EXISTENCE OF NASH EQUILIBRIA IN TWO-PERSON STOCHASTIC GAMES OF RESOURCE EXTRACTION

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Abstract. This paper deals with two-person stochastic games of resource extraction under both the discounted and the mean payoff criterion. Under some concavity and additivity assumptions concerning the payoff and the transition probability function a stationary Nash equilibrium is shown to exist. The proof is based on Schauder-Tychonoff's fixed point theorem, applied to a suitable payoff vector space.

1. Introduction. The games of resource extraction were introduced by Levhari and Mirman [14]. In such a dynamic game the players extract some amount of common renewable resource at each stage of the game. The amount of available resource in a next stage depends on the amount of the resource left by the players in the previous stage (investment). Extraction of small amount of the resource results in small payoffs of the players. Extraction of great amount of the resource reduces potential payoffs in the future.

In the models of Sundaram [25] or Majumdar and Sundaram [15] the strategies of the players were lower semicontinuous functions of the game state and utilities were the same for all the players. In those models the existence of Nash equilibria in pure stationary strategies was proved.

A similar model was considered by Amir [1]. He used the Topkis Theorem (see [27])—about monotonicity of a maximand of a submodular function on a lattice—and the Schauder fixed point theorem (see [23]).

The models of resource extraction games were also considered by Dutta and Sundaram [9], Cave [6], Fisher and Mirman [11], Mendelsohn and Sobel [16], Sobel [24] and Więcek [28].

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This paper also deals with such models. Unlike in the papers of Amir [1, 2], the state space of the game does not have to be unbounded from above. The main assumptions deal with the structure of transition probabilities (they have to be additive) and utility functions of the players. The existence of a stationary pure Nash equilibria in both cases of discounted and—under some additional assumptions—mean payoffs are proved.

Some models with similar transition probability structure were also considered by Nowak and Szajowski [22] and Szajowski [26]. In those papers it is assumed that there exists an absorbing state of total resource extraction (of resource level zero). This additional assumption allows for a construction of sequences converging to Nash equilibria. However, in the model from this paper it was not the case. Similar approach can be found in the paper of Nowak [20] and also in the one of Balbus and Nowak [3], in which the symmetric games were considered.

A survey on the Nash and correlated equilibria in stochastic games with infinite state space can be found in the paper of Nowak [21].

This paper is organized in the following way: in the first section we define the model, in the second one we formulate and prove the main results concerning the discounted game model, and in the third one, the results concerning the model with mean payoffs. In the last section and in the appendix the results needed for the proofs in the previous sections are discussed.

2. The model of the game—definitions, notation and assumptions. In this paper we deal with a model of a stochastic game of resource extraction:

DEFINITION 2.1. The 2-*person stochastic game of resource extraction* is defined as a 4-tuple $(S, D, \bar{u}, p)$ of the following objects:

(i) S is an interval in $(0, \infty)$ open from the left (called later *the set of all possible resource stocks* or the *state space*),

(ii) $D := \{(s, \bar{x}) : s \in S, \bar{x} = (x_1, x_2), x_i \in A_i(s) \quad \text{for} \quad i = 1, 2\}$, where $A_i(s) := [0, a_i(s)]$ represents *the set of admissible decisions* of player i in state $s \in S$. The functions $a_i : S \to R_+ := [0, +\infty)$ are Borel and satisfy

$$a_1(s) + a_2(s) < s - \inf S \quad \text{for each} \quad s \in S.$$

(iii) $\bar{u} = (u_1, u_2)$, where $u_i : R_+ \to R_+$ (for $i = 1, 2$) is a continuous utility function of player i,

(iv) p is a transition probability from the set D into S.

The game takes place in discrete time. If in some moment the state of the game is s, then player i chooses $x_i \in A_i(s) \subseteq R_+$ and obtains the payoff $u_i(x_i)$ and the state of the game changes according to the transition probability distribution $p(\cdot|s, \bar{x})$, where $\bar{x} = (x_1, x_2)$.

x_i may be interpreted as an amount of resource consumed by the player i in that moment. It must be nonnegative and is bounded from above by the quantity $a_i(s)$, which represents the maximal admissible level of consumption of player i in state s. If the game is in a state s, it reflects the situation that the amount s of the resource is present in the environment. The common level of players' consumption should not lead to overcon-

sumption, that is, the amount of the resource after players' consumption must be greater than some minimal level.

DEFINITION 2.2. Let $H^1 = S$ and for $n \geq 2$

$$H^n = D_1 \times \cdots \times D_{n-1} \times S,$$

where $D_k := D, k = 1, \cdots, n-1$. H^n is called the *space of histories* of the game *up to the n-th state.*

Let $H^\infty = D \times D \times \cdots$ This is called the *space of all histories* of the infinite horizon game.

REMARK 2.1. (a) The spaces H^n and H^∞ are endowed with the product σ-algebras.

(b) This paper deals with the models of investment/consumption in which randomized strategies do not have natural interpretation. Therefore we shall restrict attention to non-randomized strategies.

(c) The players have full information about the entire history of the game at any stage.

DEFINITION 2.3. A strategy of player i is defined as a sequence

$$\pi_i = (\pi_{i,1}, \pi_{i,2}, \cdots),$$

where $\pi_{i,n}$ is a Borel mapping from H^n into $[0, +\infty)$ such that for every

$$h^n = (s_1, x_{1,1}, x_{2,1}, s_2, \cdots, s_n) \in H^n,$$

we have

$$\pi_{i,n}(h^n) \in A_i(s_n).$$

The set of strategies of player i will be denoted by Π_i.

DEFINITION 2.4. A strategy $\pi_i = (\pi_{i,1}, \pi_{i,2}, \cdots)$ of player i such that $\pi_{i,n}$ depends on n and the state s_n (in the current moment n) only will be called *Markovian.*

DEFINITION 2.5. A Markovian strategy of player i of the form $\pi_i = (f, f, \cdots)$, where f associates with any $s \in S$ a point $f(s) \in A_i(s)$ will be called *stationary.*

The set of stationary strategies of player i will be denoted by F_i.

For every strategy profile $\bar{\pi} = (\pi_i)_{i=1}^2$ of both players and any initial state of the game $s_1 = s \in S$ the stochastic process $\{(s_n, \bar{x}_n)\}$ (where $\bar{x}_n = (x_{1,n}, x_{2,n})$) is defined on H^∞ in the canonical way (see Chapter 7 in [4]), where the random variables s_n and $\bar{x}_n$ are the state of the game and the profile of players' strategies on n-th step of the game, respectively.

REMARK 2.2. Existence of the stochastic process $\{(s_n, \bar{x}_n)\}$ and the unique probability measure on H^∞ generated by the initial state of the game s and the transition probabilities follow from the Ionescu-Tulcea Theorem (see Proposition V.1.1 in Neveu [19]).

Let us denote the above mentioned probability measure by $P_s^{\bar{\pi}}$ and the expected value operator associated with this measure by $E_s^{\bar{\pi}}$.

DEFINITION 2.6. Put $\Pi = \Pi_1 \times \Pi_2$, choose $\bar{\pi} \in \Pi$. Let β be a discount factor. Assuming that $\beta \in (0,1)$, define the β-*discounted expected payoff* of player i in the infinite horizon game as

$$\gamma_i(\bar{\pi})(s) := E_s^{\bar{\pi}} \Big(\sum_{n=1}^{\infty} \beta^{n-1} u_i(x_{i,n}) \Big),$$

where $E_s^{\bar{\pi}}$ is the expected value operator with respect to the probability measure $P_s^{\bar{\pi}}$ and $x_{i,n}$ is the i-th coordinate of the vector $\bar{x}_n$.

DEFINITION 2.7. The *expected mean payoff* of player i is defined as

$$\phi_i(\bar{\pi})(s) := \liminf_{m \to \infty} \frac{E_s^{\bar{\pi}}(\sum_{n=1}^{m} u_i(x_{i,n}))}{m}.$$

DEFINITION 2.8. A strategy profile $\bar{\pi}^* \in \Pi$ is called a *Nash equilibrium* in the discounted infinite horizon stochastic game iff

$$\gamma_1(\bar{\pi}^*)(s) \geq \gamma_1(\pi_1, \pi_2^*)(s),$$
$$\gamma_2(\bar{\pi}^*)(s) \geq \gamma_2(\pi_1^*, \pi_2)(s)$$

for all $\pi_i \in \Pi_i, s \in S$.

A Nash equilibrium in the infinite horizon stochastic game with the expected mean payoff functions is defined analogously.

In this paper we make the following additional assumptions:

ASSUMPTION 2.1. For every $(s, x_1, x_2) \in D$ the transition probability $p(\cdot|s, x_1, x_2)$ has the form

$$p(\cdot|s, x_1, x_2) = l(s - x_1 - x_2)H_1(\cdot|s) + (1 - l(s - x_1 - x_2))H_2(\cdot|s),$$

where

(i) $l : S \to [0,1]$ is increasing, concave and twice differentiable,

(ii) $H_1(\cdot|s), H_2(\cdot|s)$ are transition probabilities from S into S such that there is a probability measure μ on the set S, for which

$$H_i(\cdot|s) \ll \mu(\cdot), \quad i = 1, 2.$$

The term $l(s - x_1 - x_2)$ indicates (indirectly) how much of the resource is left after players' consumption in a previous stage. So, the influence of the transition probability H_1 on the probabiliy p increases with the common players' investment and the influence of H_2 increases with the consumption.

Additionally we assume that the utility functions satisfy:

ASSUMPTION 2.2. u_i are increasing, nonnegative, bounded, twice differentiable (at zero— from the right-hand side) and $u_i''(x) < 0$ for each $x \in (0, +\infty)$.

By $b > 0$ we shall denote a common upper bound of the functions u_i, $i = 1, 2$.

3. Nash equilibria in discounted stochastic games

THEOREM 3.1. *Suppose that Assumptions 2.1 and 2.2 hold in the considered model of the stochastic game with the β-discounted expected payoff. Then there exists a stationary Nash equilibrium (f_1^*, f_2^*) in this game.*

For the proof two additional definitions and a lemma are needed. First, consider an auxiliary game:

DEFINITION 3.1. Let $\Gamma(v_1, v_2, s)$ be a one-shot game, where the payoff function of player i is:

$$r_{v_i}^i(s, x_1, x_2) := (1 - \beta)u_i(x_i) + \beta \int_S v_i(s')p(ds'|s, x_1, x_2),$$

where

$$(v_1, v_2) \in B := B_1 \times B_2,$$
$$B_i := \{v : S \to R_+ : ||v||_\infty \le b\},$$

and

$$||v||_\infty := \underset{s \in S}{\mathrm{esssup}} |v(s)|.$$

It is obvious that $B_i \subseteq L^\infty(\mu)$. Endow $L^\infty(\mu)$ with the weak*-topology. It is well known that a sequence $\{f_n\} \subset L^\infty(\mu)$ converges to $f \in L^\infty(\mu)$ in weak*-topology iff for all $g \in L^1(\mu)$

$$\lim_{n \to \infty} \int_S f_n(s)g(s)\mu(ds) = \int_S f(s)g(s)\mu(ds).$$

REMARK 3.1. It is obvious that B is convex. From the Banach-Alaoglu Theorem (see [8]), it follows that B_i is compact in the relative weak*-topology. Hence $B = B_1 \times B_2$ is also compact.

DEFINITION 3.2. Let $(v_1, v_2) \in B$. Put

$$w_i(s) := r_{v_i}^i(s, \tilde{x}_1(s), \tilde{x}_2(s)),$$
$$\tilde{x}_1(s) := \underset{x_1 \in [0, a_1(s)]}{\mathrm{argmax}} \ r_{v_1}^1(s, x_1, \tilde{x}_2(s)),$$
$$\tilde{x}_2(s) := \underset{x_2 \in [0, a_2(s)]}{\mathrm{argmax}} \ r_{v_2}^2(s, \tilde{x}_1(s), x_2).$$

Define

$$W(v_1, v_2) := (w_1, w_2).$$

The above definition is correct, because if $s \in S$ is fixed, then the game $\Gamma(v_1, v_2, s)$ fulfils the assumptions of Lemma 5.3. Thus there exists a unique Nash equilibrium $(\tilde{x}_1(s), \tilde{x}_2(s))$ in $\Gamma(v_1, v_2, s)$.

LEMMA 3.1. *The mapping W is continuous when B is endowed with the product weak*-topology.*

Proof. Consider a sequence $\{(v_1^n, v_2^n)\} \subset B$ converging to some $(v_1, v_2) \in B$. We will show that the sequence $\{W(v_1^n, v_2^n)\}$ converges to $W(v_1, v_2) \in B$.

Denote:

$$\tilde{x}_1^n(s) := \underset{x_1 \in [0, a_1(s)]}{\mathrm{argmax}} \ r_{v_1^n}^1(s, x_1, \tilde{x}_2^n(s)),$$
$$\tilde{x}_2^n(s) := \underset{x_2 \in [0, a_2(s)]}{\mathrm{argmax}} \ r_{v_2^n}^2(s, \tilde{x}_1^n(s), x_2),$$

$$\tilde{x}_1(s) := \operatorname*{argmax}_{x_1 \in [0, a_1(s)]} r_{v_1}^1(s, x_1, \tilde{x}_2(s)),$$

$$\tilde{x}_2(s) := \operatorname*{argmax}_{x_2 \in [0, a_2(s)]} r_{v_2}^2(s, \tilde{x}_1(s), x_2).$$

$(\tilde{x}_1^n(s), \tilde{x}_2^n(s))$ is a Nash equilibrium in the game $\Gamma(v_1^n, v_2^n, s)$, and $(\tilde{x}_1(s), \tilde{x}_2(s))$ in the game $\Gamma(v_1, v_2, s)$. Lemma 5.3 implies that these Nash equilibria are unique.

Note that for fixed $s \in S$ the sequence $r_{v_i^n}^i(s, x_1, x_2)$ converges to $r_{v_i}^i(s, x_1, x_2)$ (as $n \to \infty$) uniformly in x_1 and x_2:

$$
\begin{aligned}
|r_{v_1^n}^1(s, x_1, x_2) - r_{v_1}^1(s, x_1, x_2)| = \Big| & (1 - \beta) u_1(x_1) + \beta \Big[\int_S v_1^n(s') l(s - x_1 - x_2) H_1(ds'|s) \\
& + \int_S v_1^n(s')(1 - l(s - x_1 - x_2)) H_2(ds'|s) \Big] \\
& - (1 - \beta) u_1(x_1) - \beta \Big[\int_S v_1(s') l(s - x_1 - x_2) H_1(ds'|s) \\
& + \int_S v_1(s')(1 - l(s - x_1 - x_2)) H_2(ds'|s) \Big] \Big| \\
= \beta \Big| & l(s - x_1 - x_2) \int_S [v_1^n(s') - v_1(s')] H_1(ds'|s) \\
& + (1 - l(s - x_1 - x_2)) \int_S [v_1^n(s') - v_1(s')] H_2(ds'|s) \Big|. \quad (1)
\end{aligned}
$$

By the Radon-Nikodym Theorem, for any fixed state $s \in S$, the probability measure $H_j(\cdot|s)$ has a density function, say g_j. Therefore (1) can be rewritten in the following way:

$$
\begin{aligned}
|r_{v_1^n}^1(s, x_1, x_2) - r_{v_1}^1(s, x_1, x_2)| = \beta \Big| & l(s - x_1 - x_2) \int_S [v_1^n(s') - v_1(s')] g_1(s') \mu(ds') \\
& + (1 - l(s - x_1 - x_2)) \int_S [v_1^n(s') - v_1(s')] g_2(s') \mu(ds') \Big|. \quad (2)
\end{aligned}
$$

Since $l(s - x_1 - x_2) \in [0, 1]$ and the integrals in (2) converge to zero as $n \to \infty$ (by the convergence of the sequence $\{(v_1^n)\}$ to v_1 in the weak*-topology) we can conclude that $r_{v_1^n}^1(s, x_1, x_2)$ converges uniformly to $r_{v_1}^1(s, x_1, x_2)$. The case of the sequence $\{r_{v_2^n}^2(s, x_1, x_2)\}$ follows along the same lines.

Fix $s \in S$. For any $n \geq 1$, $x_1 \in [0, a_1(s)]$ and $x_2 \in [0, a_2(s)]$ we have

$$r_{v_1^n}^1(s, x_1, \tilde{x}_2^n(s)) \leq r_{v_1^n}^1(s, \tilde{x}_1^n(s), \tilde{x}_2^n(s)), \quad (3)$$

$$r_{v_2^n}^2(s, \tilde{x}_1^n(s), x_2) \leq r_{v_2^n}^2(s, \tilde{x}_1^n(s), \tilde{x}_2^n(s)). \quad (4)$$

Let $(x_1^0(s), x_2^0(s))$ be any accumulation point of the sequence $\{(\tilde{x}_1^n(s), \tilde{x}_2^n(s))\}$. Then there exists a subsequence $\{(\tilde{x}_1^{k_n}(s), \tilde{x}_2^{k_n}(s))\}$ such that

$$\lim_{n \to \infty} \tilde{x}_1^{k_n}(s) = x_1^0(s), \qquad \lim_{n \to \infty} \tilde{x}_2^{k_n}(s) = x_2^0(s).$$

Obviously, with $\tilde{x}_1^{k_n}(s)$ and $\tilde{x}_2^{k_n}(s)$ the inequalities (3) and (4) hold. From the unifom convergence of the sequences $\{r_{v_1^n}^1(s, x_1, x_2)\}$ and $\{r_{v_2^n}^2(s, x_1, x_2)\}$ we can conclude:

$$r_{v_1}^1(s, x_1, x_2^0(s)) \leq r_{v_1}^1(s, x_1^0(s), x_2^0(s)), \qquad r_{v_2}^2(s, x_1^0(s), x_2) \leq r_{v_2}^2(s, x_1^0(s), x_2^0(s)),$$

for any $x_1 \in [0, a_1(s)]$ and $x_2 \in [0, a_2(s)]$. It means that $(x_1^0(s), x_2^0(s))$ is a Nash equilibrium in the game $\Gamma(v_1, v_2, s)$. Lemma 5.3 implies that the equilibrium is unique, so we obtain

$$\lim_{n \to \infty} (\tilde{x}_1^n(s), \tilde{x}_2^n(s)) = (\tilde{x}_1(s), \tilde{x}_2(s))$$

and

$$w_i(s) = r_{v_i}^i(s, \tilde{x}_1(s), \tilde{x}_2(s)) = \lim_{n \to \infty} r_{v_i^n}^i(s, \tilde{x}_1^n(s), \tilde{x}_2^n(s)) = \lim_{n \to \infty} w_i^n(s).$$

By the Lebesgue dominated convergence theorem $w_i^n \to w$ in the weak*-topology on B_i. Thus the result follows. ∎

Proof of Theorem 3.1. The above lemmas and the Schauder-Tychonoff Fixed Point Theorem (see [23]) imply the existence of $v_i \in B_i$ such that

$$W(v_1, v_2) = (v_1, v_2) \quad \mu\text{-a.e.} \tag{5}$$

Let $(f_1^*(s), f_2^*(s))$ be the unique Nash equilibrium in the game $\Gamma(v_1, v_2, s), s \in S$. Then from (5), it follows that

$$v_1(s) = \max_{x_1 \in [0, a_1(s)]} \left[(1 - \beta) u_1(x_1) + \beta \int_S v_1(s') p(ds'|s, x_1, f_2^*(s)) \right],$$

$$v_2(s) = \max_{x_2 \in [0, a_2(s)]} \left[(1 - \beta) u_2(x_2) + \beta \int_S v_2(s') p(ds'|s, f_1^*(s), x_2) \right]$$

for all $s \in S \setminus E$ with $\mu(E) = 0$.

Let (v_1^*, v_2^*) be the pair of the Nash equilibrium payoffs in the game $\Gamma(v_1, v_2, s)$ for every $s \in S$. We have $v_i^*(s) = v_i(s)$ for each $s \in S \setminus E$, so the pair $(f_1^*(s), f_2^*(S))$ is also the Nash equilibrium point in the game $\Gamma(v_1^*, v_2^*, s)$ for $s \in S \setminus E$. Observe that since $p(\cdot|s, x_1, x_2) \ll \mu(\cdot)$ for every $s \in S, x_i \in A_i(s), i = 1, 2$. We have

$$v_1^*(s) = \max_{x_1 \in [0, a_1(s)]} \left[(1 - \beta) u_1(x_1) + \beta \int_S v_1(s') p(ds'|s, x_1, f_2^*(s)) \right]$$

$$= \max_{x_1 \in [0, a_1(s)]} \left[(1 - \beta) u_1(x_1) + \beta \int_S v_1^*(s') p(ds'|s, x_1, f_2^*(s)) \right], \tag{6}$$

$$v_2^*(s) = \max_{x_2 \in [0, a_2(s)]} \left[(1 - \beta) u_2(x_2) + \beta \int_S v_2(s') p(ds'|s, f_1^*(s), x_2) \right]$$

$$= \max_{x_2 \in [0, a_2(s)]} \left[(1 - \beta) u_2(x_2) + \beta \int_S v_2^*(s') p(ds'|s, f_1^*(s), x_2) \right] \tag{7}$$

for each $s \in S$.

Let $w_i^*(s) = \frac{v_i^*(s)}{1 - \beta}$. The above equations may be rewritten in the form

$$w_1^*(s) = \max_{x_1 \in [0, a_1(s)]} \left[u_1(x_1) + \beta \int_S w_1^*(s') p(ds'|s, x_1, f_2^*(s)) \right], \tag{8}$$

$$w_2^*(s) = \max_{x_2 \in [0, a_2(s)]} \left[u_2(x_2) + \beta \int_S w_2^*(s') p(ds'|s, f_1^*(s), x_2) \right], \tag{9}$$

for every $s \in S$.

From Blackwell's work on discounted dynamic programming [5] and the above equations ((8) and (9)), it follows that

$$w_1^*(s) = \gamma_1(f_1^*, f_2^*)(s) = \max_{\pi_1 \in \Pi_1} \gamma_1(\pi_1, f_2^*)(s)$$

and

$$w_2^*(s) = \gamma_2(f_1^*, f_2^*)(s) = \max_{\pi_2 \in \Pi_2} \gamma_2(f_1^*, \pi_2)(s)$$

for each $s \in S$. So the pair (f_1^*, f_2^*) is also a Nash equilibrium in the stochastic game with the β-discounted expected payoff. ∎

4. Nash equilibrium in stochastic games with expected mean payoffs. Let us make an additional assumption on our game model:

ASSUMPTION 4.1. (i) The probabilities $H_i(\cdot|s)$ do not depend on s (for simplicity of notation we shall denote them by $H_i(\cdot)$).

(ii) There exist a probability measure ν on S and a constant $\delta \in (0,1)$ such that for any measurable set $A \subseteq S$ we have

$$H_i(A) \geq \delta\nu(A)$$

for any $i \in \{1, 2\}$.

REMARK 4.1. From the above assumption we conclude immediately that $p(A|s, x_1, x_2) \geq \delta\nu(A)$ and

$$H_i \ll \mu := \frac{H_1 + H_2}{2}.$$

THEOREM 4.1. *If Assumptions 2.1, 2.2 and 4.1 hold in our model of the infinite horizon game with the expected mean payoffs of the players, then a Nash equilibrium exists.*

Proof. Note that for the discounted game in which the transition probabilities are of the form

$$\tilde{p}(\cdot|s, x_1, x_2) = \frac{p(\cdot|s, x_1, x_2) - \delta\nu(\cdot)}{1 - \delta}$$

and the discount factor is equal $\beta = 1 - \delta$, both Assumptions 2.1 and 2.2 are satisfied.

Theorem 3.1 implies that in such a game there exists a stationary Nash equilibrium (f_1^*, f_2^*) and the Bellman optimality equations are satisfied:

$$v_1(s) = \max_{x_1 \in [0, a_1(s)]} \left[u_1(x_1) + (1 - \delta) \int_S v_1(s') \tilde{p}(ds'|s, x_1, f_2^*(s)) \right], \qquad (10)$$

$$v_2(s) = \max_{x_2 \in [0, a_2(s)]} \left[u_2(x_2) + (1 - \delta) \int_S v_2(s') \tilde{p}(ds'|s, f_1^*(s), x_2) \right]. \qquad (11)$$

Denote:

$$d_i := \delta \int_S v_i(s') \nu(ds').$$

Equations (10) and (11) can be rewritten in the following form:

$$v_1(s) = \max_{x_1 \in [0, a_1(s)]} \left[u_1(x_1) + \int_S v_1(s') p(ds'|s, x_1, f_2^*(s)) - d_1 \right],$$

$$v_2(s) = \max_{x_2 \in [0, a_2(s)]} \left[u_2(x_2) + \int_S v_2(s') p(ds'|s, f_1^*(s), x_2) - d_2 \right].$$

These are the optimality equations for the game with the expected mean payoffs (see [13]). Thus (f_1^*, f_2^*) is a stationary Nash equilibrium in the infinite game with the expected mean payoffs and the Nash equilibrium payoff of player i in this game is $\phi_i(f_1^*, f_2^*)(s) = d_i$ (it does not depend on s). ∎

REMARK 4.2. (a) The transformation to the discounted game used in the proof was introduced in dynamic programming by Dynkin and Yushkevich in their book [10].
(b) Assumption 4.1 implies that the Markov process generated by the strategies in the Nash equilibrium (f_1^*, f_2^*) is geometrically ergodic (see [13]) and has a stationary distribution. The existence of a stationary distribution is an important question in studying various economic models, see [2, 7].

5. The uniqueness of the Nash equilibria in a one-shot game.
In this section we shall consider the problem of uniqueness of the Nash equilibrium in an auxiliary game.

In the lemmas below we shall consider functions

$$w_i : X_1 \times X_2 \to R$$

for $i = 1, 2$, where X_1, X_2 are compact intervals. The function w_i is strictly concave in the i-th variable and its second partial derivatives are continuous.

REMARK 5.1. From the well known result of Nash (see [18]) the two-person one-shot game with the payoff functions w_i and the actions sets X_i has a Nash equilibrium.

ASSUMPTION 5.1. Assume that for any $\bar{x} = (x_1, x_2) \in X_1 \times X_2$ and $i = 1, 2$ it holds

$$\left| \frac{\partial^2 w_i}{\partial x_i^2}(\bar{x}) \right| > \left| \frac{\partial^2 w_i}{\partial x_i \partial x_{3-i}}(\bar{x}) \right|. \tag{12}$$

For any $\bar{x} = (x_1, x_2) \in X_1 \times X_2$ define the norm

$$||\bar{x}||_\infty = \max_{i \in \{1,2\}} |x_i|.$$

The first lemma in this section is a result of Moré (see [17]). The proof of this lemma, based on the Rolle Theorem can be found in the paper of Gabay and Moulin [12].

LEMMA 5.1 (Moré). *If Assumption 5.1 holds and $\bar{x}, \bar{y} \in X_1 \times X_2$ are different, then*

$$\frac{\partial w_i}{\partial x_i}(\bar{x}) = \frac{\partial w_i}{\partial x_i}(\bar{y})$$

implies that

$$|x_i - y_i| < ||\bar{x} - \bar{y}||_\infty.$$

The following lemma is a slight modification of Theorem 4.1 from [12].

LEMMA 5.2. *Let Assumption 5.1 be fulfilled and w_i be strictly concave in x_i. Then in the two-person game in which $w_i(x_1, x_2)$ are the payoffs of the players, there exists a unique Nash equilibrium.*

Proof. The strict concavity of w_i in x_i implies that for any i and fixed x_i, $w_{3-i}(\bar{x})$ attains its maximum at a unique point x_{3-i}. Thus we may define the mapping:

$$A : X_1 \times X_2 \to X_1 \times X_2$$

such that for $(x_1, x_2) = A(y_1, y_2)$ we have

$$x_1 = \operatorname*{argmax}_{x \in X_1} w_1(x, y_2), \qquad x_2 = \operatorname*{argmax}_{x \in X_2} w_2(y_1, x). \tag{13}$$

Put $A_i(y_1, y_2) := x_i$ for $i = 1, 2$. We will prove now that

$$||A(\bar{x}) - A(\bar{y})||_\infty < ||\bar{x} - \bar{y}||_\infty \tag{14}$$

for $\bar{x}, \bar{y} \in X_1 \times X_2$.

Consider the case in which both $A_i(\bar{x})$ and $A_i(\bar{y})$ are inside the interval X_i. Then

$$\frac{\partial w_1}{\partial x_1}(A_1(\bar{x}), x_2) = \frac{\partial w_1}{\partial x_1}(A_1(\bar{y}), y_2) = 0$$

and Lemma 5.1 implies

$$|A_1(\bar{x}) - A_1(\bar{y})| < ||(A_1(\bar{x}), x_2) - (A_1(\bar{y}), y_2)||_\infty = |x_2 - y_2|. \tag{15}$$

A similar inequality can be obtained for A_2:

$$|A_2(\bar{x}) - A_2(\bar{y})| < ||(x_1, A_2(\bar{x})) - (y_1, A_2(\bar{y}))||_\infty = |x_1 - y_1|. \tag{16}$$

From (15) and (16) it can be easily concluded that (14) holds.

If any of the values $A_i(\bar{x})$ or $A_i(\bar{y})$ belongs to the edge of the interval X_i, we may define (thanks to the fact that the derivatives w_i are finite and X_i are closed) the function $\tilde{w}_i : R \times R \to R$ such that $\tilde{w}_i(\bar{x}) = w_i(\bar{x})$ for $\bar{x} \in X_1 \times X_2$ with the same properties like w_i, that is strict concavity in x_i and Assumption 5.1.

For these functions we conduct similar reasoning, as in the case of $A_i(\bar{x})$ and $A_i(\bar{y})$ inside the set $X_1 \times X_2$.

Define $\tilde{A}$ for the function $\tilde{w}_i$ just as A was defined for w_i. Note that

$$A_i(\bar{x}) \in [\tilde{A}_i(\bar{x}), \tilde{A}_i(\bar{y})].$$

This implies that

$$|A_i(\bar{x}) - A_i(\bar{y})| \le |\tilde{A}_i(\bar{x}) - \tilde{A}_i(\bar{y})| < ||\bar{x} - \bar{y}||_\infty,$$

and this ends the proof of (14).

Note that there exists a Nash equilibrium in the considered game (see Remark 5.1). Assume that there exist two different Nash equilibria: $\bar{x}^* \in X_1 \times X_2$ and $\bar{y}^* \in X_1 \times X_2$. Note that these are fixed points of the mapping A, i.e. $\bar{x}^* = A(\bar{x}^*)$ and $\bar{y}^* = A(\bar{y}^*)$. Thus

$$||\bar{x}^* - \bar{y}^*||_\infty = ||A(\bar{x}^*) - A(\bar{y}^*)||_\infty < ||\bar{x}^* - \bar{y}^*||_\infty$$

and so we have obtained a contradiction. Therefore the game has a unique Nash equilibrium. ∎

LEMMA 5.3. *For a fixed $s \in S \subseteq (0, \infty)$ consider a two-person game with the payoff of player i given by*

$$u_i(x_i) + k_i l(s - x_1 - x_2),$$

where $x_i \in [0, a_i(s)]$, $a_1(s) + a_2(s) < s$ and

(i) *$u_i : [0, \infty) \to [0, \infty)$ fulfils $u_i''(x) < 0$ for all $x \in (0, +\infty)$ and it is increasing and twice differentiable,*

(ii) *$k_i \in R$,*

(iii) *$l : [0, \infty) \to [0, \infty)$ is concave, increasing and twice differentiable.*

This game has a unique Nash equilibrium.

Proof. (a) Consider the case that $k_1 \leq 0$ or $k_2 \leq 0$. Assume that $k_2 \leq 0$ (for $k_1 \leq 0$ the proof is similar). Note that the payoff of player 2: $u_2(x_2) + k_2 l(s - x_1 - x_2)$ is increasing in x_2. Thus, the optimal choice of player 2 (independent of the choice of player 1) is $x_2^* = a_2(s)$.

If $k_1 > 0$, then the payoff of player 1, $u_1(x_1) + k_1 l(s - x_1 - x_2)$, is strictly concave in x_1, thus there exists a unique point

$$x_1^* = \operatorname*{argmax}_{x_1 \in [0, a_1(s)]} [u_1(x_1) + k_1 l(s - x_1 - x_2^*)].$$

(x_1^*, x_2^*) is then a unique Nash equilibrium.

(b) If $k_i \leq 0$ for $i = 1, 2$ then a unique Nash equilibrium is (x_1^*, x_2^*), where $x_1^* = a_1(s)$, $x_2^* = a_2(s)$.

(c) Let us consider the case that $k_i > 0, i = 1, 2$.

Note that the payoff of player i is strictly concave in x_i and both payoffs fulfil Assumption 5.1. Thus the uniqueness of the Nash equilibrium is implied by Lemma 5.2. ∎

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