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ON SOME GENERALIZATION OF THE t-TRANSFORMATION

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Abstract. Using the Nevanlinna representation of the reciprocal of the Cauchy transform of probability measures, we introduce a two-parameter transformation $U^{\mathbb{T}}$ of probability measures on the real line $\mathbb{R}$, which is another possible generalization of the t-transformation. Using that deformation we define a new convolution by deformation of the free convolution.

The central limit measure with respect to the $\mathbb{T}$-deformed free convolutions is still a Kesten measure, but the Poisson limit depends on the two parameters and is different from the Poisson measures for (a, b)-deformation.

We also show that the $\mathbb{T}$-deformed free convolution is different from the convolution obtained as the deformed conditionally free convolution of Bożejko, Leinert and Speicher. Thus the $\mathbb{T}$ does not satisfy the Bożejko property.

1. Introduction. In this paper we are going to introduce a transformation of probability measures on the real line $\mathbb{R}$, which is another possible generalization of the t-transformation investigated in [BW1] (see also [BW2], [Wo1]). We will call it the $\mathbb{T}$-*transformation*.

The definition still bases on the reciprocals of the Cauchy transforms as for the t-transformation, but we will use two parameters, the diagonal graph of which gives the original t-transformation. To define this transformation we will use the Nevanlinna representation of the reciprocal of the Cauchy transform of probability measures, in contrast to the (a, b)-transformation, [KY2], where we have used the Maassen representation.

Using that deformation we define a new convolution by deformation of free convolution in Section 4.

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The central limit measure with respect to the $\mathbb{T}$-deformed free convolution is the same as the one for the original t-deformation and (a,b)-deformation. Because the $\mathbb{T}$-deformation does not commute with the dilation of measures, proof of the central limit theorem in not as immediate as was for the (a,b)-deformation. We prove the central limit theorem for the $\mathbb{T}$-deformation of free convolution in Section 5.

The Poisson limit depends on the two parameters and is different from the Poisson measure for the (a,b)-deformation. In Section 6 we calculate the Poisson measure for the $\mathbb{T}$-deformed free convolution.

The subsequent sections are devoted to the study of the deformation of the conditionally free convolution. In Section 7 we show that the $\mathbb{T}$-deformed free convolution is different from the convolution obtained as the deformed conditionally free convolution.

In the last section we prove the central limit theorem and Poisson limit theorem for the N-fold $\mathbb{T}$-deformed convolution arising from the conditionally free convolution and show that the central limit measure is still the Kesten measure κ_t. An explicit formula of the Poisson measure is also given.

2. Preliminaries

2.1. The Cauchy transform. Let μ be a probability measure on the real line $\mathbb{R}$, $\mu \in \mathrm{Prob}(\mathbb{R})$. We then denote by $G_\mu(z)$ the Cauchy transform of the measure μ, for $z \in \mathbb{C}^+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$, defined as follows

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu(x)}{z-x}.$$

The Cauchy transform $G_\mu(z)$ is analytic in the open upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ and takes values in the open lower half plane $\mathbb{C}^- = \{z \in \mathbb{C}\colon \operatorname{Im} z < 0\}$, $G_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^-$.

Remark 1. For compactly supported probability measures μ the Cauchy transform $G_\mu(z)$ and the generating function of moments of μ for z in some neighbourhood of zero

$$M_\mu(z) = \sum_{n=0}^{\infty} m_\mu(n) z^n$$

are related by

$$M_\mu(z) = \frac{1}{z}\, G_\mu\left(\frac{1}{z}\right). \tag{2.1}$$

Theorem 1 (Nevanlinna theorem). *A function F is the reciprocal of the Cauchy transform of some probability measure μ, that is, $F(z) = \frac{1}{G_\mu(z)}$, if and only if there exist a unique real number α and a unique positive measure ρ such that for $z \in \mathbb{C}^+$*

$$\frac{1}{G_\mu(z)} = F(z) = \alpha + z + \int_{-\infty}^{\infty} \frac{1+x\,z}{x-z}\, \mathrm{d}\rho(x). \tag{2.2}$$

We will call the representation (2.2) *the* Nevanlinna representation.

For a proof of this fact see for instance [Do], [Bh].

PROPOSITION 1. *The constant α in the Nevanlinna representation of the reciprocal of the Cauchy transform of the measure μ is equal to* $\mathrm{Re}\big(\frac{1}{G_\mu(i)}\big)$. *We will call it the* Nevanlinna constant *of the measure μ.*

Proof. Using the Nevanlinna representation for $z = i$ we obtain

$$F_\mu(i) = \alpha + i + \int_{-\infty}^{\infty} \frac{1 + x\,i}{x - i}\,\mathrm{d}\rho(x),$$

and because

$$\frac{1 + x\,i}{x - i} = \frac{x + i + x^2\,i - x}{x^2 + 1} = \frac{i + x^2\,i}{x^2 + 1} = i,$$

we have

$$F_\mu(i) = \frac{1}{G_\mu(i)} = \alpha + i + i\,\rho(\mathbb{R}).$$

Taking the real part we obtain $\mathrm{Re}\big(\frac{1}{G_\mu(i)}\big) = \alpha$. ∎

We will also use the representation of the Cauchy transform coming from the following lemma proved by Maassen ([Ma], see also [Ak], [AG]) which characterizes the reciprocals of the Cauchy transforms of measures with finite variance:

LEMMA 1. *A holomorphic function $F : \mathbb{C}^+ \to \mathbb{C}^+$ is the reciprocal of the Cauchy transform of a probability measure μ with finite second moment if and only if there exists a positive finite Borel measure τ on $\mathbb{R}$ and real constant α_0 such that for $z \in \mathbb{C}^+$*

$$F(z) = z - \alpha_0 + \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau(x)}{x - z}, \quad z \in \mathbb{C}^+. \tag{2.3}$$

REMARK 2. The representation (2.3) of the reciprocal $F(z)$ of the Cauchy transform of some probability measure μ with finite second moment will be called the *Maassen representation*. The constant α_0 is equal to the first moment of the measure μ.

It is known, see Akhiezer [Ak] and Akhiezer and Glazman [AG], that we have the following connection between measures in Nevanlinna and Maassen representations:

REMARK 3. Let the reciprocal of the Cauchy transform of a probability measure μ be represented in both Nevanlinna form (2.2) and in the Maassen form (2.3). Then

$$\mathrm{d}\tau(x) = \int_{-\infty}^{x} (1 + u^2)\,\mathrm{d}\rho(u).$$

REMARK 4. Assume that the measure μ has the first moment equal to zero and the Maassen representation

$$F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1}{x - z}\,\mathrm{d}\tau(x).$$

The Nevanlinna constant of the measure μ is equal to

$$\mathrm{Re}\left(\frac{1}{G_\mu(i)}\right) = \mathrm{Re}\int_{-\infty}^{\infty} \frac{1}{x - i}\,\mathrm{d}\tau(x) = \int_{-\infty}^{\infty} \frac{x}{x^2 + 1}\,\mathrm{d}\tau(x).$$

Lemma 2. *Assume that the measure μ is symmetric and determined by its moments and has the Maassen representation*

$$F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1}{x-z}\,\mathrm{d}\tau(x).$$

Then the Maassen measure τ is also symmetric and determinate. Moreover, the Nevanlinna constant of the measure μ is equal to zero.

Proof. For the Cauchy transform of a symmetric measure μ we have

$$G_\mu(z) = -G_\mu(-z), \quad z \in \mathbb{C}\setminus\mathbb{R}$$

which means that for the Maassen representation we also have

$$z + \int_{-\infty}^{\infty} \frac{1}{x-z}\,\mathrm{d}\tau(x) = z - \int_{-\infty}^{\infty} \frac{1}{x+z}\,\mathrm{d}\tau(x),$$

which implies that

$$\int_{-\infty}^{\infty} \frac{1}{x-z}\,\mathrm{d}\tau(x) = -\int_{-\infty}^{\infty} \frac{1}{x+z}\,\mathrm{d}\tau(x),$$

that is, for $z \in \mathbb{C}^+$

$$G_\tau(z) = -G_\tau(-z),$$

so the measure τ is symmetric. Using theorems from [ST] we obtain that the measure τ is also determinate.

Moreover, by Remark 4 we have

$$\mathrm{Re}\biggl(\frac{1}{G_\mu(i)}\biggr) = \int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}\tau(x) = 0. \ \blacksquare$$

Recall that for a probability measure μ and positive number λ we denote by $\mathbb{D}_\lambda\mu$ the *dilation* by λ, i.e.

$$\mathbb{D}_\lambda\mu(A) = \mu(\lambda^{-1}A).$$

Let us note that we have the following relation between moments of a measure μ and of its dilation $\mathbb{D}_\lambda\mu$

$$m_{\mathbb{D}_\lambda\mu}(n) = \lambda^n m_\mu(n). \tag{2.4}$$

Moreover, we have the following

Remark 5. The Cauchy transform of a probability measure μ and the Cauchy transform of its dilation are related by

$$G_{\mathbb{D}_\lambda\mu}(z) = \frac{1}{\lambda} G_\mu\biggl(\frac{z}{\lambda}\biggr).$$

Proposition 2. *Assume that a probability measure μ has its first moment equal to* 0 *and the following Maassen representation for $z \in \mathbb{C}^+$:*

$$F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1}{x-z}\,\mathrm{d}\tau_\mu(x).$$

Then

$$\operatorname{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\right) = \lambda^2 \int_{-\infty}^{\infty} \frac{x}{x^2+1}\, \mathrm{d}\mathbb{D}_\lambda\tau_\mu(x).$$

PROPOSITION 3. *Assume that a probability measure μ has mean zero and the following Maassen representation*

$$F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1}{x-z}\, \mathrm{d}\tau_\mu(x)$$

with the measure τ_μ such that the absolute first moment exists (that is, $\int_{-\infty}^{\infty} |x|\, \mathrm{d}\tau_\mu(x) < \infty$). Then for $\lambda > 0$

$$\left|\operatorname{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\right)\right| \le \lambda^3 \int_{-\infty}^{\infty} |x|\, \mathrm{d}\tau_\mu(x).$$

In particular, for $\lambda \to 0$

$$\operatorname{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\right) = o(\lambda^2).$$

Proof. From Proposition 2 we obtain that

$$\operatorname{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\right) = \lambda^2 \int_{-\infty}^{\infty} \frac{x}{x^2+1}\, \mathrm{d}\mathbb{D}_\lambda\tau_\mu(x).$$

Since

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \frac{x}{x^2+1}\, \mathrm{d}\mathbb{D}_\lambda\tau_\mu(x)\right| = \left|\int_{-\infty}^{\infty} \frac{\lambda x}{\lambda^2 x^2+1}\, \mathrm{d}\tau_\mu(x)\right| &\le \lambda \int_{-\infty}^{\infty} \left|\frac{x}{\lambda^2 x^2+1}\right| \mathrm{d}\tau_\mu(x)\\ &\le \lambda \int_{-\infty}^{\infty} |x|\, \mathrm{d}\tau_\mu(x),\end{aligned}$$

we have

$$\left|\operatorname{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\right)\right| \le \lambda^3 \int_{-\infty}^{\infty} |x|\, \mathrm{d}\tau_\mu(x). \quad \blacksquare$$

2.2. Free convolution. We are going to recall some basic facts on the free convolution [VDN], [HiP].

The free convolution $\mu_1 \boxplus \mu_2$ is linearized by the series of *free cumulants* $R^{\boxplus}_{\mu_1}(n)$ and $R^{\boxplus}_{\mu_2}(n)$. These series are related to the respective measures by the Speicher [Sp] moment-cumulant formulae

$$m_{\mu_i}(n) = \int x^n\, \mathrm{d}\mu_i(x) = \sum_{\pi\in\mathrm{NC}(n)} R^{\boxplus}_{\mu_i}(\pi) = \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=\{B_1,\ldots,B_k\}}} \prod_{j=1}^{k} R^{\boxplus}_{\mu_i}(|B_j|), \tag{2.5}$$

where $\mathrm{NC}(n)$ is the set of noncrossing partitions of the set $\{1,\ldots,n\}$. Solving the above equation for $R^{\boxplus}_{\mu_i}(n)$ we obtain

$$R^{\boxplus}_{\mu_i}(n) = m_{\mu_i}(n) - \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi\neq(1,\ldots,n)}} R^{\boxplus}_{\mu_i}(\pi) = m_{\mu_i}(n) - \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi\neq(1,\ldots,n)}} \prod_{i=1}^{k} R^{\boxplus}_{\mu_i}(|B_i|), \tag{2.6}$$

where $(1,\ldots,n)$ denotes the partition with one block containing all the points.

REMARK 6. The first free cumulant of a probability measure μ is equal to the mean of μ. The second free cumulant of a probability measure μ is equal to the variance of μ, $R^{\boxplus}_{\mu}(2) = \sigma^2_{\mu}$.

The cumulants are associated to the respective $R^{\boxplus}$-transform, which is a function analytic in a neighbourhood of zero U through the relation

$$R^{\boxplus}_{\mu}(z) = \sum_{n=1}^{\infty} R^{\boxplus}_{\mu}(n) z^{n-1}, \quad z \in U.$$

In terms of the $R^{\boxplus}$-transform this relation corresponds to the following relation

$$\frac{1}{G_{\mu}(z)} = z - R^{\boxplus}_{\mu}(G_{\mu}(z)), \tag{2.7}$$

where $G_{\mu}(z)$ is the Cauchy transform of the measure μ. The fact, that $R^{\boxplus}$-transform is analytic in a neighbourhood of zero follows from Bercovici and Voiculescu's result [BV1].

2.3. Conditionally free convolution and product of states. In the papers [BLS], [BS1] the authors present the construction of the conditionally free products and convolutions. We recall their definitions and essential properties.

In analogy to the free convolution $\mu_1 \boxplus \mu_2$, which is linearized by the series of free cumulants $R^{\boxplus}_{\mu_1}(n)$ and $R^{\boxplus}_{\mu_2}(n)$, the conditionally free convolution of pairs of measures $(\mu_1, \nu_1) \boxed{c}$ (μ_2, ν_2) is linearized by pairs of series $(R^{\boxed{c}}_{(\mu_1,\nu_1)}(n), R^{\boxplus}_{\nu_1}(n))$ and $(R^{\boxed{c}}_{(\mu_2,\nu_2)}(n), R^{\boxplus}_{\nu_2}(n))$. We call $(R^{\boxed{c}}_{(\mu_i,\nu_i)}(n)$ the *conditionally free cumulants.*

These series are related to the respective measures by the moment-cumulant formulae

$$m_{\nu_i}(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi = (\pi_1, \ldots, \pi_k)}} \prod_{l=1}^{k} R^{\boxplus}_{\nu_i}(|\pi_l|), \tag{2.8}$$

$$m_{\mu_i}(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi = (\pi_1, \ldots, \pi_k)}} \prod_{\pi_l \text{ outer}} R^{\boxed{c}}_{(\mu_i,\nu_i)}(|\pi_l|) \prod_{\pi_l \text{ inner}} R^{\boxplus}_{\nu_i}(|\pi_l|), \tag{2.9}$$

where a block π_l is called inner when there exists another block π_j with $a, b \in \pi_j$ with the property that $a < p < b$ for all $p \in \pi_l$. All blocks which are not enveloped in such a way are called outer. Equivalently one can say that outer blocks π_j have depth $d(\pi_j) = 0$ and inner π_l have $d(\pi_l) > 0$.

The cumulants are associated to respective R-transforms through the relations

$$R^{\boxplus}_{\nu_i}(z) = \sum_{n=1}^{\infty} R^{\boxplus}_{\nu_i}(n) z^{n-1}, \qquad R^{\boxed{c}}_{(\mu_i,\nu_i)}(z) = \sum_{n=1}^{\infty} R^{\boxed{c}}_{(\mu_i,\nu_i)}(n) z^{n-1}.$$

In terms of the R-transforms the relations (2.8) and (2.9) correspond to the following relations

$$\frac{1}{G_{\mu_i}(z)} = z - R^{\boxed{c}}_{(\mu_i,\nu_i)}(G_{\nu_i}(z)), \tag{2.10}$$

$$\frac{1}{G_{\nu_i}(z)} = z - R^{\boxplus}_{\nu_i}(G_{\nu_i}(z)), \tag{2.11}$$

where $G_\mu(z)$ is the Cauchy transform of the measure μ. The functions $R^{\boxed{c}}_{(\mu_i,\nu_i)}(z), R^{\boxplus}_{\nu_i}(z)$ are analytic in some neighbourhood of zero.

The moments of the N-fold conditionally free convolution can be calculated from the formula

$$m_{\mu_i}(n) = \sum_{\substack{\pi\in \mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} \prod_{\pi_l \text{ outer}} \Big[\sum_{i=1}^{N} R^{\boxed{c}}_{(\mu_i,\nu_i)}(|\pi_l|)\Big] \prod_{\pi_l \text{ inner}} \Big[\sum_{i=1}^{N} R^{\boxplus}_{\nu_i}(|\pi_l|)\Big].$$

The conditionally free convolution of pairs of probability measures (μ_i,ν_i) is a single probability measure μ, however, as noted in [BLS], if we associate to it the free convolution $\nu = \nu_1 \boxplus \cdots \boxplus \nu_N$, we obtain an associative convolution of pairs of measures. The associativity property allows us to reduce the N-fold convolution to the binary convolution, thus

$$(\mu,\nu) = (\boxed{c_N}(\mu_i,\nu_i), \boxplus\nu_i) = (\mu_1,\nu_1)\,\boxed{c}\,(\mu_2,\nu_2)\,\boxed{c}\cdots\boxed{c}\,(\mu_N,\nu_N).$$

3. Definition and properties of $\mathbb{T}$-transformation. By the Nevanlinna theorem, (Theorem 1), a function $F(z)$ is the reciprocal of the Cauchy transform of a probability measure if and only if there exists a positive measure ρ and a real number $a = \mathrm{Re}(F(i))$ such for every $z \in \mathbb{C}^+$

$$F(z) = a + z + \int_{-\infty}^{\infty} \frac{1 + x\,z}{x - z}\,\mathrm{d}\rho(x).$$

PROPOSITION 4. *Let μ be a probability measure on the real line $\mathbb{R}$. Let t be a real number and s be a positive real number and let $\mathbb{T} = (t,s)$. The function $G_{\mu^{(\mathbb{T})}}(z)$ defined for $z \in \mathbb{C}^+$, a real number t and a non-negative number s by the formula*

$$\frac{1}{G_{\mu^{(\mathbb{T})}}(z)} = t\,a + z + s\int_{-\infty}^{\infty} \frac{1 + x\,z}{x - z}\,\mathrm{d}\rho(x) = t\,a + z + \int_{-\infty}^{\infty} \frac{1 + x\,z}{x - z}\,\mathrm{d}(s\rho)(x)$$

is the reciprocal of the Cauchy transform of a unique probability measure which we will denote by $U^{\mathbb{T}}\mu$ or $\mu^{(\mathbb{T})}$.

REMARK 7. This construction generalizes the t-transformation of measures, where $t = s > 0$ we have $U_t\mu = U^{\mathbb{T}}\mu$, see [BW1], [BW2], [Wo1].

REMARK 8. Lemma 2 shows that for symmetric measures the Nevanlinna constant is equal to zero. It implies that for symmetric measures the $\mathbb{T}$-transformation is equal to the t- and the (a,b)-transformation.

COROLLARY 1. *We obtain the following expression for the Cauchy transform of the $\mathbb{T}$-transformation of a measure μ*

$$\frac{1}{G_{\mu^{(\mathbb{T})}}(z)} = \frac{s}{G_\mu(z)} + z(1-s) + a(t-s), \quad z \in \mathbb{C}^+$$

where $a = \mathrm{Re}\big(\frac{1}{G_\mu(i)}\big)$ is the so-called Nevanlinna constant of the measure μ.

Proof. For $\mathbb{T}=(t,s)$ from the Nevanlinna theorem it follows that for $z\in\mathbb{C}^+$

$$\begin{aligned}\frac{1}{G_{\mu^{(\mathbb{T})}}(z)} &= t\,a+z+s\int_{-\infty}^{\infty}\frac{1+x\,z}{x-z}\,\mathrm{d}\rho(x)\\ &= s\bigg(a+z+\int_{-\infty}^{\infty}\frac{1+x\,z}{x-z}\,\mathrm{d}\rho(x)\bigg)+a(t-s)+(1-s)z\\ &= \frac{s}{G_\mu(z)}+z(1-s)+a(t-s)=\frac{s}{G_\mu(z)}+(1-s)G_{\delta_{-\frac{a(t-s)}{1-s}}}(z). \ \blacksquare\end{aligned}$$

DEFINITION 1. For $\mathbb{T}=(t,s)\in\mathbb{R}\times\mathbb{R}_+$ the measure $\mu^{(\mathbb{T})}$ will be called the $\mathbb{T}$-transformation of the measure μ and the transformation $U^{\mathbb{T}}:\mu\to\mu^{(\mathbb{T})}$ will be called the $\mathbb{T}$-transformation. We have for $z\in\mathbb{C}^+$

$$\frac{1}{G_{U^{\mathbb{T}}(\mu)}(z)}=\frac{s}{G_\mu(z)}+z(1-s)-\mathrm{Re}\bigg(\frac{1}{G_\mu(i)}\bigg)(s-t). \tag{3.1}$$

REMARK 9. The Dirac measure δ_0 is the only invariant measure of the $\mathbb{T}$-transformation.

REMARK 10. For $\mathbb{T}=(t,s)$ we have the following relation between the Nevanlinna constant of a measure μ and of its $\mathbb{T}$-transformation

$$\mathrm{Re}\bigg(\frac{1}{G_{U^{\mathbb{T}}\mu}(i)}\bigg)=t\;\mathrm{Re}\bigg(\frac{1}{G_\mu(i)}\bigg).$$

Now we would like to describe the change of moments, which is done by our transformation. Let μ be a compactly supported probability measure and

$$m_\mu(n)=\int x^n\,\mathrm{d}\mu(x),\quad m_{\mu^{(\mathbb{T})}}(n)=\int x^n\,\mathrm{d}\mu^{(\mathbb{T})}(x).$$

LEMMA 3. *For any positive integer n we have*

(1) for the original measure

$$\begin{aligned}m_\mu(n)=\frac{1}{s}\,m_{\mu^{(\mathbb{T})}}(n)&+\bigg(\frac{1}{s}-1\bigg)\sum_{k=1}^{n-1}m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k)\\ &-\mathrm{Re}\bigg(\frac{1}{G_\mu(i)}\bigg)\bigg(1-\frac{t}{s}\bigg)\sum_{k=0}^{n-1}m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k-1),\end{aligned}$$

(2) for the deformed measure

$$\begin{aligned}m_{\mu^{(\mathbb{T})}}(n)=s\,m_\mu(n)&+(s-1)\sum_{k=1}^{n-1}m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k)\\ &-\mathrm{Re}\bigg(\frac{1}{G_\mu(i)}\bigg)(t-s)\sum_{k=0}^{n-1}m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k-1).\end{aligned}$$

Proof. We use the relation between the Cauchy transform of a measure μ and the generating function of its moments (2.1)

$$M_\mu(z)=\frac{1}{z}G_\mu\bigg(\frac{1}{z}\bigg)$$

for z in some neighbourhood of 0. From the definition of the $\mathbb{T}$-transformation (3.1) we have

$$\frac{1}{\frac{1}{z}M_{\mu^{(\mathbb{T})}}(\frac{1}{z})} = \frac{s}{\frac{1}{z}M_{\mu}(\frac{1}{z})} + z(1-s) - \mathrm{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t),$$

$$M_\mu(z) = s\,M_{\mu^{(\mathbb{T})}}(z) + \left((1-s) - \mathrm{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t)z\right)M_{\mu^{(\mathbb{T})}}(z)M_\mu(z),$$

and using the Leibniz formula for differentiation we obtain the formula at $z=0$:

$$\frac{d^n}{d\,z^n}\frac{M_\mu(0)}{n!} = s\,\frac{d^n}{d\,z^n}\frac{M_{\mu^{(\mathbb{T})}}(0)}{n!} + (1-s)\sum_{k=0}^{n}\frac{d^k}{d\,z^k}\frac{M_\mu(0)}{k!}\,\frac{d^{n-k}}{d\,z^{n-k}}\frac{M_{\mu^{(\mathbb{T})}}(0)}{(n-k)!}$$
$$- \mathrm{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t)\sum_{k=0}^{n-1}\frac{d^k}{d\,z^k}\frac{M_\mu(0)}{k!}\,\frac{d^{n-k-1}}{d\,z^{n-k-1}}\frac{M_{\mu^{(\mathbb{T})}}(0)}{(n-k-1)!}$$

which implies that

$$\begin{aligned} m_\mu(n) &= s\,m_{\mu^{(\mathbb{T})}}(n) + (1-s)\sum_{k=0}^{n} m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k) \\ &\quad - \mathrm{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t)\sum_{k=1}^{n-1} m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k-1) \\ &= \frac{1}{s}\,m_{\mu^{(\mathbb{T})}}(n) + \left(\frac{1}{s}-1\right)\sum_{k=1}^{n-1} m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k) \\ &\quad - \mathrm{Re}\left(\frac{1}{G_\mu(i)}\right)\left(1-\frac{t}{s}\right)\sum_{k=0}^{n-1} m_\mu(k)\,m_{\mu^{(\mathbb{T})}}(n-k-1). \end{aligned}$$

It is obvious that the second formula is equivalent to the first one. ■

PROPOSITION 5. *Let* $\mathbb{T}=(t,s), \widetilde{\mathbb{T}}=(\tilde{t},\tilde{s})$. *For a probability measure* μ, *real numbers* $t,\tilde{t}$ *and real nonnegative numbers* $s,\tilde{s}$ *the following properties are satisfied:*

(1) $U^{\mathbb{T}}$ *is a multiplicative semigroup:*

$$U^{\mathbb{T}}U^{\widetilde{\mathbb{T}}}(\mu) = U^{\mathbb{T}\circ\widetilde{\mathbb{T}}}(\mu)$$

where $\mathbb{T}\circ\widetilde{\mathbb{T}} = (t,s)\circ(\tilde{t},\tilde{s}) = (t\,\tilde{t}, s\,\tilde{s})$;

(2) for $t\neq 0, s\neq 0$ *let* $\hat{T}=(1/t,1/s)$. *Then* $U^{\mathbb{T}}$ *and* $U^{\hat{\mathbb{T}}}$ *are inverses of each other;*

(3) dilation of measures does not commute with the transformation $U^{\mathbb{T}}$

$$\mathbb{D}_\lambda\,U^{\mathbb{T}}\mu \neq U^{\mathbb{T}}\,\mathbb{D}_\lambda\mu\,;$$

(4) $U^{\mathbb{T}}(\delta_a) = \delta_{ta}$.

Proof. (1) For $z \in \mathbb{C}^+$ we have

$$\begin{aligned}\frac{1}{G_{U^{\mathbb{T}}U^{\tilde{\mathbb{T}}}(\mu)}(z)} &= \frac{s}{G_{U^{\tilde{\mathbb{T}}}(\mu)}(z)} + z(1-s) - \operatorname{Re}\left(\frac{1}{G_{U^{\tilde{\mathbb{T}}}(\mu)}(i)}\right)(s-t)\\ &= s\left(\frac{\tilde{s}}{G_\mu(z)} + z(1-\tilde{s}) - \operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)(\tilde{s}-\tilde{t})\right) + z(1-s) - \operatorname{Re}\left(\frac{1}{G_{U^{\tilde{\mathbb{T}}}(\mu)}(i)}\right)(s-t)\\ &= \frac{s\,\tilde{s}}{G_\mu(z)} + z(s - s\,\tilde{s} + 1 - s) - s\operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)(\tilde{s}-\tilde{t}) - \operatorname{Re}\left(\frac{1}{G_{U^{\tilde{\mathbb{T}}}(\mu)}(i)}\right)(s-t).\end{aligned}$$

By Lemma 10 we have

$$\operatorname{Re}\left(\frac{1}{G_{U^{\tilde{\mathbb{T}}}(\mu)}(i)}\right) = \tilde{t}\operatorname{Re}\left(\frac{1}{G_\mu(i)}\right),$$

hence we obtain

$$\frac{1}{G_{U^{\mathbb{T}}U^{\tilde{\mathbb{T}}}(\mu)}(z)} = \frac{s\,\tilde{s}}{G_\mu(z)} + z(1-s\,\tilde{s}) - \operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)(s\,\tilde{s} - t\,\tilde{t}) = \frac{1}{G_{U^{\mathbb{T}\circ\tilde{\mathbb{T}}}(\mu)}(z)},$$

so $U^{\mathbb{T}}$ is a multiplicative semigroup.

(2) This is obvious.

(3) By Remark 5 showing the relation between the Cauchy transforms of a measure μ and its dilation we have

$$G_{\mathbb{D}_\lambda\mu}(z) = \frac{1}{\lambda}G_\mu\left(\frac{z}{\lambda}\right),$$

hence

$$\operatorname{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\right) = \lambda\operatorname{Re}\left(\frac{1}{G_\mu(\frac{i}{\lambda})}\right),$$

which implies

$$\frac{1}{G_{U^{\mathbb{T}}\,\mathbb{D}_\lambda\mu}(z)} = \frac{\lambda\,s}{G_\mu(\frac{z}{\lambda})} + z(1-s) - \lambda\operatorname{Re}\left(\frac{1}{G_\mu(\frac{i}{\lambda})}\right)(s-t).$$

On the other hand

$$\begin{aligned}\frac{1}{G_{\mathbb{D}_\lambda\,U^{\mathbb{T}}\mu}(z)} &= \frac{\lambda}{G_{U^{\mathbb{T}}\mu}(\frac{z}{\lambda})} = \lambda\left(\frac{s}{G_\mu(\frac{z}{\lambda})} + \frac{z}{\lambda}(1-s) - \operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t)\right)\\ &= \frac{\lambda\,s}{G_\mu(\frac{z}{\lambda})} + z(1-s) - \lambda\operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t)\end{aligned}$$

and because in general

$$\operatorname{Re}\left(\frac{1}{G_\mu(\frac{i}{\lambda})}\right) \neq \operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)$$

dilation and $\mathbb{T}$-transformation do not commute. Instead we have the following relation

$$\frac{1}{G_{U^{\mathbb{T}}\,\mathbb{D}_\lambda\mu}(z)} - \frac{1}{G_{\mathbb{D}_\lambda\,U^{\mathbb{T}}\mu}(z)} = \lambda(s-t)\left(\operatorname{Re}\left(\frac{1}{G_\mu(\frac{i}{\lambda})}\right) - \operatorname{Re}\left(\frac{1}{G_\mu(i)}\right)\right).$$

(4) We have

$$\frac{1}{G_{U^{\mathbb{T}}\delta_a}(z)} = z - t\,a = \frac{1}{G_{\delta_{ta}}(z)}. \ \blacksquare$$

EXAMPLE 1. We calculated the $\mathbb{T}$-transformation of a probability measure, which is supported in δ_0 and δ_1. We will use these results later on. Let

$$\mu = (1-p)\delta_0 + p\,\delta_1, \quad p \geq 0.$$

Then

$$U^{\mathbb{T}}(\mu) = P\,\delta_A + (1-P)\delta_B$$

with

$$\begin{aligned}
A &= \frac{(s-1)p^3 + (3-2s)p^2 + (s+t-4)p + 2}{2(2-2p+p^2)} \\
&\quad - \frac{\sqrt{((s-1)p^3 + (3-2s)p^2 + (s+t-4)p + 2)^2 - 4p(s-t)(p-1)(2-2p+p^2)}}{2(2-2p+p^2)}, \\
B &= \frac{(s-1)p^3 + (3-2s)p^2 + (s+t-4)p + 2}{2(2-2p+p^2)} \\
&\quad + \frac{\sqrt{((s-1)p^3 + (3-2s)p^2 + (s+t-4)p + 2)^2 - 4p(s-t)(p-1)(2-2p+p^2)}}{2(2-2p+p^2)}, \\
P &= \frac{1}{2} + \frac{2 - p^3(1+s) + p^2(2s+3) - (4+s+t)p}{2\sqrt{((s-1)p^3 + (3-2s)p^2 + (s+t-4)p + 2)^2 - 4p(s-t)(p-1)(2-2p+p^2)}}.
\end{aligned}$$

4. $\mathbb{T}$-deformation of convolution. Consider two probability measures μ, ν on the real line, a real number t and a positive number s and a convolution $\oplus$ (for instance classical convolution, free or Boolean convolution). We define the $\mathbb{T}$-deformation of the convolution $\oplus$ by the relation

$$\mu \oplus_{\mathbb{T}} \nu = (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\mu) \oplus U^{\mathbb{T}}(\nu)). \tag{4.1}$$

PROPOSITION 6. *If a given convolution $\oplus$ is associative, then for $t \in \mathbb{R}$, $s > 0$ the $\mathbb{T}$-deformation of the convolution $\oplus$ is also associative.*

Proof. For measures μ, ν, θ we have

$$\begin{aligned}
(\mu \oplus_{\mathbb{T}} \nu) \oplus_{\mathbb{T}} \theta &= (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\mu \oplus_{\mathbb{T}} \nu) \oplus U^{\mathbb{T}}(\theta)) \\
&= (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}((U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\mu) \oplus U^{\mathbb{T}}(\nu))) \oplus U^{\mathbb{T}}(\theta)) \\
&= (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\mu) \oplus U^{\mathbb{T}}(\nu) \oplus U^{\mathbb{T}}(\theta)) = \mu \oplus_{\mathbb{T}} (\nu \oplus_{\mathbb{T}} \theta). \ \blacksquare
\end{aligned}$$

Suppose that for a given convolution $\oplus$ we have a cumulant transformation, that is, a map $\mu \mapsto \{R_\mu(n)\}$ which satisfies the following linearization property

$$R^{\oplus}_{\mu \oplus \nu}(n) = R^{\oplus}_{\mu}(n) + R^{\oplus}_{\nu}(n).$$

Then there exist cumulants for $\mathbb{T}$-transformation. Indeed, we have

Remark 11. If the convolution $\oplus$ has cumulant transformation R then for any $t \in \mathbb{R}$, $s > 0$ and any positive integer n define

$$R^{\oplus_{\mathbb{T}}}_{\mu}(n) := R^{\oplus}_{U^{\mathbb{T}}(\mu)}(n).$$

Then

$$R^{\oplus_{\mathbb{T}}}_{\mu \oplus_{\mathbb{T}} \nu}(n) = R^{\oplus_{\mathbb{T}}}_{\mu}(n) + R^{\oplus_{\mathbb{T}}}_{\nu}(n).$$

But as in the case of t-transformation cumulants are not uniquely defined, since for any sequence $\sigma = (\sigma_n)$ the new cumulant defined as $\widetilde{R^{\oplus_{\mathbb{T}}}}_{\mu}(n) = \sigma_n R^{\oplus_{\mathbb{T}}}_{\mu}(n)$ also has the linearization property.

5. Central limit theorem for $\mathbb{T}$-deformed free convolution. Now we will study the limit of a sequence of measures of the form

$$\mathbb{D}_\lambda(\mu) \boxplus_{\mathbb{T}} \ldots \boxplus_{\mathbb{T}} \mathbb{D}_\lambda(\mu).$$

Lemma 4. *Let μ be a compactly supported probability measure on the real line with mean zero and the following Maassen representation*

$$\frac{1}{G_\mu(z)} = z + \int \frac{1}{x-z}\, \mathrm{d}\tau_\mu(x), \quad z \in \mathbb{C}^+$$

with the measure τ_μ such that its absolute first moment exists. Then

$$m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(n) = \begin{cases} o(\lambda^2), & n = 1, \\ s\,\lambda^2\, m_\mu(2) + o(\lambda^4), & n = 2, \\ s\,\lambda^n\, m_\mu(n) + o(\lambda^{n-1}), & n \geq 3. \end{cases} \tag{5.1}$$

Proof. Using the relation between moments of measure and of its dilation (2.4) and Lemma 3 we obtain

$$\begin{aligned} m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(n) &= s\, m_{\mathbb{D}_\lambda \mu}(n) + (s-1)\sum_{k=1}^{n-1} m_{\mathbb{D}_\lambda \mu}(k) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(n-k) \\ &\quad - \mathrm{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda \mu}(i)}\right)(t-s)\sum_{k=0}^{n-1} m_{\mathbb{D}_\lambda \mu}(k) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(n-k-1) \\ &= s\,\lambda^n\, m_\mu(n) + (s-1)\sum_{k=1}^{n-1} \lambda^k\, m_\mu(k) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(n-k) \\ &\quad - \mathrm{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda \mu}(i)}\right)(t-s)\sum_{k=0}^{n-1} \lambda^k\, m_\mu(k) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(n-k-1). \end{aligned}$$

By Proposition 3 we have

$$\mathrm{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda \mu}(i)}\right) = o(\lambda^2).$$

In particular for $\lambda \to 0$ and for $n = 1$ we have

$$m_{U^{\mathbb{T}}(\mathbb{D}_\lambda \mu)}(1) = s\,\lambda\, m_\mu(1) - \mathrm{Re}\left(\frac{1}{G_{\mathbb{D}_\lambda \mu}(i)}\right)(t-s) = o(\lambda^2).$$

For $n = 2$ we have

$$\begin{aligned} m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(2) &= s\, m_{\mathbb{D}_\lambda\mu}(2) + (s-1) m_{\mathbb{D}_\lambda\mu}(1) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1) \\ &\quad - \operatorname{Re}\biggl(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\biggr)(t-s)(m_{\mathbb{D}_\lambda\mu}(0) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1) + m_{\mathbb{D}_\lambda\mu}(1) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(0)) \\ &= s\,\lambda^2\, m_\mu(2) + \biggl(\operatorname{Re}\biggl(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\biggr)\biggr)^2 (t-s)^2 = s\,\lambda^2\, m_\mu(2) + o(\lambda^4). \end{aligned}$$

Hence by induction for $\lambda \to 0$ and n

$$\begin{aligned} & m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(n+1) = s\,\lambda^{n+1}\, m_\mu(n+1) + (s-1)\sum_{k=1}^{n} \lambda^k\, m_\mu(k) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(n+1-k) \\ & - \operatorname{Re}\biggl(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\biggr)(t-s)\sum_{k=0}^{n} \lambda^k\, m_\mu(k) m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(n-k) \\ & = s\,\lambda^{n+1}\, m_\mu(n+1) + (s-1)\sum_{k=1}^{n} \lambda^k\, m_\mu(k)(s\,\lambda^{n+1-k}\, m_\mu(n+1-k) + o(\lambda^{n-k})) \\ & - o(\lambda^2)(t-s)\sum_{k=0}^{n} \lambda^k\, m_\mu(k)(s\,\lambda^{n-k}\, m_\mu(n-k) + o(\lambda^{n-1-k})) \\ & = s\,\lambda^{n+1}\, m_\mu(n+1) + o(\lambda^n). \ \blacksquare \end{aligned}$$

LEMMA 5. *Let μ be a compactly supported probability measure on the real line with mean zero, variance equal to* 1 *and the Maassen measure τ such that its first absolute moment exists. Then*

$$\begin{aligned} R^{\mathbb{T}}_{\mathbb{D}_\lambda\mu}(1) &= R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1) = o(\lambda^2), \\ R^{\mathbb{T}}_{\mathbb{D}_\lambda\mu}(2) &= R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(2) = s\,\lambda^2 + o(\lambda^4), \\ R^{\mathbb{T}}_{\mathbb{D}_\lambda\mu}(k) &= R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(k) = o(\lambda^2) \quad \textit{for } k \ge 3. \end{aligned} \tag{5.2}$$

Proof. Let us observe that for a measure μ with mean zero and variance equal to 1 the measure $\mathbb{D}_\lambda\mu$ has the first moment $m_{\mathbb{D}_\lambda\mu}(1) = 0$ and the second moment $m_{\mathbb{D}_\lambda\mu}(2) = \lambda^2$. The measure $U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)$ has the first moment

$$m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1) = -\operatorname{Re}\biggl(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\biggr)(t-s) = o(\lambda^2)$$

and the second moment

$$m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(2) = s\,\lambda^2 + o(\lambda^4).$$

Because of the moment-cumulant formulae for the free convolution (2.6), for the $\mathbb{T}$-deformation of dilation of measures we have

$$R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1) = o(\lambda^2), \qquad R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(2) = s\,\lambda^2 + o(\lambda^4).$$

Moreover, for $k = 3$ by the moment-cumulant formula

$$R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(3) = m_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(3) - 3R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(2) R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1) - R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(1)^3,$$

hence by Lemma 4 we have

$$\begin{aligned}R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(3) &= \lambda^3 m_\mu(3)+o(\lambda^2)-3s\,\lambda^2\,\mathrm{Re}\biggl(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\biggr)(t-s)-\biggl(\mathrm{Re}\biggl(\frac{1}{G_{\mathbb{D}_\lambda\mu}(i)}\biggr)\biggr)^3(t-s)^3\\ &= o(\lambda^2),\end{aligned}$$

and by induction: if in π there exists a block B such that $|B|\ge 3$, then $R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(\pi) = o(\lambda^2)$. If not, there must be at least two blocks, $|B_1|+|B_2|\ge 2$, hence

$$\prod_{B_i\in\pi} R^{\boxplus}_{U^{\mathbb{T}}(\mathbb{D}_\lambda\mu)}(B_i) = o(\lambda^3).$$

This fact ends the proof. ■

THEOREM 2 (Central limit theorem for the $\mathbb{T}$-deformed free convolution). *Let $\mathbb{T}=(t,s)$ and μ be a compactly supported probability measure on the real line with mean zero, variance equal to* 1 *and the Maassen representation*

$$\frac{1}{G_\mu(z)} = z + \int \frac{1}{x-z}\,\mathrm{d}\tau_\mu(x),$$

where the first absolute moment of τ_μ exists. Then the sequence

$$\mathbb{D}_{1/\sqrt{N}}\mu \boxplus_{\mathbb{T}} \ldots \boxplus_{\mathbb{T}} \mathbb{D}_{1/\sqrt{N}}\mu$$

is weak-$\star$ convergent to the Kesten measure κ_s with parameter s.

Proof. The sequence of N-fold $\mathbb{T}$-convolution of the measure μ is of the form

$$\mu_N = \mathbb{D}_{1/\sqrt{N}}\mu \boxplus_{\mathbb{T}} \ldots \boxplus_{\mathbb{T}} \mathbb{D}_{1/\sqrt{N}}\mu = (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu \boxplus \ldots \boxplus U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu).$$

Let us denote for simplicity

$$\nu_N = \underbrace{U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu \boxplus \cdots \boxplus U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu}_{N\text{ times}}.$$

Then $\mu_N = (U^{\mathbb{T}})^{-1}\nu_N$ and because by definition for an arbitrary measure ν

$$R^{\mathbb{T}}_\nu(z) = R^{\boxplus}_{U^{\mathbb{T}}\nu}(z) = \sum_{n=1}^{\infty} R^{\boxplus}_{U^{\mathbb{T}}\nu}(n)\,z^{n-1},$$

we have

$$R^{\boxplus}_{\nu_N}(z) = R^{\boxplus}_{U^{\mathbb{T}}\mu_N}(z) = R^{\mathbb{T}}_{\mu_N}(z).$$

Because

$$R^{\boxplus}_{\nu_N}(z) = N\,R^{\boxplus}_{U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu}(z) = N\,R^{\mathbb{T}}_{\mathbb{D}_{1/\sqrt{N}}\mu}(z)$$

by Lemma 5

$$\begin{aligned}R^{\boxplus}_{\nu_N}(1) &= -N\cdot o\biggl(\frac{1}{N}\biggr) \overset{N\to\infty}{\longrightarrow} 0,\\ R^{\boxplus}_{\nu_N}(2) &= N\cdot\biggl(s\,\frac{1}{N}+o\biggl(\frac{1}{N^2}\biggr)\biggr) \overset{N\to\infty}{\longrightarrow} s,\\ R^{\boxplus}_{\nu_N}(k) &= N\cdot o\biggl(\frac{1}{N}\biggr) \overset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

and therefore

$$R^{\boxplus}_{\nu_N}(z) \to R^{\boxplus}_{\omega_s}(z) = s,$$

and ω_s is the semicircular law with mean 0 and variance s. Hence the sequence

$$\mu_N = \mathbb{D}_{1/\sqrt{N}}\mu \boxplus_{\mathbb{T}} \ldots \boxplus_{\mathbb{T}} \mathbb{D}_{1/\sqrt{N}}\mu$$

is convergent to $\xi = (U^{\mathbb{T}})^{-1}\omega_s$. Because the Cauchy transform of such a Wigner measure is equal to

$$G_{\omega_s}(z) = \frac{z - \sqrt{z^2 - 4\,s}}{2\,s}$$

we have $\mathrm{Re}\big(\frac{1}{G_{\omega_s}(i)}\big) = 0$. Therefore

$$G_{\xi}(z) = \frac{1}{\frac{\frac{1}{s}}{G_{\omega_s}(z)} + z\left(1 - \frac{1}{s}\right) - \mathrm{Re}\big(\frac{1}{G_{\omega_s}(i)}\big)\left(\frac{1}{s} - \frac{1}{t}\right)} = \frac{s}{\frac{1}{G_{\omega_s}(z)} + z(s-1)},$$

hence the measure ξ is equal to the t-transformation of the measure ω_s, which means that ξ is equal to the Kesten measure κ_s with parameter s. ∎

6. Poisson type limit theorem for $\mathbb{T}$-deformed free convolution. For a number $\lambda > 0$ consider the sequence of measures

$$\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1.$$

We define the Poisson measure as the limit (if it exists)

$$p_\lambda = \lim_{N\to\infty} \mu_N \boxplus_{\mathbb{T}} \ldots \boxplus_{\mathbb{T}} \mu_N.$$

Recall that from Example 1 it follows that the Cauchy transform of the $\mathbb{T}$-transformed measure μ_N is equal to

$$G_{U^{\mathbb{T}}(\mu)}(z) = \frac{(p^2 - 2\,p + 2)(z + p - 1)}{(p^2 - 2\,p + 2)z^2 + ((1-s)p^3 + p^2(2s-3) + (4 - s - t)p - 2)z + p(s-t)(p-1)}$$

for $p = \frac{\lambda}{N}$. The related cumulants and R-transform are defined as follows

$$R^{\mathbb{T}}_{\mu}(n) := R^{\boxplus}_{U^{\mathbb{T}}(\mu)}(n), \qquad R^{\mathbb{T}}_{\mu}(z) := \sum_{n=1}^{\infty} R^{\mathbb{T}}_{\mu}(n)\, z^{n-1} = R^{\boxplus}_{U^{\mathbb{T}}(\mu)}(z).$$

We can use cumulants $R^{\mathbb{T}}_{\mu}(n)$ and calculate

$$R^{\mathbb{T}}_{\mu_N \boxplus_{\mathbb{T}} \ldots \boxplus_{\mathbb{T}} \mu_N}(z) = N\, R^{\mathbb{T}}_{\mu_N}(z) = N\, R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)}(z).$$

By definition

$$R^{\mathbb{T}}_{\mu_N}(z) = R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)}(z) = R^{\boxplus}_{P_N\delta_{A_N} + Q_N\delta_{B_N}}(z),$$

where $P_N, A_N, Q_N = 1 - P_N, B_N$ are defined in Example 1 for $p = \frac{\lambda}{N}$. We have

$$\frac{1}{z} = G^{-1}_{U^{\mathbb{T}}(\mu_N)}(z) - R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)}(z), \tag{6.1}$$

where $G^{-1}_{U^{\mathbb{T}}(\mu_N)}(z)$ is the inverse of $G_{U^{\mathbb{T}}(\mu_N)}(z)$ with respect to composition of functions. To calculate it we have to solve a quadratic equation. A straightforward calculation gives

$$G^{-1}_{U^{\mathbb{T}}(\mu_N)}(z) = \frac{2-2\,p+p^2+2\,z-4\,p\,z+3\,p^2\,z-p^3\,z+p\,s\,z-2\,p^2\,s\,z+p^3\,s\,z+p\,t\,z}{2\,z(2-2\,p+p^2)} - \frac{\sqrt{SQRT_p}}{2\,z(2-2\,p+p^2)},$$

with

$$SQRT_p = (-2+2\,p-p^2-2\,z+4\,p\,z-3\,p^2\,z+p^3\,z-p\,s\,z+2\,p^2\,s\,z-p^3\,s\,z-p\,t\,z)^2 - 4(2\,z-2\,p\,z+p^2\,z)\,(2-4\,p+3\,p^2-p^3-p\,s\,z+p^2\,s\,z+p\,t\,z-p^2\,t\,z).$$

It gives for the respective Voiculescu's $R^{\boxplus}$-transform

$$R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)}(z) = \frac{2-2\,p+p^2+2\,z-4\,p\,z+3\,p^2\,z-p^3\,z+p\,s\,z-2\,p^2\,s\,z+p^3\,s\,z+p\,t\,z}{2\,z(2-2\,p+p^2)} - \frac{1}{z} - \frac{\sqrt{SQRT_p}}{2\,z(2-2\,p+p^2)}.$$

For $p=\frac{\lambda}{N}$ it is equal to

$$\begin{aligned} R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)}(z) &= \frac{-2\,N^3+2\,N^3\,z+2\,N^2\,\lambda-4\,N^2\,z\,\lambda+N^2\,s\,z\,\lambda}{2\,N\,z(2\,N^2-2\,N\,\lambda+\lambda^2)} \\ &\quad + \frac{N^2\,t\,z\,\lambda-N\,\lambda^2+3\,N\,z\,\lambda^2-2\,N\,s\,z\,\lambda^2-z\,\lambda^3+s\,z\,\lambda^3}{2\,N\,z(2\,N^2-2\,N\,\lambda+\lambda^2)} \\ &\quad - \frac{\sqrt{SQRT_{\frac{\lambda}{N}}}}{2\,N\,z(2\,N^2-2\,N\,\lambda+\lambda^2)}, \end{aligned}$$

with

$$\begin{aligned} SQRT_{\frac{\lambda}{N}} &= 4(-1+z)^2N^6+4(-2+6\,z+s\,z+t\,z-4\,z^2+3\,s\,z^2-t\,z^2)\lambda\,N^5 \\ &+(8-32\,z-12\,s\,z-4\,t\,z+28\,z^2-32\,s\,z^2+s^2\,z^2+8\,t\,z^2+2\,s\,t\,z^2+t^2\,z^2)\lambda^2N^4 \\ &-2(2-12\,z-7\,s\,z-t\,z+14\,z^2-19\,s\,z^2+2\,s^2\,z^2+3\,t\,z^2+2\,s\,t\,z^2)\lambda^3N^3 \\ &+(1-10\,z-8\,s\,z+17\,z^2-26\,s\,z^2+6\,s^2\,z^2+2\,t\,z^2+2\,s\,t\,z^2)\lambda^4N^2 \\ &-2\,z(-1-s+3\,z-5\,s\,z+2\,s^2\,z)\lambda^5N+(s-1)^2\,z^2\,\lambda^6. \end{aligned}$$

For the measure

$$\mu^N := \mu_N \boxplus_{\mathbb{T}} \dots \boxplus_{\mathbb{T}} \mu_N = (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\mu_N)\boxplus\cdots\boxplus U^{\mathbb{T}}(\mu_N))$$

we get

$$\begin{aligned} R^{\boxplus}_{U^{\mathbb{T}}(\mu^N)}(z) &= R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)\boxplus\cdots\boxplus U^{\mathbb{T}}(\mu_N)}(z) = N\,R^{\boxplus}_{U^{\mathbb{T}}(\mu_N)}(z) \\ &= \frac{-2\,N^3+2\,N^3\,z+2\,N^2\,\lambda-4\,N^2\,z\,\lambda+N^2\,s\,z\,\lambda+N^2\,t\,z\,\lambda-N\,\lambda^2}{2\,z(2\,N^2-2\,N\,\lambda+\lambda^2)} \\ &+ \frac{3\,N\,z\,\lambda^2-2\,N\,s\,z\,\lambda^2-z\,\lambda^3+s\,z\,\lambda^3-\sqrt{SQRT_{\frac{\lambda}{N}}}}{2\,z(2\,N^2-2\,N\,\lambda+\lambda^2)}. \end{aligned}$$

Taking the limit as $N \to \infty$ we obtain

$$R^{\boxplus}_{U^{\mathbb{T}}(p_\lambda)}(z) = \lim_{N\to\infty} R^{\boxplus}_{U^{\mathbb{T}}(\mu^N)}(z) = \frac{(z(s-t)+(s+t))}{2}\frac{\lambda}{1-z},$$

which gives that

$$\begin{aligned} R^{\mathbb{T}}_{p_\lambda}(z) = R^{\boxplus}_{U^{\mathbb{T}}(p_\lambda)}(z) &= \frac{(z(s-t)+(s+t))}{2}\frac{\lambda}{1-z} \\ &= \frac{(z(s-t)+(s+t))}{2}\lambda\sum_{n=1}^{\infty} z^{n-1} = \frac{(s+t)}{2}\lambda + \lambda\, s\sum_{n=2}^{\infty} z^{n-1}, \end{aligned}$$

and because

$$R^{\mathbb{T}}_{\mu}(z) = \sum_{n=1}^{\infty} R^{\mathbb{T}}_{\mu}(n)\, z^{n-1}$$

we obtain

$$R^{\mathbb{T}}_{p_\lambda}(n) = \begin{cases} \dfrac{(s+t)}{2}\lambda & \text{if } n=1, \\ \lambda\, s & \text{if } n \geq 2. \end{cases}$$

Using the relation (6.1) once again we can calculate the Cauchy transform of the measure $U^{\mathbb{T}}(p_\lambda)$. Indeed, we have

$$\frac{1}{G_{U^{\mathbb{T}}(p_\lambda)}(z)} = z - R^{\boxplus}_{U^{\mathbb{T}}(p_\lambda)}(G_{U^{\mathbb{T}}(p_\lambda)}(z)),$$

hence

$$\frac{1}{G_{U^{\mathbb{T}}(p_\lambda)}(z)} = z - \frac{(G_{U^{\mathbb{T}}(p_\lambda)}(z)(s-t)+(s+t))}{2}\frac{\lambda}{1-G_{U^{\mathbb{T}}(p_\lambda)}(z)},$$

which implies that

$$G_{U^{\mathbb{T}}(p_\lambda)}(z) = \frac{2+2\,z-s\,\lambda-t\,\lambda-\sqrt{(2-2\,z+s\,\lambda+t\,\lambda)^2-16\,s\,\lambda}}{2(2\,z+s\,\lambda-t\,\lambda)}.$$

Because

$$(U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(p_\lambda)) = p_\lambda$$

we can apply the definition of the $\mathbb{T}$-transformation (3.1) and obtain

$$\frac{1}{G_{p_\lambda}(z)} = \frac{1}{s\,G_{U^{\mathbb{T}}(p_\lambda)}(z)} + z\left(1-\frac{1}{s}\right) - \mathrm{Re}\left(\frac{1}{G_{U^{\mathbb{T}}(p_\lambda)}(i)}\right)\left(\frac{1}{s}-\frac{1}{t}\right).$$

Since

$$\begin{aligned} A = \mathrm{Re}\left(\frac{1}{G_{U^{\mathbb{T}}(p_\lambda)}(i)}\right) &= \frac{1}{2}-\frac{s\,\lambda}{4}-\frac{t\,\lambda}{4} \\ &+\frac{1}{4}\sqrt{\frac{\lambda(12\,s-4\,t-(s+t)^2\,\lambda)+\sqrt{16(2+(s+t)\lambda)^2+\lambda^2(4(-3\,s+t)+(s+t)^2\,\lambda)^2}}{2}}, \end{aligned}$$

we obtain

$$G_{p_\lambda}(z) = \frac{st(4A(s-t)-t(-2+(2-4s)z+s\lambda+t\lambda+\sqrt{(2-2z+s\lambda+t\lambda)^2-16s\lambda}))}{4(-1+s)st^2z^2+bz+c},$$

where

$$b = 2(A(-2+4\,s)(s-t)t - s^2\,t^2\,\lambda + t^3\,\lambda + s\,t^2(2+\lambda - t\,\lambda)),$$
$$c = 2(s-t)(2\,A^2(s-t) + t^2\,\lambda - A\,t(-2+s\,\lambda + t\,\lambda)).$$

Using the Stieltjes inversion formula we can calculate the measure p_λ explicitly. The limiting measure p_λ has the following form

$$p_\lambda = \mathrm{d}\widehat{p_\lambda}(x) + \mathrm{d}\widetilde{p_\lambda}(x)(x) = w_1\delta_{z_1} + w_2\delta_{z_2} + \widetilde{p_\lambda}(x)\,\mathrm{d}x$$

where

$$\begin{aligned}
\mathrm{d}\widetilde{p_\lambda}(x) &= \frac{1}{\pi}\frac{s\,t^2\,\sqrt{16\,s\,\lambda - (2-2\,x+s\,\lambda+t\,\lambda)^2}}{4(-1+s)s\,t^2\,x^2 + b\,x + c}\,\mathrm{d}x,\\
A &= \frac{1}{2} - \frac{s\,\lambda}{4} - \frac{t\,\lambda}{4}\\
&\quad + \frac{1}{4}\sqrt{\frac{\lambda(12\,s - 4\,t - (s+t)^2\,\lambda) + \sqrt{16(2+(s+t)\lambda)^2 + \lambda^2(4(-3\,s+t)+(s+t)^2\,\lambda)^2}}{2}},\\
b &= 2(A(-2+4\,s)(s-t)t - s^2\,t^2\,\lambda + t^3\,\lambda + s\,t^2(2+\lambda - t\,\lambda)),\\
c &= 2(s-t)(2\,A^2(s-t) + t^2\,\lambda - A\,t(-2+s\,\lambda+t\,\lambda))\\
z_1 &= \frac{t(2\,A(1-2\,s)(s-t) - 2\,s\,t + (-1+s)t(s+t)\lambda)}{4(-1+s)\,s\,t^2}\\
&\quad - \frac{\sqrt{16(1-s)s^2\,t^4\,\lambda + t^2(2(A(s-t)+s\,t) + (-1+s)t(s+t)\lambda)^2}}{4(-1+s)\,s\,t^2}\\
z_2 &= \frac{t(2\,A(1-2\,s)(s-t) - 2\,s\,t + (-1+s)t(s+t)\lambda)}{4(-1+s)\,s\,t^2}\\
&\quad + \frac{\sqrt{16(1-s)s^2\,t^4\,\lambda + t^2(2(A(s-t)+s\,t) + (-1+s)t(s+t)\lambda)^2}}{4(-1+s)\,s\,t^2}\\
w_1 &= \frac{4\,A(s-t) - t(-2+(2-4\,s)z_2 + s\,\lambda + t\,\lambda + \sqrt{(2-2\,z_2+s\,\lambda+t\,\lambda)^2 - 16\,s\,\lambda})}{4(-1+s)t(z-z_2)}\\
w_2 &= \frac{4\,A(s-t) - t(-2+(2-4\,s)z_1 + s\,\lambda + t\,\lambda + \sqrt{(2-2\,z_1+s\,\lambda+t\,\lambda)^2 - 16\,s\,\lambda})}{4(-1+s)t(z-z_1)}.
\end{aligned}$$

REMARK 12. The above Poisson measure is a distribution of the non-standardized (mean is non-zero) free Meixner type, see [BB].

7. Another approach to defining deformed convolution. Let us use the notion of conditionally free convolution to define another deformed convolution $\boxed{\mathbb{T}}$, which we will call the $\mathbb{T}$-conditional convolution of measures.

DEFINITION 2. Using the $\mathbb{T}$-deformation of probability measures

$$\frac{1}{G_{U^{\mathbb{T}}(\mu)}(z)} = \frac{s}{G_\mu(z)} + z(1-s) - \mathrm{Re}\left(\frac{1}{G_\mu(i)}\right)(s-t)$$

for compactly supported probability measures μ_1, μ_2 we define the corresponding convolution

$$\mu_1 \boxed{\scriptstyle\mathbb{T}} \mu_2 = (\mu_1, U^{\mathbb{T}}\mu_1) \boxed{\scriptstyle c} (\mu_2, U^{\mathbb{T}}\mu_2).$$

Similarly to the r-deformation V_r, see [BKW], [KY1], the $\mathbb{T}$-deformation does not behave well in this application. Indeed, we have the following

LEMMA 6. *In general, for $t \in \mathbb{R}$, $s \in \mathbb{R}_+$ there exist probability measures μ_1, μ_2 such that*

$$U^{\mathbb{T}}(\mu_1 \boxed{\scriptstyle\mathbb{T}} \mu_2) \neq U^{\mathbb{T}}\mu_1 \boxplus U^{\mathbb{T}}\mu_2.$$

Proof. Let us take the following measure

$$\mu = \frac{1}{2}(\delta_0 + \delta_1).$$

By Example 1 we have that the Cauchy transform of $\mathbb{T}$-transformation of μ is equal to

$$G_{U^{\mathbb{T}}\mu}(z) = \frac{10\,z - 5}{10\,z^2 - (5 + s + 4\,t)z - 2(s - t)}.$$

The Voiculescu $R^{\boxplus}$-transform of $U^{\mathbb{T}}\mu$ is equal to

$$R^{\boxplus}_{U^{\mathbb{T}}\mu}(z) = G^{-1}_{U^{\mathbb{T}}\mu}(z) - \frac{1}{z},$$

and a simple calculation shows that

$$G^{-1}_{U^{\mathbb{T}}\mu}(z) = \frac{10 + (5 + s + 4\,t)z - \sqrt{40\,z(-5 + 2\,s\,z - 2\,t\,z) + (10 + (5 + s + 4\,t)\,z)^2}}{20\,z},$$

hence

$$R^{\boxplus}_{U^{\mathbb{T}}\mu}(z) = \frac{-10 + (5 + s + 4\,t)z - \sqrt{40\,z(-5 + 2\,s\,z - 2\,t\,z) + (10 + (5 + s + 4\,t)\,z)^2}}{20\,z}.$$

Thus for $\nu = U^{\mathbb{T}}\mu \boxplus U^{\mathbb{T}}\mu$ we have

$$R^{\boxplus}_{\nu}(z) = \frac{-10 + (5 + s + 4\,t)z - \sqrt{40\,z(-5 + 2\,s\,z - 2\,t\,z) + (10 + (5 + s + 4\,t)\,z)^2}}{10\,z},$$

which implies that

$$G_\nu(z) = \frac{-5 + s + 4\,t - \sqrt{s^2 + s(-90 + 8\,t - 20\,z) + (5 + 4\,t - 10\,z)^2}}{2(5\,z^2 - (5 + s + 4\,t)z - 4(s - t))}.$$

On the other hand, the conditionally free transform of pair $(\mu, U^{\mathbb{T}}\mu)$ can be calculated from (2.10), that is defined by

$$\frac{1}{G_\mu(z)} = z - R^{\boxed{\scriptstyle c}}_{(\mu, U^{\mathbb{T}}\mu)}(G_{U^{\mathbb{T}}\mu}(z)).$$

Thus

$$R^{\boxed{\scriptstyle c}}_{(\mu, U^{\mathbb{T}}\mu)}(z) = G^{-1}_{U^{\mathbb{T}}\mu}(z) - \frac{1}{G_\mu(G^{-1}_{U^{\mathbb{T}}\mu}(z))},$$

which gives

$$R^{\boxed{\scriptstyle c}}_{(\mu, U^{\mathbb{T}}\mu)}(z) = -\frac{10 + (-5 - 9s + 4t)z + \sqrt{40z(-5 + 2sz - 2tz) + (10 + (5 + s + 4t)z)^2}}{20sz}.$$

Thus for $\eta = \mu \boxed{\mathbb{T}} \mu$ we have

$$R^{\boxed{c}}_{(\eta,\nu)}(z) = -\frac{10 + (-5 - 9\,s + 4\,t)\,z + \sqrt{40\,z(-5 + 2\,s\,z - 2\,t\,z) + (10 + (5 + s + 4\,t)\,z)^2}}{10\,s\,z},$$

hence

$$\frac{1}{G_\eta(z)} = z - R^{\boxed{c}}_{(\eta,\nu)}(G_\nu(z)),$$

which gives

$$\frac{1}{G_\eta(z)} =$$

$$\frac{25 - 40\,s - 9\,s^2 + 40\,t - 32\,s\,t + 16\,t^2 - 100\,z - 70\,s\,z + 10\,s^2\,z - 80\,t\,z + 40\,s\,t\,z + 100\,z^2}{10\,s(-5 + s + 4\,t - \sqrt{s^2 + s(-90 + 8\,t - 20\,z) + (5 + 4\,t - 10\,z)^2})}$$

$$+ \frac{(5 - 4\,t + s(9 - 10\,z))\sqrt{s^2 + s(-90 + 8\,t - 20\,z) + (5 + 4\,t - 10\,z)^2}}{10\,s(-5 + s + 4\,t - \sqrt{s^2 + s(-90 + 8\,t - 20\,z) + (5 + 4\,t - 10\,z)^2})}$$

$$- \frac{\sqrt{\frac{2(5+s+4t-10z)^2 \times (25-50s+s^2+8st+16t^2-(s+4t-5)\sqrt{s^2+s(8t-20z-90)+(5+4t-10z)^2}-50z-10sz-40tz+50z^2)}{((4t-5z)(-1+z)+s(4+z))^2}}}{10s(-5 + s + 4t - \sqrt{s^2 + s(-90 + 8t - 20z) + (5 + 4t - 10z)^2})}$$

$$\times ((4\,t - 5\,z)(-1 + z) + s(4 + z)).$$

We will show that

$$U^{\mathbb{T}}(\mu \boxed{\mathbb{T}} \mu) \neq U^{\mathbb{T}}\mu \boxplus U^{\mathbb{T}}\mu,$$

that is

$$\eta \neq (U^{\mathbb{T}})^{-1}\nu.$$

Because

$$\mathrm{Re}\left(\frac{1}{G_\nu(i)}\right) = \frac{5 - s - 4\,t - \frac{\sqrt{75+90\,s-s^2-40\,t-8\,s\,t-16\,t^2+\sqrt{400(5+s+4\,t)^2+(75+90\,s-s^2-8(5+s)t-16\,t^2)^2}}}{\sqrt{2}}}{10},$$

we have

$$\frac{1}{G_{(U^{\mathbb{T}})^{-1}\nu}(z)} = \frac{-2\,t(s + 4\,t - 5) - 20(s - 1)z - 2\,s\,\sqrt{s^2 + s(8\,t - 20\,z - 90) + (5 + 4\,t - 10\,z)^2}}{20}$$

$$+ \sqrt{2}(s - t)\times \frac{\sqrt{75-s^2+s(90-8t)-8t(5+2t)+\sqrt{400(5+s+4t)^2+(75+90s-s^2-8(5+s)t-16t^2)^2}}}{20},$$

hence

$$G_\eta(z) \neq G_{(U^{\mathbb{T}})^{-1}\nu}(z). \ \blacksquare$$

8. The N-fold convolution. In this section we will only consider the $\boxed{\mathbb{T}}$-convolution of N measures as

$$\mathop{\boxed{\mathbb{T}^c_N}}_{i=1}^{N}\mu_i = \mathop{\boxed{c_N}}_{i=1}^{N}(\mu_i, U^{\mathbb{T}}\mu_i).$$

THEOREM 3 (Central limit theorem). *Let $\mathbb{T} = (t,s)$ and μ be a compactly supported probability measure on the real line with mean zero and variance equal to 1 and the Maassen representation*

$$\frac{1}{G_\mu(z)} = z + \int \frac{1}{x-z}\,\mathrm{d}\tau_\mu(x),$$

when the first absolute moment of τ_μ exists. Then the sequence

$$\mathop{\boxed{\mathbb{T}^c_N}}_{i=1}^{N}\mathbb{D}_{1/\sqrt{N}}\mu = \mathop{\boxed{c_N}}_{i=1}^{N}(\mathbb{D}_{1/\sqrt{N}}\mu, U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu)$$

is weak-$\star$ convergent to the Kesten measure κ_s as $N \to \infty$.

Proof. The sequence of N-fold $\boxed{\mathbb{T}}$-convolutions of the measure μ is of the form

$$\mu_n = \mathop{\boxed{\mathbb{T}^c_N}}_{i=1}^{N}\mathbb{D}_{1/\sqrt{N}}\mu$$

and by Lemma 4 we know that

$$m_{\mathbb{D}_{1/\sqrt{N}}\mu}(1) = 0, \qquad m_{\mathbb{D}_{1/\sqrt{N}}\mu}(2) = \frac{1}{N} + o\left(\frac{1}{N^2}\right),$$
$$m_{U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu}(1) = o\left(\frac{1}{N}\right), \qquad m_{U^{\mathbb{T}}\mathbb{D}_{1/\sqrt{N}}\mu}(2) = \frac{s}{N} + o\left(\frac{1}{N^2}\right),$$

thus by a similar argument as in the case of the central limit theorem in [BLS], the sequence μ_n is weak-$\star$ convergent to the pair (κ_s, ω_s), defined by the requirement $R^{\boxed{c}}_{(\kappa_s,\omega_s)}(z) = z$, where ω_s denotes the Wigner measure with variance s. Using the relation between the Cauchy transforms and the conditional cumulant transform (2.10), (2.11) we obtain

$$\frac{1}{G_{\kappa_s}(z)} = z - G_{\omega_s}(z),$$
$$\frac{1}{G_{\omega_s}(z)} = z - s\,G_{\omega_s}(z) = \cfrac{1}{z - \cfrac{s}{z - \cfrac{s}{z - \cfrac{s}{z - \ddots}}}}$$

hence

$$\frac{1}{G_{\kappa_s}(z)} = \cfrac{1}{z - \cfrac{1}{z - \cfrac{s}{z - \cfrac{s}{z - \ddots}}}}$$

and κ_s is indeed the Kesten measure. ∎

THEOREM 4 (Poisson type limit theorem). *For $\lambda > 0$ define for all N*

$$\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1, \qquad N \geq 1.$$

Then for $\mathbb{T} = (t, s)$ we have

$$\lim_{N\to\infty} \mathop{\boxed{\scriptstyle \mathbb{T}^c_N}}_{i=1}^{N} \mu_N = \mathop{\boxed{\scriptstyle c_N}}_{i=1}^{N} (\mu_N, U^{\mathbb{T}}\mu_N) = (p^c_\lambda, \nu)$$

in the weak-$\star$ topology, where

$$\mathrm{d}\widetilde{\nu}(x) = \frac{\sqrt{16\, s\, \lambda - (2 - 2\, x + s\, \lambda + t\, \lambda)^2}}{2\, \pi (2\, x + s\, \lambda - t\, \lambda)} \chi_{[\frac{2+s\,\lambda+t\,\lambda}{2} - 2\sqrt{s\,\lambda}, \frac{2+s\,\lambda+t\,\lambda}{2} + 2\sqrt{s\,\lambda}]}(x)\mathrm{d}x,$$

$$\widehat{\nu} = \max(1 - s\,\lambda, 0)\delta_{\frac{(t-s)\lambda}{2}}$$

and

$$\mathrm{d}\widetilde{p^c_\lambda}(x) = \frac{\sqrt{4\,\lambda\, s - (x - \frac{2+s\lambda+t\lambda}{2})^2}}{\pi(2(s-1)x^2 + (2 + (2 - 3s + t)\lambda)x + (s-t)\lambda^2)} \chi_{\left[\frac{2+(s+t)\lambda}{2} - 2\sqrt{s\lambda}, \frac{2+(s+t)\lambda}{2} + 2\sqrt{s\lambda}\right]}(x)\mathrm{d}x,$$

$$\mathrm{d}\widehat{p^c_\lambda}(x) = \begin{cases} 0 & \textit{when } g(x) \textit{ has no real roots,} \\ \max\left(0, 1 - \dfrac{s\,\lambda}{(\lambda - \frac{2+s\,\lambda+t\,\lambda}{2})^2}\right)\delta_y & \textit{when } s = 1, \\ w_1\, \delta_{y_1} + w_2\, \delta_{y_2} & \textit{when } g(x) \textit{ has two real roots } y_1, y_2, \end{cases}$$

$$w_i = \frac{1}{\sqrt{\left(\lambda - \frac{2+s\,\lambda+t\,\lambda}{2}\right)^2 - 4\,\lambda(s-1)}} \max\left(0, \frac{\lambda}{|y_i - \lambda|} - s|y_i - \lambda|\right).$$

Proof. This is a similar argument as in the case of the Poisson limit theorem in [BLS]. ∎

REMARK 13. The above Poisson measure is also an example of a non-standardized (mean is non-zero) free Meixner law, see [BB].

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References

[Ak] N. I. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, Edinburgh and London, 1965.

[AG] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Ungar, New York, 1963.

[BV1] H. Bercovici and D. Voiculescu, *Lévy-Hinčin type theorems for multiplicative and additive free convolution*, Pacific J. Math. 153 (1992), 217–248.

[Bh] R. Bhatia, *Matrix Analysis*, Graduate Text in Mathematics 169, Springer-Verlag, New York, 1997.

[BB] M. Bożejko and W. Bryc, *On a class of free Lévy laws related to a regression problem*, J. Funct. Anal. 236 (2006), 59–77.

[BKW] M. Bożejko, A. D. Krystek and Ł. J. Wojakowski, *Remarks on the r- and Δ-convolutions*, Math. Z. 253 (2006), 177–196.

[BLS] M. Bożejko, M. Leinert and R. Speicher, *Convolution and limit theorems for conditinally free random variables*, Pac. J. Math. 175 (1996), 357–388.

[BS1] M. Bożejko and R. Speicher, *Ψ-independent and symmetrized white noises* in: L. Accardi (ed.), Quantum Probability and Related Topics VI, World Scientific, Singapore, 1991, 219–236.

[BW1] M. Bożejko and J. Wysoczański, *New examples of convolutions and non-commutative central limit theorems*, in: Banach Cent. Publ. 43 (1998), 95–103.

[BW2] M. Bożejko and J. Wysoczański, *Remarks on t-transformations of measures and convolutions*, Ann. Inst. Henri Poincaré Probab. Stat. 37 (2001), 737–761.

[Do] W. F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Springer-Verlag, 1974.

[HiP] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy*, American Mathematical Society, 2000.

[KY1] A. D. Krystek and H. Yoshida, *The combinatorics of the r-free convolution*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003), 619–628.

[KY2] A. D. Krystek and H. Yoshida, *Generalized t-transformatons of probability measures and deformed convolution*, Probability and Mathematical Statistics 24 (2004), 97–119.

[Ma] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. 106 (1992), 409–438.

[Mł] W. Młotkowski, *Operator-valued version of conditionally free product*, Stud. Math. 153 (2002), 13–29.

[SY] N. Saitoh and H. Yoshida, *The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory*, Probability and Mathematical Statistics 21 (2001), 159–170.

[ST] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Mathematical Surveys 1, American Mathematical Society, Providence, 1943.

[Sp] R. Speicher, *Multiplicative functions on the lattice of non–crossing partitions and free convolution*, Math. Ann. 298 (1994), 611–628.

[V1] D. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, in: Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Math. 1132, Springer, 1986, 556–588.

[V2] D. Voiculescu, *Addition of certain non-commuting random variables*, Journal of Functional Analysis 66 (1986), 323–346.

[VDN] D. Voiculescu, K. Dykema and A. Nica, *Free Random Variables*, CRM Monograph Series. 1, AMS, 1992.

[Wo1] Ł. J. Wojakowski, *Probability interpolating between free and boolean*, Dissertationes Mathematicae 446, 2007.