalued excess demand E.

In the case of angle-compatibility rule $\bar{\alpha}$ is equal to 0.285π and the condition (10) is satisfied by the excess demand vectors in $\mathrm{I} \cup \mathrm{III}$ and by the vector $e(p,0) + e(p, 2\frac{p_1}{p_2}) \in C^{\alpha}[E(p)]$ for $p \in \mathrm{II}$. Thus, by Theorem 8, there exists a globally asymptotically stable price mechanism which is α-compatible with the multivalued excess demand E for $\alpha > 0.285\pi$.

Acknowledgements. The author is greatly indebted to L. E. Rybiński and K. Przesławski for many helpful comments and stimulating conversations.

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