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DEMAND CONTINUITY AND EQUILIBRIUM IN BANACH COMMODITY SPACES

ANTHONY HORSLEY and A. J. WROBEL

Department of Economics, London School of Economics
Houghton Street, London WC2A 2AE, United Kingdom
E-mail: LSEecon123@mac.com

Dedicated to the memory of Professor Jerzy Łoś

Abstract. Norm-to-weak* continuity of excess demand as a function of prices is proved by using our two-topology variant of Berge's Maximum Theorem. This improves significantly upon an earlier result that, with the extremely strong finite topology on the price space, is of limited interest, except as a vehicle for proving equilibrium existence. With the *norm* topology on the price space, our demand continuity result becomes useful in applications of equilibrium theory, especially to problems with continuous commodity spectra. Some auxiliary results are also given, including closedness of the total production set and additivity of the asymptotic cone operation. Both are needed in proving equilibrium existence by the use of the Debreu-Gale-Nikaido Lemma.

1. Introduction. Although the properties of demand in infinite-dimensional commodity and price spaces have attracted much interest, hitherto the results on its price-continuity that are needed for establishing equilibrium existence by the direct excess-demand approach have been unsatisfactory. For example, both Aliprantis and Brown [1, p. 204], who initiated this line of research, and Araujo [2] report negative findings, whilst Florenzano [10, p. 216] manages only a continuity result with the finite topology on the price space, obliging her to use finite-dimensional price sets.[1] These failures led

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[1]The direct approach to equilibrium existence consists in extending the methods developed originally for a finite-dimensional commodity space, and continuity of demand in prices is needed if the excess-demand method is adopted. For want of a satisfactory result on the demand derived from the optimizing behaviour of individual consumers and producers, Aliprantis and Brown [1] take continuous demand as a primitive rather than derived concept—except in [1, Example 4.8, p. 205], where they, too, resort to using finite-dimensional price sets.

others to use finite-dimensional approximations of the commodity space as well as of the price space,[2] because this method does not require demand continuity: see, e.g., [5], [8] or [21]. The resort to approximation is often packaged with an interpretation of the infinite-dimensional commodity space as an idealized description of a "large but finite" number of commodities: see, e.g., [21, p. 512]. Sometimes this may be appropriate, but in problems for which infinite-dimensional modelling is tailor-made, and where it has turned out to be most successful, the spectra of commodities are genuinely continuous, e.g., the flows of goods in continuous-time pricing of public utilities. In such contexts, it is mistaken to hold that all meaningful results can be captured by the approximation approach. Discretization rules out techniques that yield key calculus results, such as the continuity of the equilibrium price density [16] and its uses in the marginal valuation of capital and other fixed inputs in [14] and [17]. It also rules out the sensitivity analysis that is needed for any implementation of the equilibrium solution: in the case in point, demand continuity properties are essential for deciding whether small deviations from the equilibrium price system will or will not result in large shifts of demand.

For demand continuity to be of interest in applications, the topologies used on the price and commodity spaces must be kept, respectively, as weak and as strong as possible. If, by contrast, an extremely strong topology is used on the *price* space as in [10], then demand continuity becomes a weak result that has little value except as a vehicle for an equilibrium existence proof. For a more detailed account of [10], as well as of [1], see Section 6.

What we establish is norm-to-weak* continuity of demand, which is the best general property available when the commodity space, L, is the Banach dual of a price space L' (on which the demand map is defined). It is essential that this result be applicable to preferences that are weakly* upper semicontinuous (w*-u.s.c.) but not necessarily weakly* lower semicontinuous (w*-l.s.c.), since even some of the simplest functional forms for utility are not weakly* continuous. For example, an additively separable, strictly concave utility function on L^∞_+ is *not* w*-l.s.c. (although it is Mackey-continuous and hence w*-u.s.c.): see [5, Appendix II]. *Lower* semicontinuity of preferences should therefore be assumed for a topology that is significantly stronger than the weak* topology—and the best choice is the finite topology of the *commodity* space, denoted by $\mathcal{T}_{\mathrm{Fin}}(L)$. This gives a very large class of continuous preferences, which obviously includes all the norm-continuous ones. The $\mathcal{T}_{\mathrm{Fin}}$-continuity condition is actually no more restrictive than it is in the finite-dimensional case (so the only truly "infinite-dimensional" restriction on preferences here is that of w*-u.s. continuity).[3]

The case of a preference order $\preccurlyeq$ that is w*-u.s.c. but only $\mathcal{T}_{\mathrm{Fin}}$-l.s.c. requires a variant of Berge's Maximum Theorem with two topologies on the set of actions, which is here the consumption set. Such extensions, given in [18], are applied to prove demand continuity

[2] Florenzano [10, Proof of Proposition 3, p. 216] works with demand as a map of a finite-dimensional price set into the infinite-dimensional commodity space, as do Aliprantis and Brown [1, Example 4.8, p. 205] when dealing with derived demand.

[3] A function $U\colon L \to \mathbb{R}$ is continuous for $\mathcal{T}_{\mathrm{Fin}}$ if (and only if) its restriction to any affine subspace of a finite dimension d is continuous for the usual topology of $\mathbb{R}^d$.

(Theorem 5) as well as another result used in the direct proof of equilibrium existence (Lemma 7).

The main reason for using the weak* topology (w*) is that it is weak enough to make the consumption set compact. Furthermore, in the context of demand continuity, the parameter set is the price space L' with the norm topology, and w* is also weak enough to make the budget correspondence norm-to-w* upper hemicontinuous (u.h.c.). The other topology on the consumption set is purely auxiliary in that it enters the assumptions but *not* the conclusion—which is that the excess-demand correspondence is norm-to-w* u.h.c. The role of this auxiliary topology is only to make the preferences l.s.c. whilst making the budget correspondence lower hemicontinuous (l.h.c.) when the price space L' carries the norm topology. Since $\mathcal{T}_{\mathrm{Fin}}(L)$ meets the latter condition despite its strength—the budget correspondence is actually even weak-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c., as the Proof of Theorem 5 shows—it is the best choice for the auxiliary topology.

In the context of demand continuity, the price-space topology should be kept as weak as possible (i.e., just strong enough to make the budget correspondence u.h.c. with w* on the consumption set, and l.h.c. with $\mathcal{T}_{\mathrm{Fin}}(L)$ thereon). We achieve this by using the norm topology of L'. This is what allows us to improve on the analysis of Florenzano [10, Proof of Proposition 3], who establishes demand continuity, but only when the *price* space carries the finite topology $\mathcal{T}_{\mathrm{Fin}}(L')$, which is even stronger than the strongest vector topology $\mathcal{T}_{\mathrm{SV}}(L')$. As with the two norms, the two finite topologies (on the price and commodity spaces) should not be mistaken for each other: whereas the use of $\mathcal{T}_{\mathrm{Fin}}(L')$ as in [10] severely weakens the demand continuity result, our use of $\mathcal{T}_{\mathrm{Fin}}(L)$ can only strengthen it (albeit perhaps not significantly by comparison with using the norm of L for this purpose).

The stronger continuity property of demand does not, however, strengthen the equilibrium existence result itself (Theorem 8): this does not differ significantly from [10, Propositions 3 and 4], except for minor improvements. Given here mainly for completeness, it establishes the existence of an equilibrium with a price system $p^\star$ in the norm-dual L^* of L (which is larger than the predual L', unless the space is reflexive). However, for the continuity properties of demand to be relevant for investigating the impact of price deviations, the exact equilibrium price $p^\star$ must be known to belong not just to L^* but actually to the smaller price space L' (since demand is defined only on L'). Although no such price representation result is given here, under appropriate assumptions it holds for both (i) the commodity space of all essentially bounded functions L^∞, with $L' = L^1$ (the space of integrable functions) and (ii) the commodity space of measures $\mathcal{M}$, with $L' = \mathcal{C}$ (the space of continuous functions on a compact space of commodity characteristics): see [5] and [21], respectively.[4]

The analysis is complemented by examples showing that demand may be undefined at a $p \in L^* \setminus L'$ and, also, that demand can be weak-to-weak* discontinuous (as a map of L' into L): see Section 7.

Our own interest in Bewley's model [5] comes from our use of it in continuous-time peak-load pricing. This has the potential for implementation by public utilities and com-

[4]For the case of $L = L^\infty$, see also [15]. And Richard's result [28] applies to both cases.

petitive industries: see [14], [15], [16], [17] and the references therein. In this context, however, demand continuity would be of even greater interest if it could be established for the Mackey topology on the commodity space L^∞ (paired with the price space L^1), but this is an open question. If true, it would mean, for example, that the disequilibrium resulting from a price deviation which is small in the $L^1[0,T]$-norm could be corrected by rationing users without much loss of utility or output (on the assumption that their utility and production functions are Mackey continuous, but that much is needed anyway to guarantee that $p^\star \in L^1$). It is also of interest to examine demand continuity for the supremum norm on the commodity space $L^\infty[0,T]$: such a property would mean that the extra cost of meeting demand out of equilibrium could be "absorbed" by the supplier (since this is the norm that makes his cost function continuous in peak-load pricing). For such a continuity property, the price space has to be restricted further, and its norm strengthened to the supremum norm, on a suitable *subspace* of $L^1[0,T]$ such as $\mathcal{C}[0,T]$. For this use of the supremum norm to be possible, the equilibrium price function $p^\star$ must be known to be at least bounded; and in [15] we identify cases in which $p^\star$ is actually in $\mathcal{C}[0,T]$ when the commodity space is $L^\infty[0,T]$. That the usual norm of the price space L^1 is not strong enough to make demand a norm-to-norm continuous map of L^1 into L^∞ is clear from simple counterexamples, as well as from a general discontinuity result of Araujo [2, Theorem 3(b)].[5]

However, Araujo's conclusion [2, p. 319] that "it is not a good idea to try to prove the existence of equilibria by means of a globally defined (i.e., on the whole dual) demand function" is mistaken, at least in so far as he specifically refers to Bewley's model: although the demand map (from L^1 to L^∞) is norm-to-norm discontinuous, this simply has little or no bearing on this approach to equilibrium existence. A sufficient property of demand is its norm-to-weak* continuity, although genuine technical difficulties do arise in exploiting it. The base Δ^* of the polar P^* of the production cone is not norm-compact, nor is the demand map defined on the whole of Δ^* because this is a subset of L^* and not of L', on which demand *is* defined. And although Δ^* *is* weakly* compact, its intersection with L' is not: the weak* closure of $\Delta^* \cap L'$ equals the larger price set Δ^*. There is, nevertheless, a useful extension of the Debreu-Gale-Nikaido Lemma—given by Florenzano [10]—that does apply to this setting. Its application can prove only the existence of an equilibrium price $p^\star$ in L^* (and not in L'), but the problem of price representation is conceptually separate from that of its existence; and in principle $p^\star$ can be shown to belong to L' by an additional argument. Such an argument is well known for the case of $L = L^\infty$ with $L' = L^1$ (and is based on the Hewitt-Yosida decomposition of $L^{\infty *}$).[6]

Some other technical results needed to realize the full potential of the direct approach are also provided. As is recognized in [5, p. 520] and [8], for the Adequacy Assumption it is best to use the largest cone contained in the total production set Y: this helps both to weaken the assumption and to limit the range of relevant prices to a compact

[5]This exploits the separability of L^1 and the nonseparability of L^∞ for their respective norms. By contrast, L^∞ is separable for the weak* topology (when L^1 is separable for the norm).

[6]So far as we know, no corresponding argument exists for $L = \mathcal{M}$ with $L' = \mathcal{C}$.

set Δ^*.[7] However, if this cone is to be used for an equilibrium existence proof based on the Debreu-Gale-Nikaido Lemma, one needs to know that it is weakly* closed. This is established here: Y is shown to be closed (Lemma 2), and it follows that so is the cone in question, which therefore equals the asymptotic cone, as Y. One also needs to know that it (as Y) is equal to the sum of the asymptotic cones of the individual production sets, and this is shown in Lemma 4.[8]

2. Model and assumptions. The commodity space, L, is taken to be the norm-dual (equal to the order-dual) of a Banach lattice L'; i.e., $L = L'^*$. The nonnegative cone in L' is denoted by L'_+, and the norm of a $p \in L'$ is $\|p\|'$. The dual nonnegative cone in L is L_+. The (dual) norm of an $x \in L$ is denoted by $\|x\|$. The norm-dual L^* of L, which contains L', is used as the price space; and $\langle p \,|\, x \rangle$ denotes the value of a commodity bundle $x \in L$ at a price system $p \in L^*$. The weak* topology of L is denoted by w^* for brevity; the full notation is $\mathrm{w}(L, L')$. As for the weak* topology of L^*, this is always denoted by $\mathrm{w}(L^*, L)$ for clarity. Also, the finite topology on the commodity space L—in which a set is closed if and only its intersection with any affine subspace of a finite dimension d is closed for the usual topology of $\mathbb{R}^d$—is denoted by $\mathcal{T}_{\mathrm{Fin}}(L)$. This is abbreviated to $\mathcal{T}_{\mathrm{Fin}}$ (which *never* means $\mathcal{T}_{\mathrm{Fin}}(L')$).

The (finite) sets of producers and households (or consumers) are denoted by Pr and Ho. The production set of producer $i \in \mathrm{Pr}$ is denoted by Y_i, and the consumption set of household $h \in \mathrm{Ho}$ is X_h. Consumer preferences, taken to be complete and transitive, are given by a total (a.k.a. complete) weak preorder $\preccurlyeq_h$ on X_h, for each h. The corresponding strict preference is denoted by $\prec_h$. The household's initial endowment is x_h^{En}; the household's share in the profits of producer i is $\varsigma_{hi} \geq 0$, with $\sum_h \varsigma_{hi} = 1$ for every i. (The ranges of running indices in summations, etc., are always taken to be the largest possible with any specified restrictions.)

The attainable consumption and production sets consist of those points of X_h or Y_i that appear in some feasible allocation. Formally, with $x^{\mathrm{En}} := \sum_h x_h^{\mathrm{En}}$ denoting the total initial endowment, the attainable consumption and production sets are

$$X_h^{\mathrm{At}} := X_h \cap \Big(- \sum_{h' : \, h' \neq h} X_{h'} + x^{\mathrm{En}} + \sum_i Y_i \Big) \tag{1}$$

$$Y_i^{\mathrm{At}} := Y_i \cap \Big(\sum_h X_h - x^{\mathrm{En}} - \sum_{i' : \, i' \neq i} Y_{i'} \Big). \tag{2}$$

The complete list of assumptions follows.

[7]A similar restriction on the relevant range of prices can be obtained on the consumption side by assuming the properness of preferences: see [26] and [8, pp. 2–3]. However, it is shown in [22, Section 3] that this use of properness is formally equivalent to assuming that the production cone has a nonempty interior (for the norm topology). A distinctive feature of L^∞ is that its nonnegative orthant has a nonempty interior.

[8]These results are obtained by using the "localization" of weak* closedness property to bounded parts of convex sets, known as the Krein-Smulian Theorem. The technique is also instrumental in establishing weak* upper semicontinuity of concave functions: see [13].

Set Closedness: The sets Y_i and X_h are w*-closed (for each i and h).
Set Convexity: The sets Y_i and X_h are convex.
Preference Continuity: For each h the preorder $\preccurlyeq_h$ is:

1. w*-upper semicontinuous, i.e., for every x' the set $\{x \in X_h : x' \preccurlyeq_h x\}$ is w*-closed; and
2. $\mathcal{T}_{\mathrm{Fin}}$-lower semicontinuous, i.e., for every x' the set $\{x \in X_h : x \preccurlyeq_h x'\}$ is $\mathcal{T}_{\mathrm{Fin}}$-closed.

Preference Convexity: For each h, if $x \prec_h x'$, then $x \prec_h \epsilon x' + (1-\epsilon)x$ for every number ϵ with $0 < \epsilon \leq 1$.[9]
Nonsatiation: For every h and $x \in X_h^{\mathrm{At}}$ there exists $x' \in X_h$ with $x \prec_h x'$.
Inaction Feasibility: $0 \in Y_i$ for every i.
Boundedness: For every norm-bounded set $B \subset L$, the set

$$Y_i \cap \Big(L_+ - B - \sum_{i' : i' \neq i} Y_{i'}\Big)$$

is norm-bounded (for each i); and X_h is contained in L_+ (for each h).[10]
Adequacy: For each h,

$$(X_h - x_h^{\mathrm{En}}) \cap \operatorname{cor} \operatorname{as} Y \neq \emptyset \tag{3}$$

where $Y := \sum_i Y_i$, i.e., a feasible trade for the consumer belongs to the core (a.k.a. the algebraic interior) of the asymptotic cone of the total production set.

Comments:

- The Adequacy Assumption (3) guarantees that feasible allocations exist, i.e., that X_h^{At} and Y_i^{At} are nonempty.
- The cone $\operatorname{as} Y$ can be characterized as the largest cone (with vertex at 0) that is contained in Y; it is further discussed in Section 4.
- Part of the Adequacy Assumption is that $\operatorname{cor} \operatorname{as} Y \neq \emptyset$. For a *convex* set A, its core is equal to the interior of A for each of the following: $\mathcal{T}_{\mathrm{Fin}}$ (the finite topology), $\mathcal{T}_{\mathrm{SV}}$ (the strongest vector topology), and $\mathcal{T}_{\mathrm{SLC}}$ (the strongest locally convex topology, a.k.a. the natural or convex-core topology): see, e.g., [24, (1.3) and Section 3: p. 108]. In a Banach space L, the core of a convex, norm-closed set A is also equal to the norm-interior of A (in L): see, e.g., [12, p. 84] or [30, II.7.1].
- The (algebraic) polar A° of a cone $A \subset L$ with a nonempty norm-interior is a cone in L^* with a w(L^*, L)-compact base. This is essential for the fixed-point argument in the equilibrium existence proof, where such a base Δ^* for the price cone $P^* := (\operatorname{as} Y)^\circ \setminus \{0\}$ is specified by (14).
- For demand continuity, a significantly weaker form of the Adequacy Assumption is sufficient (Theorem 5). This is because, in the continuity proof, the assumption is

[9] This condition is also known as semi-strict quasi-convexity. It implies quasi-convexity (i.e., the convexity of $\{x : x' \preccurlyeq x\}$) if $\preccurlyeq$ is $\mathcal{T}_{\mathrm{Fin}}$-u.s.c.: see, e.g., [7, pp. 59–60].

[10] When $L = L^\infty$, this Boundedness Assumption is equivalent to that of [5, p. 520], since norm-boundedness and order-boundedness are the same in this case.

needed only to make the budget correspondence lower hemicontinuous by guaranteeing that each consumer's income is (strictly) above the survival minimum at all price systems from the relevant range, i.e., that

$$\forall p \in P' \ \exists x_h \in X_h \quad \langle p \,|\, x_h - x_h^{\mathrm{En}} \rangle < 0 \tag{4}$$

where $P' := P^* \cap L'$. This obviously holds if

$$\exists x_h \in X_h \ \forall p \in P' \quad \langle p \,|\, x_h - x_h^{\mathrm{En}} \rangle < 0, \tag{5}$$

i.e., if a feasible trade is (strictly) negative as a linear functional on P'. And (3) implies the stronger property of negativity on P^*, i.e., it implies that

$$\exists x_h \in X_h \ \forall p \in P^* \quad \langle p \,|\, x_h - x_h^{\mathrm{En}} \rangle < 0. \tag{6}$$

For a proof, see (12).

- For the case of $Y = -L_+$ in a Banach lattice L, Condition (3) implies a specific restriction on the space itself: $\operatorname{cor} L_+ \neq \emptyset$ if and only if L is the space $\mathcal{C}(K)$ of all continuous real-valued functions on a compact K. This is the Kakutani-Krein-Krein Theorem: see, e.g., [30, V.8.5 with V.8.4].[11] The existence of a $y \in L$ that is strictly positive on $L_+^* \setminus \{0\}$, as is required for (6), can be a significantly weaker condition than the nonemptiness of $\operatorname{cor} L_+$. This is because, in any Banach lattice L, strictly positive elements are the same as quasi-interior points of L_+;[12] and the latter exist whenever L is separable:[13] see, e.g., [30, V.7.6]. Therefore (6) is a useful condition when $L = L^\varrho(\Xi, \mathcal{A}, \mu)$ for a $\varrho < +\infty$ (where μ is a sigma-finite measure on a countably generated sigma-algebra $\mathcal{A}$, or on its completion).
- The assumption that L has a (Banach) predual can be avoided by replacing L' with some separating subspace of the norm-dual L^* and using the weak topology $\mathrm{w}(L, L')$ instead of the weak* topology on L. For example, when $L = L^1$ one can set $L' = L^* = L^\infty$, and work with $\mathrm{w}(L^1, L^\infty)$ using the Dunford-Pettis Compactness Criterion.
- The space $\mathcal{M}(K)$ has generally no element that is strictly positive on $\mathcal{M}_+^* \setminus \{0\}$. But it has elements that are strictly positive on $\mathcal{C}_+ \setminus \{0\}$ when K is a metric compact: any measure that is positive on every open subset of K is an example. So, although (6) cannot hold in this case, Condition (5) can still be useful.
- The Adequacy Assumption keeps the value of the initial endowment above the minimum; any profit income plays no part in the argument (except for being nonnegative). For best results, all the productive factors should be included in the list of commodities, to represent the rents on any fixed factors as endowment rather

[11] More precisely, L can be equivalently renormed so as to be isomorphic, as a normed lattice, to $\mathcal{C}(K)$. For the case of L^∞, note that: (i) renorming is unnecessary, (ii) the K in question is extremally disconnected.

[12] If L is an order-complete Banach lattice of minimal type (e.g., L^1 or L^ϱ for a $\varrho < \infty$), then strictly positive elements (or, equivalently, quasi-interior points of L_+) are also the same as weak order units: see, e.g., [30, V.7.7].

[13] More generally, quasi-interior points exist (and are dense in L_+) for any separable, completely metrizable and locally convex space (a separable Fréchet space) L ordered by a closed cone that generates it (i.e., a cone L_+ such that $L_+ - L_+ = L$).

than profit income. This can be achieved by "conification", which formally converts a technology with decreasing returns to scale into one with constant returns. This procedure—detailed in, e.g., [27, Section 5]—enlarges the commodity space by introducing "entrepreneurial" factors, one for each production set Y_i that is not a cone from the start.[14] The added factors are in fixed supply: there is, say, a unit of each, which is owned by the consumers in amounts proportional to their shares in the firm. (Each factor is taken to be of use only for the firm in question, and so it does not enter consumer preferences.) The original production set Y_i is embedded into the enlarged commodity space by setting the additional coordinates of each input-output vector at -1 for the i-th entrepreneurial factor (and at 0 for the others). Finally, the i-th production set is redefined as the closure of the cone generated (in the enlarged space) by the embedded original set.

- Although the *lower* semicontinuity of preferences need be (and is) assumed only for $\mathcal{T}_{\mathrm{Fin}}$, little would be lost by way of applications had l.s. continuity been assumed for the norm of L. (By contrast, in the price space L' the distinction between the norm topology and $\mathcal{T}_{\mathrm{Fin}}(L')$ is significant, as is pointed out in the Introduction.)
- Note, however, that $\mathcal{T}_{\mathrm{Fin}}$ is *not* a vector topology, unless dim L is countable (which is never the case for an infinite-dimensional Banach space L): see, e.g., [24, Section 3: p. 108]. When the vector-space property, or local convexity, is also needed, the best replacement is $\mathcal{T}_{\mathrm{SV}}$, or $\mathcal{T}_{\mathrm{SLC}}$. Even with $\mathcal{T}_{\mathrm{SLC}}$ on L, every concave utility function $U\colon\ L \to \mathbb{R}$ (defined and finite on the whole commodity space) is continuous: see, e.g., [3, V.3.3 (d)].

3. Compactness of attainable sets

LEMMA 1. *The attainable sets X_h^{At} and Y_i^{At} are* $\mathrm{w}(L, L')$*-compact, for each h and i. Equivalently, the set of all feasible allocations is weakly* compact.*

Proof. First consider the case of $X_h = L_+$ for each h. Then:

$$X_h^{\mathrm{At}} = L_+ \cap \Big(-L_+ + x^{\mathrm{En}} + \sum_i Y_i\Big),$$

$$Y_i^{\mathrm{At}} = Y_i \cap \Big(L_+ - x^{\mathrm{En}} - \sum_{i' :\, i' \neq i} Y_{i'}\Big).$$

So Y_i^{At} is norm-bounded, by the Boundedness Assumption with $B = \{x^{\mathrm{En}}\}$. Furthermore, note that

$$X_h^{\mathrm{At}} = L_+ \cap \Big(-L_+ + x^{\mathrm{En}} + \sum_i Y_i^{\mathrm{At}}\Big).$$

It follows that this set is also norm-bounded: use [19, 3.2.6 with 3.2.3]. (This applies because L is a normed lattice, so the cone L_+ is self-allied for the norm topology. The result is also given in [30, V.3.1: Corollary 2], where the property of L_+ is referred to as "normality".) It follows *a fortiori* that the attainable sets are also norm-bounded in the

[14] McKenzie [27] also shows how to weaken the adequacy assumption, in another respect, by using the concept of an irreducible economy.

general case of $X_h \subseteq L_+$ (since they can only be smaller than in the case of $X_h = L_+$). So X_h^{At} and Y_i^{At} are weakly* compact relatively to L (by the Banach-Alaoglu Theorem). That they are actually compact (or, equivalently, closed) can be shown in two ways: one consists in using Lemma 2 (below) to show that the sums of the weakly* closed sets in (1) and (2) are also closed. For an alternative proof, note that the set of all feasible allocations, $\mathfrak{A}$, is contained in the Cartesian product of X_h^{At} and Y_i^{At} (over all h's and i's); and so $\mathfrak{A}$ is weakly* compact relatively to $L^{\mathrm{Ho} \cup \mathrm{Pr}}$. Since $\mathfrak{A}$ is also weakly* closed in this space, it is weakly* compact. It follows that so are X_h^{At} and Y_i^{At}, since they are weakly* continuous images of $\mathfrak{A}$, viz., its coordinate projections. ∎

4. Total production set and its asymptotic cone. When the commodity space is finite-dimensional, the Boundedness Assumption is equivalent to positive semi-independence of the asymptotic cones of the production sets together with the cone $-L_+$, and it is well known to imply that the total production set is closed and, also, that the asymptotic cone operation is additive: see, e.g., [7, p. 23] and [29, 9.1.1]. Both results are next extended to the case of a dual Banach commodity space by using the Krein-Smulian Theorem. The closed-sum result (Lemma 2) is the more important of the two,[15] since the additivity result can be made superfluous by transforming the production sets into cones in the way described towards the end of Section 2.

LEMMA 2. *The set* $Y := \sum_i Y_i$ *is* w*-*closed.*

Proof. Take any bounded and w*-closed subset, B, of L. Since Y is convex, it suffices to show that $Y \cap B$ is w*-closed and apply the Krein-Smulian Theorem: see, e.g., [9, V.7.5] or [12, 18E]. For any net (y^{n}) in $Y \cap B$ convergent weakly* to some $y \in L$, decompose each of its terms into the sum $y^{\mathrm{n}} = \sum_i y_i^{\mathrm{n}}$ for some $y_i^{\mathrm{n}} \in Y_i$. By the Boundedness Assumption the net (y_i^{n}) is bounded; so one can assume that it converges weakly* to some y_i, for each i. (If not, replace it with a w*-convergent subnet, which exists by the Banach-Alaoglu Theorem.) Since Y_i is w*-closed, $y_i \in Y_i$. It follows that $y = \sum_i y_i \in \sum_i Y_i$. ∎

A vector $v \in L$ is called a *direction of recession* in a convex set $S \subseteq L$, at a point $s \in S$, if $s + \alpha v \in S$ for every $\alpha \in \mathbb{R}_+$. The *recession cone* $\operatorname{rec} S$ of S consists of all those directions of recession common to every point $s \in S$, i.e., $\operatorname{rec} S = \{v : v + S \subseteq S\}$. The *asymptotic cone* $\operatorname{as} S$ is the recession cone of the algebraic closure of S.[16] The distinction between $\operatorname{rec} S$ and $\operatorname{as} S$ disappears when S is closed for any vector topology $\mathcal{T}$ on L: the

[15] For Lemma 2, it suffices to assume that the set $Y_i \cap (-B - \sum_{i' : i' \neq i} Y_{i'})$ be norm-bounded (for every bounded B). So Lemma 2 extends, to the case of any (finite) number of subsets of a dual Banach space L, the w*-closedness result given in [23] for the sum of two sets. In the case of a Banach space, the equicontinuity condition of [23] is the same as the above one for two sets, and the hypercompleteness assumption holds by the Krein-Smulian Theorem. The criterion of [20, Proposition 5] for the closed sum of two cones is similar: "Property (G)" holds if the cones are allied; and alliedness can be shown to imply the above boundedness condition by using [19, 3.2.5].

[16] This is the same as the closure of S for $\mathcal{T}_{\mathrm{SLC}}$ or $\mathcal{T}_{\mathrm{SV}}$ if the core of S is nonempty: this follows from [12, 11A], given that $\operatorname{cor} S$ is the interior of S for $\mathcal{T}_{\mathrm{SLC}}$ (when S is convex).

directions of recession are then the same at every $s \in S$, i.e.,

$$\operatorname{as} S = \operatorname{rec} S = \bigcap_{\alpha > 0} \frac{1}{\alpha}(S - s) \quad \text{for each } s \in S.$$

It follows that $\operatorname{as} S$ is $\mathcal{T}$-closed and, also, that if $0 \in S$ then $\operatorname{as} S$ is the largest cone contained in S: see, e.g., [3, I.3.5], [6, p. 1909] or [12, (8.5)]. Furthermore, if (s^{n}) and (ϵ^{n}) are nets in S and $\mathbb{R}_+$ with $\epsilon^{\mathrm{n}} \to 0$ and $\epsilon^{\mathrm{n}} s^{\mathrm{n}} \to v$ for $\mathcal{T}$, then $v \in \operatorname{as} S$: see, e.g., [11, 1.1] or [12, 8C: Lemma (c)].

Corollary 3. *The cone* $\operatorname{as} Y := \operatorname{as}(\sum_i Y_i)$ *is* w*-*closed.*

Lemma 4. $\operatorname{as}(\sum_i Y_i) = \sum_i \operatorname{as} Y_i$.

Proof. Take any $v \in \operatorname{as}(\sum_i Y_i)$; this means that $nv \in \sum_i Y_i$ for each $n \in \mathbb{N}$ (since $0 \in Y_i$). So

$$v = \sum_i \frac{y_i^n}{n} \tag{7}$$

for some $y_i^n \in Y_i$. By using the Boundedness Assumption as in the proof of Lemma 2, the sequence (y_i^n/n) is shown to be bounded; so it can be assumed to converge weakly* to some v_i, for each i.[17] Since $1/n \to 0$ (and $y_i^n/n \to v_i$), $v_i \in \operatorname{as} Y_i$. And $v = \sum_i v_i$ by passage to the limit in (7) as $n \to \infty$. This shows that $\operatorname{as}(\sum_i Y_i) \subseteq \sum_i \operatorname{as} Y_i$; the reverse inclusion holds obviously. ∎

5. Norm-to-weak* continuity of truncated demand. The truncated consumption and production sets are defined as[18]

$$X_h^{\mathrm{Tr}} := (X_h^{\mathrm{At}} + \{x : \|x\| \le 1\}) \cap X_h \tag{8}$$

$$Y_i^{\mathrm{Tr}} := (Y_i^{\mathrm{At}} + \{y : \|y\| \le 1\}) \cap Y_i. \tag{9}$$

Since X_h^{At}, Y_i^{At} and the closed unit ball of L are all w*-compact, so are the sets X_h^{Tr} and Y_i^{Tr}. Also, by construction, X_h^{At} and Y_i^{At} are contained in the norm-interiors of X_h^{Tr} relative to X_h and of Y_i^{Tr} relative to Y_i. For completeness, the truncated supply and demand correspondences are next spelt out. At $p \in L^*$ the profit of producer i is

$$\Pi_i^{\mathrm{Tr}}(p) := \sup\{\langle p \,|\, y\rangle : y \in Y_i^{\mathrm{Tr}}\} \tag{10}$$

and his supply correspondence (the set of optimal input-output bundles) is

$$\hat{Y}_i^{\mathrm{Tr}}(p) := \{y \in Y_i^{\mathrm{Tr}} : \langle p \,|\, y\rangle = \Pi_i^{\mathrm{Tr}}(p)\}.$$

Household h's income and its budget set are (both at the maximum of its profit income)

$$\hat{M}_h^{\mathrm{Tr}}(p) := \langle p \,|\, x_h^{\mathrm{En}}\rangle + \sum_i \varsigma_{hi} \Pi_i^{\mathrm{Tr}}(p) \tag{11}$$

$$\hat{B}_h^{\mathrm{Tr}}(p) := \{x \in X_h^{\mathrm{Tr}} : \langle p \,|\, x\rangle \le \hat{M}_h^{\mathrm{Tr}}(p)\}.$$

[17] If it does not converge, replace it by a convergent subnet (which does exist, although a convergent subsequence need not exist unless L' is norm-separable).

[18] Our use of a single truncation, extending the technique of [7, pp. 87–88] to infinite-dimensional commodity spaces, simplifies the arguments of [10] and [31], which use a sequence (or a family) of truncations.

The household's demand is

$$\hat{X}_h^{\mathrm{Tr}}(p) := \{x \in \hat{B}_h^{\mathrm{Tr}}(p) : \forall x' \in \hat{B}_h^{\mathrm{Tr}}(p)\ x' \preccurlyeq_h x\}$$

and so the (truncated) *excess demand* correspondence is

$$\hat{E}^{\mathrm{Tr}}(p) := \sum_h (\hat{X}_h^{\mathrm{Tr}}(p) - x_h^{\mathrm{En}}) - \sum_i \hat{Y}_i^{\mathrm{Tr}}(p).$$

Note that $\hat{E}^{\mathrm{Tr}}(p)$ can be empty at some $p \in L^* \setminus L'$: see Example 9. However, $\hat{E}^{\mathrm{Tr}}$ is effectively defined on P', i.e., $\hat{E}^{\mathrm{Tr}}(p) \neq \emptyset$ for $p \in P'$: this is part of Theorem 5 below.

Recall that the *polar cone* of $\operatorname{as} Y$ is

$$(\operatorname{as} Y)^\circ = \{p \in L^* : \forall y \in \operatorname{as} Y\ \langle p \,|\, y\rangle \leq 0\}$$

and denote for brevity

$$P^* := (\operatorname{as} Y)^\circ \setminus \{0\},$$
$$P' := P^* \cap L' = ((\operatorname{as} Y)^\circ \cap L') \setminus \{0\}.$$

Comment: By definition, A° is the *algebraic* polar of a cone $A \subset L$, i.e., A° consists of all the linear functionals that are nonnegative on A. However, $A^\circ \subset L^*$ if A has a nonempty norm-interior (as is the case with $\operatorname{as} Y$ here). Also, $A^\circ \neq \{0\}$ by a separation argument if: (i) $A \neq L$, (ii) A is convex, and (iii) either $\operatorname{cor} A \neq \emptyset$ or A is $\mathcal{T}_{\mathrm{SLC}}$-closed (or both, as is the case here).

For clarity, note the distinction between hemicontinuity (of a correspondence) and semicontinuity (of an order or a real-valued function). This is by now standard in mathematical economics, but usage of these terms has varied, and in [25] "semicontinuity" means what we mean by hemicontinuity.

THEOREM 5. *The truncated excess demand, $p \mapsto \hat{E}^{\mathrm{Tr}}(p)$, is a norm-to-weak* upper hemicontinuous correspondence from P' into L, with nonempty, convex and weakly* compact values.*

Proof. Except where other topologies are specified, in this proof the space L' is topologized by its norm $\|\cdot\|'$, and L by w^*. The real line $\mathbb{R}$ carries its usual topology. Since Y_i^{Tr} is w^*-compact (and since the norm topology of L' is the topology of uniform convergence on w^*-compact subsets of L), the duality form $(y, p) \mapsto \langle p \,|\, y\rangle$ is (jointly) continuous on $Y_i^{\mathrm{Tr}} \times L'$ (for $\|\cdot\|' \times \mathrm{w}^*$). An application of Berge's Maximum Theorem [4, p. 115] shows that $\hat{Y}_i^{\mathrm{Tr}}\colon P' \twoheadrightarrow Y_i^{\mathrm{Tr}}$ is norm-to-w^* u.h.c. (with nonempty, convex and compact values), and that $\Pi_i^{\mathrm{Tr}}\colon P' \to \mathbb{R}$ is norm-continuous.

To prove that $\hat{X}_h^{\mathrm{Tr}}$ is norm-to-w^* u.h.c., note first that the budget correspondence defined by

$$(p, M) \mapsto B_h(p, M) := \{x \in X_h^{\mathrm{Tr}} : \langle p \,|\, x\rangle \leq M\}$$

for $p \in L'$ and $M \in \mathbb{R}$ is u.h.c. (Since X_h^{Tr} is compact, this is equivalent to the closedness of the graph of B_h in $L' \times \mathbb{R} \times X_h^{\mathrm{Tr}}$—see, e.g., [25, 7.1.16]—and this holds because the duality form is continuous on $X_h^{\mathrm{Tr}} \times L'$.)

Next, note that the "strict" budget correspondence defined by

$$B_h^{\mathrm{S}}(p, M) := \{x \in X_h^{\mathrm{Tr}} : \langle p \,|\, x\rangle < M\}$$

is $\mathrm{w}(L', L)$-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c. What is more, it is $\mathrm{w}(L^*, L)$-l.h.c. on L^* with any topology whatsoever on X_h^{Tr}, since it has $\mathrm{w}(L^*, L)$-open sections (i.e., $\{(p, M) : x \in B_h^{\mathrm{S}}(p, M)\}$ is an open set).

It follows that B_h is $\mathrm{w}(L^*, L)$-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c. at every point $(p, M) \in L^* \times \mathbb{R}$ with $B_h^{\mathrm{S}}(p, M) \neq \emptyset$. To see this, take any $x' \in B_h^{\mathrm{S}}(p, M)$; then, as $\epsilon \to 0+$, the sequence $x^\epsilon := \epsilon x' + (1-\epsilon)x$ converges to x for $\mathcal{T}_{\mathrm{Fin}}$; and this shows that the $\mathcal{T}_{\mathrm{Fin}}$-closure of $B_h^{\mathrm{S}}(p, M)$ contains $B_h(p, M)$. Since $B_h(p, M)$ is w^*-closed, it equals the closure of $B_h^{\mathrm{S}}(p, M)$ for any topology between w^* and $\mathcal{T}_{\mathrm{Fin}}$. To complete the proof that B_h is l.h.c. ($\mathrm{w}(L^*, L)$-to-$\mathcal{T}_{\mathrm{Fin}}$, at every (p, M) with $B_h^{\mathrm{S}}(p, M) \neq \emptyset$), recall that the correspondence whose values are the closures of an l.h.c. correspondence is also l.h.c.: see, e.g., [25, 7.3.3].

Since $\hat{M}_h^{\mathrm{Tr}}(p)$ is a norm-continuous function of $p \in L'$ (because Π_i^{Tr} is), it follows that the composition

$$p \mapsto B_h(p, \hat{M}_h^{\mathrm{Tr}}(p)) =: \hat{B}_h^{\mathrm{Tr}}(p)$$

is $\|\cdot\|'$-to-w^* u.h.c. on P'. To prove that it is $\mathrm{w}(L^*, L)$-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c. on P^*, use the Adequacy Assumption to select any $x_h^{\mathrm{S}} \in X_h$ and $y_h^{\mathrm{S}} \in \operatorname{cor} \operatorname{as} Y$ with $x_h^{\mathrm{S}} = x_h^{\mathrm{En}} + y_h^{\mathrm{S}}$. Note that $x_h^{\mathrm{S}} \in X_h^{\mathrm{At}} \subseteq X_h^{\mathrm{Tr}}$ and that

$$\langle p \,|\, x_h^{\mathrm{S}} \rangle < \langle p \,|\, x_h^{\mathrm{En}} \rangle \leq \hat{M}_h^{\mathrm{Tr}}(p) \tag{12}$$

for every $p \in P^*$; so $x_h^{\mathrm{S}} \in B_h^{\mathrm{S}}(p, \hat{M}_h^{\mathrm{Tr}}(p)) \neq \emptyset$. Given the l.h.c. result for B_h, it follows that $\hat{B}_h^{\mathrm{Tr}}$ is $\mathrm{w}(L^*, L)$-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c. on P^*. *A fortiori*, it is $\|\cdot\|'$-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c. on P'.

The strict inequality of (12) is given in, e.g., [10, Proposition 2], but it is also proved here for completeness: when $A \subset L$ is a cone and $p \in A^\circ \setminus \{0\}$, choose any $v \in L$ with $\langle p \,|\, v \rangle \neq 0$. If $y^{\mathrm{S}} \in \operatorname{cor} A$, then $y^{\mathrm{S}} + \epsilon v \in A$ and $y^{\mathrm{S}} - \epsilon v \in A$ for some $\epsilon > 0$. Therefore $\langle p \,|\, y^{\mathrm{S}} \pm \epsilon v \rangle \leq 0$, and so $\langle p \,|\, y^{\mathrm{S}} \rangle \leq -\epsilon \,|\langle p \,|\, v \rangle| < 0$, as required.

Given the hemicontinuity properties of $\hat{B}_h^{\mathrm{Tr}}$, a two-topology version of Berge's Maximum Theorem [18, Corollary 2.6] shows that $\hat{X}_h^{\mathrm{Tr}}$ is $\|\cdot\|'$-to-w^* u.h.c. with nonempty and compact values. (In this application, the action set is X_h^{Tr}, ordered by $\preccurlyeq_h$ and twice topologized by $\mathcal{T}_{\mathrm{Fin}}$ and w^*, whilst the constraint correspondence is $\hat{B}_h^{\mathrm{Tr}}$ restricted to the parameter space P', topologized by $\|\cdot\|'$.) It follows that $\hat{E}^{\mathrm{Tr}}$ is also u.h.c. (being the sum of compact-valued u.h.c. terms): see, e.g., [25, 7.3.15]. ■

6. Equilibrium existence by direct excess-demand approach. In this section, we prove the existence of an equilibrium (with a price system in L^*) by using demand continuity and Florenzano's [10] successful extension of the Debreu-Gale-Nikaido Lemma (quoted here in the Appendix), which applies to a demand map defined just on the predual price space L', provided that it is norm-to-weak* continuous (or even just $\mathcal{T}_{\mathrm{Fin}}(L')$-to-weak* continuous). It therefore applies to the demand map derived from preference maximization: if the price system belongs to L', then the budget set is w^*-compact once the consumption set has been truncated to make it bounded. So the demand derived from w^*-u.s.c. preferences is defined effectively on L'.[19]

[19]The literature on this topic also contains several other extensions of the Debreu-Gale-Nikaido Lemma that do *not* apply to the demand map derived from the optimizing behaviour. This is because those extensions impose one or both of the following conditions: (i) that the

The idea of working with a demand map defined on the intersection of L'_+ with a weakly* compact base of the cone L^*_+ is contained in the setup of Aliprantis and Brown [1, p. 195] because their Density Condition holds for any Banach lattice L with a predual L'. However, their analysis takes the demand map as a primitive concept for the most part, and they themselves point out [1, p. 204] that their Continuity Condition fails for the derived demand (in Bewley's model). In other words, in contrast to the norm-to-w* continuity established here (Theorem 5), consumer demand can be $\mathrm{w}(L', L)$-to-w* discontinuous on P', as is also shown by Example 10 below. And this is because—unlike the norm topology we use—the weak topology of the price space is too weak for the purpose: the budget correspondence is not closed for $\mathrm{w}(L^*, L) \times \mathrm{w}(L, L')$. Because of the discontinuity, Aliprantis and Brown [1, Example 4.8] resort to using finite-dimensional price simplices in the case of $L = L^\infty$. Their arguments are developed by Florenzano [10, Lemma 1 and Proof of Proposition 3], who states that derived demand is upper hemicontinuous for the finite topology $\mathcal{T}_{\mathrm{Fin}}(L')$ on the price space (with w* on L).[20] She also extends the Debreu-Gale-Nikaido Lemma in a compatible way, i.e., with the finite topology on L'. This gives a foundation for the direct approach using the demand map. However, the extreme strength of the finite topology—which is strictly stronger than every vector topology, unless $\dim L'$ is finite—weakens her continuity result, and keeps her analysis close to the finite-dimensional approximation approach.

Definition 6. A *competitive equilibrium* consists of a price system, $p^\star \in L^*$, and an allocation, $x^\star_h \in X_h$ and $y^\star_i \in Y_i$ for each household h and producer i, that meet the conditions:

1. $\sum_h (x^\star_h - x^{\mathrm{En}}_h) = \sum_i y^\star_i$.
2. $\langle p^\star \mid y^\star_i \rangle = \sup_y \{\langle p^\star \mid y\rangle : y \in Y_i\} =: \Pi_i(p^\star)$.
3. $\langle p^\star \mid x^\star_h \rangle = \langle p^\star \mid x^{\mathrm{En}}_h + \sum_i \varsigma_{hi} y^\star_i \rangle =: \hat{M}_h(p^\star)$.
4. For every $x \in X_h$, if $\langle p^\star \mid x\rangle \le \langle p^\star \mid x^\star_h\rangle$ then $x \preccurlyeq_h x^\star_h$.

Once demand continuity has been established, the main technical difficulty in using it for a direct proof of equilibrium existence is that the duality form is *not jointly* continuous for the two weak* topologies—viz., $\mathrm{w}(L^*, L)$ and $\mathrm{w}(L, L')$—that have to be put, for the fixed-point argument, on the price set Δ^* and on a consumption set X^{Tr} (or a production set Y^{Tr}). This is why even Florenzano's version of the Debreu-Gale-Nikaido Lemma cannot yield equilibrium existence without additional arguments. These are made simpler and more transparent by using a two-topology variant of Berge's Maximum Theorem that applies even to a non-closed constraint correspondence (the budget here). This is set out next, with X^{Tr}_h abbreviated to X^{Tr}, etc. (since h is fixed here).

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domain of definition for the demand map be the norm-dual L^* of the commodity space, and/or (ii) that the demand map be $\mathrm{w}(P, L)$-to-w* continuous, where the price space P is either L^* or L'. Neither condition is met by the derived demand: see Examples 9 and 10. With regard to the demand's domain, a price system that belongs to L^* but not to L' can make the (truncated) budget set w*-noncompact—with the result that there may be no optimum for a consumer with w*-u.s.c. preferences (Example 9).

[20] The proof in [10, p. 216] contains a gap which can be filled by using the two-topology version of Berge's Maximum Theorem.

LEMMA 7. *Assume that* (p, x) *is in the* $\mathrm{w}(L^*, L) \times \mathrm{w}(L, L')$*-closure of the graph* $\mathrm{gr}\,\hat{X}^{\mathrm{Tr}}$ *in* $P^* \times X^{\mathrm{Tr}}$*, and that* $x \in X^{\mathrm{At}}$*. Then:*

1. $\langle p \,|\, x \rangle \geq \hat{M}^{\mathrm{Tr}}(p)$.
2. $x \in \hat{X}^{\mathrm{Tr}}(p)$ *if (and only if)* $\langle p \,|\, x \rangle = \hat{M}^{\mathrm{Tr}}(p)$.

Proof. Since $x \in X^{\mathrm{At}}$, there is an $x' \in X$ with $x' \succ x$ (by Nonsatiation). Define $x^\epsilon := \epsilon x' + (1-\epsilon)x$. Then $x^\epsilon \in X^{\mathrm{Tr}}$ for small enough $\epsilon > 0$, since the (norm) interior of X^{Tr} relative to X contains X^{At} by construction (8). Also, $x^\epsilon \succ x$ by Preference Convexity.

By assumption, there is a net $(p^{\mathfrak{n}}, x^{\mathfrak{n}})_{\mathfrak{n} \in \mathfrak{N}}$ in $\mathrm{gr}\,\hat{X}^{\mathrm{Tr}}$ with $p^{\mathfrak{n}} \to p$ for $\mathrm{w}(L^*, L)$ and $x^{\mathfrak{n}} \to x$ for $\mathrm{w}^* := \mathrm{w}(L, L')$. By the weak* u.s. continuity of preferences, $x^\epsilon \succ x^{\mathfrak{n}}$ for every $\mathfrak{n}$ far enough in the directed set $\mathfrak{N}$ (i.e., from some $\mathfrak{n}'$ on). So

$$\langle p^{\mathfrak{n}} \,|\, x^\epsilon \rangle > \hat{M}^{\mathrm{Tr}}(p^{\mathfrak{n}}). \tag{13}$$

Furthermore, $\hat{M}^{\mathrm{Tr}}$ is an $\mathrm{w}(L^*, L)$-l.s.c. function on L^*, since each Π_i^{Tr} is by definition the supremum (10) of a family of $\mathrm{w}(L^*, L)$-continuous functions. Therefore (13) implies, by passage to the limit in $\mathfrak{n}$, that

$$\langle p \,|\, x^\epsilon \rangle \geq \hat{M}^{\mathrm{Tr}}(p).$$

By passage to the limit as $\epsilon \to 0+$, this gives that $\langle p \,|\, x \rangle \geq \hat{M}^{\mathrm{Tr}}(p)$, as is required for Part 1.

Part 2 follows directly from an application of another two-topology version of Berge's Maximum Theorem [18, Theorem 2.1], given that $\hat{B}_h^{\mathrm{Tr}}$ is $\mathrm{w}(L^*, L)$-to-$\mathcal{T}_{\mathrm{Fin}}$ l.h.c. on P^* (as is shown in the proof of Theorem 5). In this case—as distinct from the Proof of Theorem 5—the parameter space is P^* topologized by $\mathrm{w}(L^*, L)$, and this is taken as the domain of the constraint correspondence $\hat{B}_h^{\mathrm{Tr}}$. The action set is again X^{Tr}, ordered by $\preccurlyeq_h$ and topologized by $\mathcal{T}_{\mathrm{Fin}}$ and w^* as before. ∎

THEOREM 8. *On the assumptions of Section 2, a competitive equilibrium with a price system* $p^\star \in L^*$ *exists.*

Proof. Fix any $y^{\mathrm{S}} \in \mathrm{cor}\,\mathrm{as}\,Y = \mathrm{int}_{L, \|\cdot\|}\,\mathrm{as}\,Y$, and define

$$\Delta^* := \{p \in (\mathrm{as}\,Y)^\circ : \langle p \,|\, y^{\mathrm{S}} \rangle = -1\}. \tag{14}$$

This is a convex and $\mathrm{w}(L^*, L)$-compact base for the cone $(\mathrm{as}\,Y)^\circ$: see, e.g., [19, 3.8.6] or [10, Proposition 2]. Set

$$\Delta' := \Delta^* \cap L'.$$

By Theorem 5, $\hat{E}^{\mathrm{Tr}}$ is a $\|\cdot\|'$-to-w^* u.h.c. correspondence from Δ' into the w^*-compact set $\sum_h (X_h^{\mathrm{Tr}} - x_h^{\mathrm{En}}) - \sum_i Y_i^{\mathrm{Tr}}$. For every $p \in \Delta'$, the set $\hat{E}^{\mathrm{Tr}}(p)$ is w^*-closed, convex and nonempty; also, $\langle p \,|\, e \rangle \leq 0$ for every $e \in \hat{E}^{\mathrm{Tr}}(p)$. Furthermore, $\mathrm{as}\,Y$ is w^*-closed by Corollary 3. Therefore, an application of Florenzano's [10, Lemma 1] extension of the Debreu-Gale-Nikaido Lemma[21] shows that, on some directed set $\mathfrak{N}$, there exist two nets, $(p^{\mathfrak{n}})_{\mathfrak{n} \in \mathfrak{N}}$ in Δ' and $(e^{\mathfrak{n}})_{\mathfrak{n} \in \mathfrak{N}}$ with $e^{\mathfrak{n}} \in \hat{E}^{\mathrm{Tr}}(p^{\mathfrak{n}})$, that converge weakly* to some $p^\star \in \Delta^*$ and $v^\star \in \mathrm{as}\,Y$, i.e., $p^{\mathfrak{n}} \to p^\star$ for $\mathrm{w}(L^*, L)$ and $e^{\mathfrak{n}} \to v^\star$ for $\mathrm{w}(L, L')$. (Note that $p^\star$ need

[21] Since the extension applies to an excess demand that is merely $\mathcal{T}_{\mathrm{Fin}}(L')$-to-$\mathrm{w}^*$ u.h.c., it applies *a fortiori* to a demand that is norm-to-weak* u.h.c.

not belong to L', so at this stage it is not clear that $\hat{E}^{\mathrm{Tr}}(p^\star) \neq \emptyset$: even this part of the equilibrium result is yet to be established.)

By Lemma 4,

$$v^\star = \sum_i v_i^\star \tag{15}$$

for some $v_i^\star \in \operatorname{as} Y_i$. Also, for every $\mathfrak{n}$, the excess demand at $p^{\mathfrak{n}}$ can be decomposed into the sum

$$e^{\mathfrak{n}} = \sum_h (x_h^{\mathfrak{n}} - x_h^{\mathrm{En}}) - \sum_i y_i^{\mathfrak{n}} \tag{16}$$

for some $x_h^{\mathfrak{n}} \in \hat{X}_h^{\mathrm{Tr}}(p^{\mathfrak{n}})$ and $y_i^{\mathfrak{n}} \in \hat{Y}_h^{\mathrm{Tr}}(p^{\mathfrak{n}})$. Since X_h^{Tr} and Y_i^{Tr} are w*-compact, it can be assumed (by passage to subnets if necessary) that the nets $(x_h^{\mathfrak{n}})$ and $(y_i^{\mathfrak{n}})$ converge weakly* to some $x_h^\star \in X_h^{\mathrm{Tr}}$ and $y_i^\star \in Y_i^{\mathrm{Tr}}$ with

$$\sum_i v_i^\star = \sum_h (x_h^\star - x_h^{\mathrm{En}}) - \sum_i y_i^\star \tag{17}$$

from (16) and (15). It remains to show that $p^\star$ supports the allocation $(y_i^\star + v_i^\star)_{i\in \mathrm{Pr}}$ and $(x_h^\star)_{h\in \mathrm{Ho}}$ as an equilibrium.

Since $x_h^\star \in X_h^{\mathrm{At}}$ by (17), Part 1 of Lemma 7 gives that

$$\langle p^\star \,|\, x_h^\star \rangle \geq \hat{M}_h^{\mathrm{Tr}}(p^\star)$$

and summation over h gives, with the definitions (11) and (10), that

$$\sum_h \langle p^\star \,|\, x_h^\star - x_h^{\mathrm{En}} \rangle \geq \sum_i \Pi_i^{\mathrm{Tr}}(p^\star) \geq \sum_i \langle p^\star \,|\, y_i^\star \rangle. \tag{18}$$

On the other hand, $\langle p^\star \,|\, v_i^\star \rangle \leq 0$ for each i because $v_i^\star \in \operatorname{as} Y_i$ and $p^\star \in (\operatorname{as} Y)^\circ = (\sum_i \operatorname{as} Y_i)^\circ = \bigcap_i (\operatorname{as} Y_i)^\circ$. So

$$\sum_i \langle p^\star \,|\, y_i^\star \rangle \geq \sum_i \langle p^\star \,|\, v_i^\star + y_i^\star \rangle = \sum_h \langle p^\star \,|\, x_h^\star - x_h^{\mathrm{En}} \rangle \tag{19}$$

where the equality follows from (17). Therefore (18) and (19) actually hold as equalities, and so do all the inequalities which have added up to (18) and (19). That is, for each h and i,

$$\langle p^\star \,|\, x_h^\star \rangle = \hat{M}_h^{\mathrm{Tr}}(p^\star), \tag{20}$$

$$\Pi_i^{\mathrm{Tr}}(p^\star) = \langle p^\star \,|\, y_i^\star \rangle, \tag{21}$$

$$\langle p^\star \,|\, v_i^\star \rangle = 0. \tag{22}$$

What (21) means is that $\langle p^\star \,|\, y_i^\star \rangle \geq \langle p^\star \,|\, y \rangle$ for every $y \in Y_i^{\mathrm{Tr}}$. To show that this holds also for every $y \in Y_i$, introduce $y^\epsilon := \epsilon y + (1-\epsilon) y_i^\star$; then $y^\epsilon \in Y_i^{\mathrm{Tr}}$ for small enough $\epsilon > 0$ (since $y_i^\star \in Y_i^{\mathrm{At}}$, which lies in the norm-interior of Y_i^{Tr} relative to Y_i, by (9)). Therefore $\langle p^\star \,|\, y_i^\star \rangle \geq \langle p^\star \,|\, y^\epsilon \rangle$; substitute for y^ϵ, cancel out the terms with the coefficient $1-\epsilon$ and divide by ϵ. This shows that $y_i^\star$ maximizes profit (at $p^\star$, on Y_i); and so does $y_i^\star + v_i^\star$ in view of (22).

It remains only to verify the preference maximization condition of Definition 6. Given (20), Part 2 of Lemma 7 shows that $x_h^\star \in \hat{X}_h^{\mathrm{Tr}}(p^\star)$, i.e., that for $x \in X_h^{\mathrm{Tr}}$

$$\langle p^\star \,|\, x \rangle \leq \langle p^\star \,|\, x_h^\star \rangle \Rightarrow x \preccurlyeq_h x_h^\star.$$

To show that this holds also for every $x \in X_h$, introduce $x^\epsilon := \epsilon x + (1-\epsilon)x_h^\star$. Suppose that $x \succ_h x_h^\star$; then also $x^\epsilon \succ_h x_h^\star$ for $\epsilon \in (0,1]$ by Preference Convexity. Also, $x^\epsilon \in X_h^{\mathrm{Tr}}$ for small enough $\epsilon > 0$ (since $x_h^\star \in X_h^{\mathrm{At}}$, which lies in the norm-interior of X_h^{Tr} relative to X_h, by (8)). So $\langle p^\star \,|\, x^\epsilon \rangle > \langle p^\star \,|\, x_h^\star \rangle$; substitute for x^ϵ, cancel out the terms with the coefficient $1-\epsilon$ and divide by ϵ to obtain that $\langle p^\star \,|\, x \rangle > \langle p^\star \,|\, x_h^\star \rangle$, as required. ∎

Comments:

- As has been pointed out, the duality form is *not jointly* continuous for the two weak* topologies—viz., $\mathrm{w}(L^*, L)$ and $\mathrm{w}(L, L')$—for which $p^{\mathfrak{n}}$ and $(y_i^{\mathfrak{n}}, x_h^{\mathfrak{n}})$ converge. (It is neither u.s.c. nor l.s.c.) This is why, although $y_i^{\mathfrak{n}}$ maximizes profit on Y_i^{Tr} at $p^{\mathfrak{n}}$, the same property for their limits $y_i^\star$ and $p^\star$ does *not* follow by continuity. Similarly (20) does *not* follow directly from the corresponding property of $(p^{\mathfrak{n}}, x_h^{\mathfrak{n}})$; another obstacle here is that $\hat{M}_h^{\mathrm{Tr}}$ is only l.s.c. for the weak* topology of L^*. (For the norm of L', it is continuous.) In other words, the topologies that must be put on the price set and the consumption set for the fixed-point argument are too weak to make the budget constraint closed.
- The equilibrium price system $p^\star \in \Delta^*$ is obtained in the proof of Theorem 8 as the limit of a net of price systems $(p^{\mathfrak{n}})$ in Δ'. Such an approach is implicitly based on the weak* denseness of Δ' in Δ^*, which indeed follows from the w*-closedness of Y and hence of $\mathrm{as}\, Y$. In precise terms, if $y^{\mathrm{S}} \in A \subset L$, A is a w*-closed cone (with the algebraic polar A°), and

$$\Delta^* = \{p \in A^\circ \cap L^* : \langle p \,|\, y^{\mathrm{S}} \rangle = -1\} \quad \text{and} \quad \Delta' = \Delta^* \cap L'$$

  then Δ' is $\mathrm{w}(L^*, L)$-dense in Δ^*.[22] This excludes, e.g., the case of a Y equal to the half-space with a normal vector $p \in L^* \setminus L'$ (so that $\Delta' = \emptyset$).

  Proof. It is shown first that $A^\circ \cap L'$ is dense in $A^\circ \cap L^*$. Suppose it is not. Then a point $p^0 \in A^\circ \cap L^*$ can be strictly separated from $A^\circ \cap L'$ by a $z^0 \in L$, i.e., $\langle p^0 \,|\, z^0 \rangle > \sup\{\langle p \,|\, z^0 \rangle : p \in A^\circ \cap L'\}$: see, e.g., [12, 11.F: Corollary] or [30, II.9.2]. Since $A^\circ \cap L'$ is a cone, it follows that the supremum equals zero, and so

$$\langle p^0 \,|\, z^0 \rangle > 0 \geq \langle p \,|\, z^0 \rangle \quad \text{for every } p \in A^\circ \cap L'. \tag{23}$$

  It only remains to deduce from the right-hand inequality that $z^0 \in A$: given that $p^0 \in A^\circ$, this will contradict the left-hand inequality. So suppose that $z^0 \notin A$. Since A is w*-closed, another separation argument shows that there exist a $p \in L'$ with $\langle p \,|\, z^0 \rangle > \sup\{\langle p \,|\, y \rangle : y \in A\}$. Since A is a cone, this implies that $\langle p \,|\, z^0 \rangle > 0 \geq \langle p \,|\, y \rangle$ for each $y \in A$, and so $p \in A^\circ \cap L'$. This contradicts the right-hand inequality of (23), thus completing the proof that $A^\circ \cap L'$ is $\mathrm{w}(L^*, L)$-dense in $A^\circ \cap L^*$. Therefore, for each $p \in \Delta^*$ there exists a net $(p^{\mathfrak{n}})_{\mathfrak{n} \in \mathfrak{N}}$ in Δ' with $p^{\mathfrak{n}} \to p$ for $\mathrm{w}(L^*, L)$. In particular $\langle p^{\mathfrak{n}} \,|\, y^{\mathrm{S}} \rangle \to \langle p \,|\, y^{\mathrm{S}} \rangle = -1$, and so $(1/\langle p^{\mathfrak{n}} \,|\, y^{\mathrm{S}} \rangle) p^{\mathfrak{n}}$ is a net in Δ' that converges weakly* to p. ∎

[22] The same holds with L^* replaced by the algebraic dual of L, though this adds nothing when $A^\circ \subset L^*$ (as is the case for $A = \mathrm{as}\, Y$ here).

7. Counterexamples. The following are counterexamples to weak-to-weak* continuity of consumer demand, and to its very existence on $\Delta^* \setminus \Delta'$.[23] In both examples, there is one differentiated good in addition to a homogeneous numeraire commodity, and $(p, 1)$ and (x, m) play the roles of the p and x of the "abstract" model. So the commodity space is $L = L^\infty[0, T] \times \mathbb{R}$ with $L' = L^1[0, T] \times \mathbb{R}$. The consumer's income comes wholly from an initial endowment m^{En} of the numeraire. The consumption set is taken to be $L^\infty_+ \times \mathbb{R}_+$, but it can be truncated to a w*-compact without changing the results. The utility function has the additively separable form with a constant marginal utility of the numeraire, i.e.,

$$U(x, m) := m + \int_0^T u(x(t))\,\mathrm{d}t$$

for $x \in L^\infty_+[0, T]$ and $m \in \mathbb{R}_+$, where u (known as the felicity function) is increasing and differentiable on $\mathbb{R}_+$, with $u(0) = 0$. For simplicity, to ensure that consumer demand is uniquely determined (i.e., is a single-valued map), assume also that u is strictly concave, i.e., that its derivative $\mathrm{d}u/\mathsf{dx}$ is a (strictly) decreasing, continuous function on $\mathbb{R}_+$. At sufficiently high income levels, this form of utility results in a cross-price independent demand for the differentiated good, with no income effect on it. Given a price function $p \in L^1$, the demand $\hat{x}(p)(t)$ can be determined from the marginal condition

$$\frac{\mathrm{d}u}{\mathsf{dx}}(\hat{x}(t)) = p(t) \tag{24}$$

at each $t \in [0, T]$, with

$$\hat{m} = m^{\mathrm{En}} - \int_0^T p(t)\hat{x}(t)\,\mathrm{d}t \tag{25}$$

as the demand for the numeraire.

Our first example shows that nonexistence of a consumer optimum can result from the presence of a nonzero purely finitely additive term in the Hewitt-Yosida decomposition of a $p \in L^{\infty*}[0, T]$. Recall that every such p can be identified with an additive set function (vanishing on Lebesgue-null sets) which has the decomposition $p = p_{\mathrm{CA}} + p_{\mathrm{FA}}$, where p_{CA} is the countably additive part (identified with its density by the Radon-Nikodym Theorem), whilst p_{FA} is the purely finitely additive (a.k.a. "singular") part: see [5] or [32] for details.

Example 9 (Nonexistence of consumer optimum when $p_{\mathrm{FA}} \neq 0$). Fix any number $\mathsf{x} > 0$, denote $\mathsf{p} := (\mathrm{d}u/\mathsf{dx})(\mathsf{x})$ for brevity, and consider the price system $(p, 1)$ with a constant $p_{\mathrm{CA}}(t) := \mathsf{p}$ for every t and with any nonzero $p_{\mathrm{FA}} \geq 0$ that is concentrated on $[t, T]$ for each $t < T$. Assume that $m^{\mathrm{En}} > T\mathsf{xp}$. If (x, m) is a consumer optimum at p, then it is also a consumer optimum at p_{CA}: see [15, Lemma 5]. At $p_{\mathrm{CA}} = \mathsf{p}1_{[0,T]}$, the demand is

$$\hat{x}(p_{\mathrm{CA}})(t) = \mathsf{x}$$

for (almost) every t, with

$$\hat{m}(p_{\mathrm{CA}}) = m^{\mathrm{En}} - T\mathsf{xp}.$$

[23] That is why the equilibrium existence proof uses a net of approximate equilibrium prices $p^{\mathrm{n}} \in \Delta'$.

At p, however, this bundle is *not* in the budget set because it costs

$$m^{\mathrm{En}} - T\mathsf{x}\mathsf{p} + \mathsf{x}\int_0^T p_{\mathrm{CA}}(t)\,\mathrm{d}t + \mathsf{x}p_{\mathrm{FA}}[0,T] = m^{\mathrm{En}} + \mathsf{x}\|p_{\mathrm{FA}}\|_\infty^* > m^{\mathrm{En}}.$$

This shows that there is *no* consumer optimum at p. Finally, note that, without changing the demand at p or p_{CA}, the consumption set can be made w*-compact by truncating it to

$$\{(x,m) \geq 0 : x \leq \mathsf{x} + 1_{[0,T]}, m \leq m^{\mathrm{En}} + 1\}.$$

Comment: A utility level arbitrarily close to that of $(\hat{x},\hat{m})(p_{\mathrm{CA}})$, in Example 9, *can* be attained within the budget constraint at p: take a sequence $t^n \nearrow T$, and $x^n := \mathsf{x}' 1_{[0,t^n]}$ with $m^n := m^{\mathrm{En}} - t^n \mathsf{x}'\mathsf{p}'$. As $n \to \infty$,

$$U(x^n, m^n) \nearrow m^{\mathrm{En}} - T\mathsf{x}'\mathsf{p}' + Tu(\mathsf{x}') = U(\hat{x}(p_{\mathrm{CA}}), \hat{m}(p_{\mathrm{CA}})).$$

But the point is that this utility limit, the supremum of U on the budget set, is not attained. Since U is Mackey-continuous and hence w*-u.s.c.—see, e.g., [5, Appendix II] or [13, Section 3]—this shows that the budget set is not w*-compact. The example can be interpreted in the context of consumption over time: the consumer should "switch off" just before the extremely concentrated charge p_{FA} around T—and there is no best time to switch off: the closer to T, the better.

Our second example shows that consumer demand can be $\mathrm{w}(L^1, L^\infty)$-to-$\mathrm{w}(L^\infty, L^1)$ discontinuous.

Example 10 (Weak-to-weak* discontinuity of demand). Fix any constant $\mathsf{x}' > 0$, and denote $\mathsf{p}' := (\mathrm{d}u/\mathrm{d}\mathsf{x})(\mathsf{x}')$ for brevity. There is a number $\delta > 0$ with $\overline{\mathsf{p}} := \mathsf{p}' + \delta < (\mathrm{d}u/\mathrm{d}\mathsf{x})(0)$ and $\underline{\mathsf{p}} := \mathsf{p}' - \delta > \lim_{\mathsf{x}\to\infty}(\mathrm{d}u/\mathrm{d}\mathsf{x})(\mathsf{x})$. One can assume that $\delta = 1$. Use the Rademacher function sequence

$$r^n(t) := \operatorname{sgn}\sin(2^n \pi t)$$

to define a sequence of price systems $(p^n, 1) \in L^1 \times \mathbb{R}$ by

$$p^n(t) = \mathsf{p}' + r^n(t)$$

for every $t \in [0,T]$ and $n \in \mathbb{N}$. As $n \to \infty$, the p^n converges for $\mathrm{w}(L^1, L^\infty)$ to the constant p' (i.e., $r^n \to 0$ weakly). As in Example 9, m^{En} is assumed to be high enough for the demand, $\hat{x}(p^n)$ and $\hat{m}(p^n)$, to be determined by (24)–(25). Then $\hat{x}(p^n)$ converges for $\mathrm{w}(L^\infty, L^1)$ as $n \to \infty$ to the constant

$$\mathsf{x}'' := \frac{\underline{\mathsf{x}} + \overline{\mathsf{x}}}{2}$$

where

$$\underline{\mathsf{x}} := \left(\frac{\mathrm{d}u}{\mathrm{d}\mathsf{x}}\right)^{-1}(\overline{\mathsf{p}}) := \left(\frac{\mathrm{d}u}{\mathrm{d}\mathsf{x}}\right)^{-1}(\mathsf{p}' + 1),$$

$$\overline{\mathsf{x}} := \left(\frac{\mathrm{d}u}{\mathrm{d}\mathsf{x}}\right)^{-1}(\underline{\mathsf{p}}) := \left(\frac{\mathrm{d}u}{\mathrm{d}\mathsf{x}}\right)^{-1}(\mathsf{p}' - 1).$$

In general $\mathsf{x}'' \neq \mathsf{x}'$ (unless $\mathrm{d}u/\mathrm{d}\mathsf{x}$, the demand curve, is linear in the relevant region). For example, if $\mathrm{d}u/\mathrm{d}\mathsf{x}$ is strictly convex (and decreasing), then $\mathsf{x}'' > \mathsf{x}'$. In such a case, the demand for the differentiated good is weak-to-weak* discontinuous, since $\hat{x}(\mathsf{p}') = \mathsf{x}'$ but $\hat{x}(p^n) \to \mathsf{x}''$ (or, put formally, $\hat{x}(\mathsf{p}'1_{[0,T]}) = \mathsf{x}'1_{[0,T]}$ but $\hat{x}(p^n) \to \mathsf{x}''1_{[0,T]}$ as $n \to \infty$).

Comments:

- If $\mathsf{x}'' = \mathsf{x}'$ in Example 10, then it is the demand for the numeraire that is discontinuous. To see this, note first that in either case (whether x'' equals x' or not) the value of the limit bundle x'' at the limit price p' is greater than the value of $\hat{x}^n$ at p^n, which is actually independent of n. That is,
$$\mathsf{p}'\mathsf{x}'' = \frac{(\underline{\mathsf{p}} + \overline{\mathsf{p}})(\underline{\mathsf{x}} + \overline{\mathsf{x}})}{4} > \frac{\underline{\mathsf{p}}\overline{\mathsf{x}} + \overline{\mathsf{p}}\underline{\mathsf{x}}}{2} = \frac{1}{T}\int_0^T p^n(t)\hat{x}(p^n)(t)\,\mathrm{d}t. \tag{26}$$
This means that the limit bundle $(\mathsf{x}'', m^{\mathrm{En}} - T(\underline{\mathsf{p}}\overline{\mathsf{x}} + \overline{\mathsf{p}}\underline{\mathsf{x}})/2)$ is outside the budget set at the limit price $(\mathsf{p}', 1)$. (By Part 2 of Lemma 7, this must be the case if the demand map is to be discontinuous along a price sequence for which the demands converge.) When $\mathsf{x}' = \mathsf{x}''$, substitution for x'' in (26) gives that
$$m^{\mathrm{En}} - T\mathsf{p}'\mathsf{x}' < m^{\mathrm{En}} - \frac{T}{2}(\underline{\mathsf{p}}\overline{\mathsf{x}} + \overline{\mathsf{p}}\underline{\mathsf{x}}),$$
i.e., that the demand for the numeraire is less at p' than at p^n (at which it is the same for each n). So it is discontinuous.
- When demand is multi-valued (at some prices), its upper hemicontinuity established in Theorem 5 does not have the same implications as the ordinary continuity (of a single-valued map): for example, it is easy to exhibit a convergent sequence of price systems for which the demands do not converge. Like Examples 9 and 10, the following example uses a u independent of t, but additionally the price systems and the demand bundles are constant on $[0, T]$: essentially there are just two commodities. Take $\mathrm{d}u/\mathrm{d}\mathsf{x}$ to be (strictly) decreasing on $\mathbb{R}_+$ *except* for being constant on an interval $[\underline{\mathsf{x}}, \overline{\mathsf{x}}]$ with $\underline{\mathsf{x}} < \overline{\mathsf{x}}$. Take any two sequences $\underline{\mathsf{x}}^n \nearrow \underline{\mathsf{x}}$ with $\underline{\mathsf{x}}^n < \underline{\mathsf{x}}$ and $\overline{\mathsf{x}}^n \searrow \overline{\mathsf{x}}$ with $\overline{\mathsf{x}}^n > \overline{\mathsf{x}}$, and set $p^n := (\mathrm{d}u/\mathrm{d}\mathsf{x})(\underline{\mathsf{x}}^n)$ for odd n and $p^n := (\mathrm{d}u/\mathrm{d}\mathsf{x})(\overline{\mathsf{x}}^n)$ for even n. Then p^n (a sequence of constants) converges to $p := (\mathrm{d}u/\mathrm{d}\mathsf{x})(\underline{\mathsf{x}}) = (\mathrm{d}u/\mathrm{d}\mathsf{x})(\overline{\mathsf{x}})$, but the corresponding sequence of demands diverges (since it alternates between $\underline{\mathsf{x}}^n$ and $\overline{\mathsf{x}}^n$). This does not contradict Theorem 5, of course: at the limit p demand equals $[\underline{\mathsf{x}}, \overline{\mathsf{x}}]$, and it *is* a u.h.c. correspondence.

Appendix A. Florenzano's extension of the Debreu-Gale-Nikaido Lemma

LEMMA 11. *Let L be a linear space carrying a vector topology $\mathcal{T}$ and a locally convex topology $\mathcal{W}$ that is weaker than $\mathcal{T}$. Assume that $A \subset L$ is a convex cone with a point y^{S} in its $\mathcal{T}$-interior, so that the polar cone A°—which is a nonempty, proper subset of the $\mathcal{T}$-continuous dual space $(L, \mathcal{T})^*$—has a $\mathrm{w}((L,\mathcal{T})^*, L)$-compact base*
$$\Delta_{\mathcal{T}} := \{p \in A^\circ : \langle p \,|\, y^{\mathrm{S}} \rangle = -1\}.$$
Assume also that A is $\mathcal{W}$-closed, so that the convex set
$$\Delta_{\mathcal{W}} := \Delta_{\mathcal{T}} \cap (L, \mathcal{W})^*$$

is $\mathrm{w}((L,\mathcal{T})^*,L)$*-dense in* $\Delta_{\mathcal{T}}$.[24] *Furthermore, assume that* E *is a* $\mathcal{T}_{\mathrm{Fin}}((L,\mathcal{W})^*)$*-to-*$\mathcal{W}$ *upper hemicontinuous correspondence from* $\Delta_{\mathcal{W}}$ *into a* $\mathcal{W}$*-compact subset of* L*, with nonempty, convex and* $\mathcal{W}$*-closed values. If also* $\langle p \mid e\rangle \le 0$ *for every* $e \in E(p)$ *and* $p \in \Delta_{\mathcal{W}}$*, then* $\Delta_{\mathcal{T}} \times A$ *intersects the* $\mathrm{w}((L,\mathcal{T})^*,L) \times \mathcal{W}$*-closure, in* $\Delta_{\mathcal{T}} \times L$*, of the graph of* E*.*

Comment: In the proof of Theorem 8, Lemma 11 is applied with $A = \mathrm{as}\, Y$, $\mathcal{W}$ equal to $\mathrm{w}^* = \mathrm{w}(L, L')$ and $\mathcal{T}$ given by $\|\cdot\|$, so that $\Delta_{\mathcal{T}} = \Delta^* \subset L^*$ and $\Delta_{\mathcal{W}} = \Delta' \subset L'$.

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