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ORTHOMODULAR LATTICES AND CLOSURE OPERATIONS IN ORDERED VECTOR SPACES

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Abstract. On a non-trivial partially ordered real vector space $(V, \leqslant)$ the orthogonality relation is defined by incomparability and $\zeta(V, \perp)$ is a complete lattice of double orthoclosed sets. We say that $A \subseteq V$ is an orthogonal set when for all $a, b \in A$ with $a \neq b$, we have $a \perp b$. In our earlier papers we defined an integrally open ordered vector space and two closure operations $A \to D(A)$ and $A \to A^{\perp\perp}$. It was proved that V is integrally open iff $D(A) = A^{\perp\perp}$ for every orthogonal set $A \subseteq V$. In this paper we generalize this result. We prove that V is integrally open iff $D(A) = W$ for every $W \in \zeta(V, \perp)$ and every maximal orthogonal set $A \subseteq W$. Hence it follows that the lattice $\zeta(V, \perp)$ is orthomodular.

1. Introduction. This paper is a complement to our earlier papers [3] and [4]. Let $(V, \leqslant)$ be a partially ordered real vector space with a positive cone $P = \{b \in V : b \geqslant 0\}$ such that $P \neq \{0\}$. A set $\{a + \omega b \in V : \omega \in \mathbb{R}\}$, $a, b \in V$, $b > 0$, is called a *P-line* (see [3]). Let us denote V_P the family of all P-lines. We define the orthogonality relation

$$a \perp b \quad \Leftrightarrow \quad a \nleqslant b \quad \text{and } b \nleqslant a \;\Leftrightarrow\; \text{there is no } g \in V_P \quad \text{such that} \quad \{a, b\} \subseteq g.$$

The pair $(V, \perp)$ is called the *orthogonality space* generated by V. For $a \in A$ and $A \subseteq V$ we denote:

$$a^{\perp} = \{b \in V : b \perp a\}, \quad A^{\perp} = \bigcap a^{\perp} \quad \text{and} \quad A^{\perp\perp} = (A^{\perp})^{\perp}.$$

The family of double orthoclosed sets $\zeta(V, \perp) = \{A \subseteq V : A = A^{\perp\perp}\}$ partially ordered by set-theoretical inclusion and equipped with the orthocomplemention $A \to A^{\perp}$, with

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l.u.b. and g.l.b. given respectively by the formulas

$$\bigvee A_j = (\bigcup A_j)^{\perp\perp}, \quad \bigwedge A_j = \bigcap A_j,$$

forms a complete orthocomplemented lattice (see [1]). This lattice is called *orthomodular* (see [1]) if the following condition is satisfied:

$$\text{If} \quad A \subseteq B \quad \text{then} \quad B = A \vee (B \wedge A^{\perp}).$$

We say that $A \subseteq V$ is an *orthogonal* set when for all $a, b \in A$ with $a \neq b$, we have $a \perp b$. Foulis and Randall [6] proved that $\zeta(V, \perp)$ is a complete orthomodular lattice iff the following condition is satisfied:

(OM) If $W = W^{\perp\perp} \subseteq V$ and A is a maximal orthogonal subset of W, then $A^{\perp\perp} = W$.

We consider two closure operations $A \to D(A)$ and $A \to A^{\perp\perp}$, where $D(A)$ is called the *causal closure* of A (see [2]) and $A^{\perp\perp}$ is called the *ortho closure* of A

$$D(A) = \{b \in V : \forall g \in V_P,\ b \in g \Rightarrow\ g \cap A \neq \emptyset\},$$

$$A^{\perp\perp} = \{b \in V : \forall g \in V_P,\ b \in g\ \Rightarrow\ g \cap A^{\perp} = \emptyset\}.$$

Let us notice that $D(A) \subseteq A^{\perp\perp}$.

The space V is called *integrally open* (see [3]) if

$$b > 0 \quad \text{and} \quad a < b \quad \Rightarrow \quad \exists n \in \mathbb{N}, \quad a + nb > 0.$$

It was shown in [3] that if V is integrally open, then the lattice $\zeta(V, \perp)$ is orthomodular. In [4] it was proved that V is integrally open iff $D(A) = A^{\perp\perp}$ for every orthogonal set $A \subseteq V$. In this paper we generalize this result. We prove (Theorem 2.1(5)) that V is integrally open iff the following condition is satisfied:

If $W = W^{\perp\perp} \subseteq V$ and A is a maximal orthogonal subset of W, then $D(A) = W$.

Hence it follows that if V is an integrally open space, then the lattice $\zeta(V, \perp)$ is orthomodular (Corollary 2.1).

The space V is *directed* (see [1]) if

$$\forall a, b \in V \quad \exists c \in V, \quad a \leqslant c \quad \text{and} \quad b \leqslant c.$$

By a *strong unit* of V we mean an element $a \in V$ such that

$$\forall b \in V \quad \exists n \in \mathbb{N}, \quad na > b.$$

It was proved in [3] that V is integrally open and directed iff every positive element of V is a strong unit. The integrally open concept is closely related to the open set in the Euclidean topology in a finite dimensional ordered vector space (the only topology in which a finite dimensional vector space is a topological vector space). For the case in which V is finite dimensional we proved in [3] that V is integrally open and directed iff $\mathrm{int}(P) = P\backslash\{0\}$, where $\mathrm{int}(P)$ denotes the interior of P in the Euclidean topology of V. In this paper we complete this result. For the case in which V is finite dimensional we prove (Theorem 3.2) that V is integrally open and directed iff $\mathrm{int}(A^{+}) = A^{+}\backslash A$ for every orthogonal set $A \subset V$ (for every maximal orthogonal set $A \subset V$). Hence it follows that if V is finite dimensional, integrally open and directed, then any maximal orthogonal set in V is closed in the Euclidean topology of V (Corollary 3.1).

2. Ortho and causal closure operations. Let $(V, \perp)$ be the orthogonality space generated by V. For $a \in A$ and $A \subseteq V$ we define:

$$a^+ = \{b \in V : a \leqslant b\}, \qquad A^+ = \bigcup a^+,$$
$$a^- = \{b \in V : b \leqslant a\}, \qquad A^- = \bigcup a^-.$$

THEOREM 2.1. *The following conditions are equivalent:*

1. *V is an integrally open space.*
2. $g \in V_P, \quad a \in V, \quad g \cap a^- \neq \emptyset \quad \Leftrightarrow \quad g \cap a^+ \neq \emptyset.$
3. $g \in V_P, \quad a \notin g \quad \Rightarrow \quad g \cap a^+$ *has no smallest element.*
4. $g \in V_P, \quad a \notin g \quad \Rightarrow \quad g \cap a^-$ *has no largest element.*
5. *If $W = W^{\perp\perp} \subseteq V$ and A is a maximal orthogonal set in W, then $D(A) = W$.*
6. *$D(A) = A^{\perp\perp}$ for every orthogonal set $A \subseteq V$.*
7. *$D(A) = V$ for every maximal orthogonal set $A \subseteq V$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) were proved in [3] and implications (6) $\Rightarrow$ (7) $\Rightarrow$ (1) were proved in [4].

(2) $\wedge$ (3) $\wedge$ (4) $\Rightarrow$ (5). Let $W = W^{\perp\perp}$ and A be a maximal orthogonal set in W. Suppose $g \in V_P$ and $g \cap W \neq \emptyset$. Since A is a maximal orthogonal set in W, $g \cap W \subseteq A^- \cup A^+$. Suppose $g \cap A^+ \cap W \neq \emptyset$ (the proof in the case $g \cap A^- \cap W \neq \emptyset$ is analogous). By condition (2), $g \cap A^-$ and $g \cap A^+$ are nonempty sets. Hence $g \cap A^+$ is a ray with an end point x. First we prove that $x \in W$. Since $x^+ \cap W \neq \emptyset$, then $x^- \cap W^\perp = \emptyset$. If there exists $y \in x^+ \cap W^\perp$, then x is the largest element of $g \cap y^-$ and we obtain a contradiction with condition (4), because $y \notin g$. Hence $x^+ \cap W^\perp = \emptyset$ and $x \in W^{\perp\perp} = W$. Now we prove that $x \in A$. Since $x \in g \cap W \subseteq A^- \cup A^+$, then one of the following cases occurs:

i) $x \in A^+$,
ii) $x \in A^-$.

Case i) There exists $s \in A$ such that x is the smallest element of $g \cap s^+$. Hence, by condition (3) $s \in g$ and $x = s \in A$.

Case ii) Since the set A is orthogonal, then there exists $t \in A$ such that x is the largest element of $g \cap t^-$. Hence, by condition (4) $t \in g$ and $x = t \in A$.

(5) $\Rightarrow$ (6). Let us notice that if A is an orthogonal set, then it is a maximal orthogonal set in $A^{\perp\perp}$. ∎

COROLLARY 2.1. *If V is an integrally open space, then $\zeta(V, \perp)$ is a complete orthomodular lattice.*

Proof. Suppose V is an integrally open space. Let $W = W^{\perp\perp} \subseteq V$ and A be a maximal orthogonal set in W. Since $D(A) \subseteq A^{\perp\perp} \subseteq W^{\perp\perp} = W$, then by Theorem 2.1(5) $D(A) = A^{\perp\perp} = W$. Hence the implication (OM) is satisfied. ∎

3. Integrally open and directed vector spaces. Let $(V,\perp)$ be the orthogonality space generated by V. For $a,b\in V$, $a\neq b$, (a,b) denotes the set of all points of the form $\alpha a+\beta b$, $\alpha>0$, $\beta>0$, $\alpha+\beta=1$. Let $S\subset V$. A point $a\in S$ is a *core point* of S if for each point $b\in V$, with $b\neq a$, there exists a point $c\in(a,b)$ such that $(a,c)\subset S$. The set of such core points is called the *core* of S (see [6]). A translation of a one dimensional subspace of V is called a *line*.

THEOREM 3.1. *The following conditions are equivalent:*

1. *V is an integrally open and directed space.*
2. *Every positive element of V is a strong unit.*
3. *The core of P is $P\setminus\{0\}$.*
4. *If $A\subseteq V$ is an orthogonal set, then the core of A^+ is $A^+\setminus A$.*
5. *If $A\subseteq V$ is a maximal orthogonal set, then the core of A^+ is $A^+\setminus A$.*
6. *If $A\subseteq V$ is a maximal orthogonal set and g is a line such that $g\cap A^+\neq\emptyset$ and $g\cap A^-\neq\emptyset$, then $g\cap A\neq\emptyset$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) was proved in [3] and the equivalence (1) $\Leftrightarrow$ (3) was proved in [4].

(3) $\Rightarrow$ (4). If A is an orthogonal set, then by (3)

$$\mathrm{core}(A^+)\subseteq A^+\setminus A\subseteq\bigcup a^+\setminus\{a\}=\bigcup\mathrm{core}(a^+)\subseteq\mathrm{core}(A^+).$$

(4) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (6). Suppose A is a maximal orthogonal set. Then $\{A^-\setminus A,\ A,\ A^+\setminus A\}$ is a partition of V. If g is a line such that $g\cap(A^+\setminus A)\neq\emptyset$ and $g\cap(A^-\setminus A)\neq\emptyset$ then by (5) $g\cap(A^-\setminus A)$ and $g\cap(A^+\setminus A)$ are nonempty open sets in the topology of line g. Hence $g\cap A\neq\emptyset$.

$\neg$(3) $\Rightarrow$ $\neg$(6). Suppose $a>0$, $b\in V$ and $g=\{a+\omega b:\omega\in\mathbb{R}\}$ is a line such that $g\cap 0^+\subseteq\{a+\omega b:\omega\geq 0\}$. If $g\cap 0^\perp\neq\emptyset$ then there exists $\gamma<0$ such that $c=a+\gamma b\in g\cap 0^\perp$. Since $c\in 0^\perp$ then $a+\lambda c\notin 0^-$ for $\lambda>0$. Hence $a+\lambda c\notin\{\mu c:\mu\leq 0\}^-$ for $\lambda>0$. Since $(1+\lambda)^{-1}(a+\lambda c)=(1+\lambda)^{-1}(a+\lambda a+\lambda\gamma b)=a+(1+\lambda)^{-1}\lambda\gamma b\in g$ and $\gamma<0$, we have $(1+\lambda)^{-1}(a+\lambda c)\notin 0^+$ for $\lambda>0$. Hence $a+\lambda c\notin\{\mu c:\mu\leq 0\}^+$ for $\lambda>0$. Thus we have

j) $\{a+\lambda c:\lambda>0\}\subseteq\{\mu c:\mu\leq 0\}^\perp$.

Since $a+\omega b<a+\omega b+\gamma^{-1}\omega a=a+\gamma^{-1}\omega(a+\gamma b)=a+\gamma^{-1}\omega c\in\{a+\lambda c:\lambda>0\}$ for $\omega<0$ and $a+\omega b>\gamma^{-1}\omega a+\omega b=\gamma^{-1}\omega(a+\gamma b)=\gamma^{-1}\omega c\in\{\mu c:\mu\leq 0\}$ for $\omega\geq 0$, and $c\in 0^\perp$, it follows that

jj) $\{a+\omega b:\omega<0\}\subseteq\{a+\lambda c:\lambda>0\}^-$ and $\{a+\omega b:\omega\geq 0\}\subseteq\{\mu c:\mu\leq 0\}^+$,

jjj) $\{a+\omega b:\omega<0\}\cap\{a+\lambda c:\lambda>0\}=\emptyset$ and $\{a+\omega b:\omega\geq 0\}\cap\{\mu c:\mu\leq 0\}=\emptyset$.

Since $c\in 0^\perp$, by *j)* $\{a+\lambda c:\lambda>0\}\cup\{\mu c:\mu\leq 0\}$ is an orthogonal set. Let A be a maximal orthogonal set such that $\{a+\lambda c:\lambda>0\}\cup\{\mu c:\mu\leq 0\}\subseteq A$. By *jjj)*, *jj)* and *j)*, $g\cap\{a+\lambda c:\lambda>0\}=\emptyset$ and $g\cap\{\mu c:\mu\leq 0\}=\emptyset$. Hence by *jj)* $g\cap A=\emptyset$. Since $a\in g\cap A^+$ and $c\in g\cap A^-$ the implication (6) is not satisfied.

If $g \cap 0^{\perp} = \emptyset$, let A be a maximal orthogonal set such that $0 \in A$. Since $0 \notin g \subseteq 0^{+} \cup 0^{-}$, we have $g \cap A = \emptyset$. Hence the implication (6) is not satisfied because $g \cap 0^{+} \neq \emptyset$ and $g \cap 0^{-} \neq \emptyset$. ■

Suppose V is a finite dimensional real vector space. Valentine [6] proved that if $S \subseteq V$ is a convex set then $\mathrm{int}(S) = \mathrm{core}(S)$, where $\mathrm{int}(S)$ denotes the interior of S in the Euclidean topology of V. Hence, in this case, Theorem 3.1 has the following formulation:

THEOREM 3.2. *If V is a finite dimensional vector space, then the following conditions are equivalent:*

1. *V is an integrally open and directed space.*
2. *Every positive element of V is a strong unit.*
3. *The interior of P is $P \setminus \{0\}$.*
4. *If $A \subseteq V$ is an orthogonal set, then $int(A^{+}) = A^{+} \setminus A$.*
5. *If $A \subseteq V$ is a maximal orthogonal set, then $int(A^{+}) = A^{+} \setminus A$.*
6. *If $A \subseteq V$ is a maximal orthogonal set and g is a line such that $g \cap A^{+} \neq \emptyset$ and $g \cap A^{-} \neq \emptyset$, then $g \cap A \neq \emptyset$.*

COROLLARY 3.1. *If V is finite dimensional, integrally open and directed, then any maximal orthogonal set in V is closed in the Euclidean topology of V.*

Proof. If $A \subseteq V$ is a maximal orthogonal set, then A is the complement set of the union $(A^{-} \setminus A) \cup (A^{+} \setminus A)$. By Theorem 3.2(5), $A^{-} \setminus A = \mathrm{int}(A^{-})$ and $A^{+} \setminus A = \mathrm{int}(A^{+})$. Hence A is a closed set in the Euclidean topology of V. ■

Let us notice that if $V = R^2$ as a lexicographically ordered vector space, then V is not integrally open (because $\mathrm{int}(P) \neq P \setminus \{0\}$) but every maximal orthogonal set in V is closed in the Euclidean topology of R^2 (because every maximal orthogonal set is a one point set).

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