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CONVOLUTIONS RELATED TO q-DEFORMED COMMUTATIVITY

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Abstract. Two important examples of q-deformed commutativity relations are: $aa^* - qa^*a = 1$, studied in particular by M. Bożejko and R. Speicher, and $ab = qba$, studied by T. H. Koornwinder and S. Majid. The second case includes the q-normality of operators, defined by S. Ôta ($aa^* = qa^*a$). These two frameworks give rise to different convolutions. In particular, in the second scheme, G. Carnovale and T. H. Koornwinder studied their q-convolution. In the present paper we consider another convolution of measures based on the so-called (p, q)-commutativity, a generalization of $ab = qba$. We investigate and compare properties of both convolutions (associativity, commutativity and positivity) and corresponding Fourier transforms.

1. Introduction

1.1. Convolutions. In the classical probability, the convolution of measures can be desribed as an associative and commutative operation (on measures), "associated" to the sum of independent random variables. The "association" goes by the distribution – the convolution is precisely the distribution of the sum of the random variables.

The situation is more complicated in the non-commutative case, but the definitions of convolutions mimic the classical one. Let $(\mathcal{A}, \phi)$ be a (non-commutative) probability space, that is a unital $*$-algebra with a state, and let $a, b \in \mathcal{A}$ be two random variables with distributions μ_a, μ_b, respectively. If *the distribution* μ_{a+b} *of the random variable* $a+b$ *depends only on* μ_a *and* μ_b, then it is called an (additive) convolution of measures μ_a and μ_b.

Note that the emphasized assumption means that we have a law to calculate all the (mixed) moments of μ_{a+b}. In the classical case, this is guaranteed by the independence

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of random variables. In the more general context, we also have several independence relations, each of which defines a specific convolution. The most profound examples of independence relations and related convolution are:

1. classical (tensor) independence $\rightarrow$ (classical) convolution $*$,
2. (Voiculescu, 1986) free independence $\rightarrow$ free convolution $\boxplus$,
3. (Speicher, Woroudi, 1997) boolean independence $\rightarrow$ boolean convolution $\uplus$,
4. (Muraki, 1995) monotone independence $\rightarrow$ monotone convolution $\triangleright$.

1.2. q-commutativity. We are used to the fact that a binary operation is commutative – the order of the terms involved does not matter. Addition or multiplication of numbers is commutative. Commutativity holds for observables in classical physics. This is also the basic assumption in the classical probability: any two random variables a, b in a probability space satisfy the relation $ab = ba$. If commutativity fails – as it happens for matrices, operators or for random variables in non-commutative probability spaces – we still want to know how much the operation fails to be commutative. The notions of commutator $[a, b] = ab - ba$ and anti-commutator $\{a, b\} = ab + ba$ appear in this context. They lead to the famous *canonical commutation and anti-commutation relations*

$$ab - ba = 1 \text{ (CCR)}, \qquad ab + ba = 1 \text{ (CAR)}.$$

Now, the idea of q-deformation enters the game. What happens if we add the (real or complex) parameter q "somewhere" to one of the commutation relations $ab = ba$ or $ab - ba = 1$?

It was probably Gauss who was the first to observe that if a and b are two elements of an asociative algebra which satisfy the relation

$$ab = qba \tag{$q1$}$$

(we shall say that they *q-commute*), then

$$(a+b)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k, \quad \text{where} \quad n \in \mathbb{N}, \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

(Here $[n]_q!$ denotes the n^{th} q-factorial, see subsection 1.3). This formula is the q-analogue of the classical Newton's binomial formula, which we get for $q = 1$, and the q-factorial $[n]_q!$ appearing above is a basic notion in the so-called q-calculus (for a nice introduction to q-calulus see [11]). This observation was the starting point to deeper studies of the relation ($q1$). (Note that if $ab = qba$ and $q \neq 1$, then a and b can no longer be realized by numbers but rather by operators. On the other hand, if $q = 1$ we recover the classical commutativity.)

The q-commutativity was studied in particular by T. H. Koornwinder [11] and S. Majid [16]. The first one focused on q-special functions involving q-commuting variables, whereas the second developed a more geometrical approach: the idea of braided algebras. Such q-commuting variables appear also in quantum groups, describing the relations between generators (for more on quantum groups see [9], [10] or [17]).

A special case of q-commutativity – the notion of q-normality of operators – was defined by S. Ôta [20] (studied also with F. H. Szafraniec [21]). An operator A in a

Hilbert space is called *q-normal* if it q-commutes with its adjoint (i.e. $AA^* = qA^*A$). Such operators are know to be unbounded (unless $q = 1$).

The idea of the q-deformation of commutation relations $ab - ba = 1$ came with the generalization of the Fock space – the mathematical description of quantum systems with unlimited number of particles. Depending on the construction, it can describe bosons' or fermions' behaviour, and two important operators – creation and anihilation of a particle – obey respectively CCR or CAR relations. In 1991, M. Bożejko and R. Speicher [4] introduced (in the context of generalized Brownian motion) the q-Fock space $\mathcal{F}_q(H)$ over a Hilbert space H, where the creation $a^*(f)$ and anihilation $a(f)$ operators satisfy the q-deformation of the canonical commutation relations:

$$a^*(f)a(g) - qa(g)a^*(f) = \langle f, g\rangle I, \quad f, g \in H. \tag{q2}$$

For $q = 1$ the q-Fock space corresponds to bosons' Fock space, CCR rule and classical probability, $q = -1$ gives fermions' Fock space and CAR, whereas $q = 0$ leads to the full Fock space and free probability.

Note that, around the same time as Bożejko and Speicher, the q-relations were also proposed by O. W. Greenberg [6] as an example for particles with "infinite statistics". Algebraic aspects of these commutation relations have been studied in [7]. In a particular case, when $f = g$, $\|f\| = 1$ and $a := a^*(f)$, the relation takes the form

$$aa^* - qa^*a = 1,$$

which dates back to the paper by M. Arik and D. D. Coon [2], and which often appears in the literature under the name of the q-(harmonic) oscillator (cf. [15], [22]).

Simple examples of elements satisfying q-normality or q-oscillator relations can be found among weighted shifts.

Example (Weighted shifts). Let us consider a Hilbert space $\ell^2(I)$, where $I = \mathbb{N}$ or $I = \mathbb{Z}$. For a given sequence $(w_n)_n$ let us define the operator

$$S : \mathcal{D}(S) \ni (a_n)_n \mapsto (w_n a_{n+1})_n \in \ell^2(I),$$

with

$$\mathcal{D}(S) = \{(a_n)_{n\in I} \in \ell^2(I) : \sum_{n\in I} |w_n a_n| < +\infty\} \subset \ell^2(I).$$

We say that S is a *unilateral* (if $I = \mathbb{N}$) or *bilateral* (if $I = \mathbb{Z}$) *weighted shift* with weights $(w_n)_n$.

If $w_n = w_0 q^{-\frac{n}{2}}$, $n \in \mathbb{Z}$, then the bilateral shift S is q-normal (i.e. $SS^* = qS^*S$). If $w_n = \sqrt{[n+1]_q}$, $n \in \mathbb{N}$, then the unilateral shift S satisfies the relation $SS^* - qS^*S = I$.

The following is a realization of the q-commutation relations.

Example (q-Fock space [4]). The q-Fock space $\mathcal{F}_q(\mathcal{H})$ is a full Fock space $\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ equipped with (and completed with respect to) the inner product

$$\langle g_1 \otimes \ldots \otimes g_m, h_1 \otimes \ldots \otimes h_n\rangle_q = \delta_{m,n} \sum_{k=1}^{n} q^{k-1} \langle g_1, h_k\rangle \langle g_2 \otimes \ldots \otimes g_m, h_1 \otimes \ldots \tilde{h}_k \ldots \otimes h_n\rangle_q.$$

We use the notation $h_1 \otimes \ldots \tilde{h}_k \ldots \otimes h_n$ to denote the tensor product $h_1 \otimes \ldots \otimes h_{k-1} \otimes h_{k+1} \otimes \ldots \otimes h_n$. For $f \in \mathcal{H}$, the creation and anihilation operators are defined as follows

$$a^*(f)h_1 \otimes \ldots \otimes h_n = f \otimes h_1 \otimes \ldots \otimes h_n,$$

$$a(f)\Omega = 0, \quad a(f)h_1 \otimes \ldots \otimes h_n = \sum_{k=1}^{n} q^{k-1}\langle f, h_k\rangle h_1 \otimes \ldots \tilde{h}_k \ldots \otimes h_n.$$

Then for any $q \in (-1, 1)$ both operators can be extended to bounded operators on $\mathcal{F}_q(\mathcal{H})$, they are adjoints of each other and they satisfy the q-deformed commutation relations

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle \mathrm{id}.$$

The realization of the q-commutativity can be obtained in braided algebra setting.

Example (Braided line [11]). The *braided line* $\mathcal{A} := \mathbb{C}_q[x]$ with $q \in (0, 1)$ is the algebra of polynomials in one variable x with a braided bialgebra structure. The braiding Ψ : $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a bijective linear mapping, defined on the basis by:

$$\Psi(x^k \otimes x^l) = q^{kl} x^l \otimes x^k, \quad k, l \in \mathbb{N}.$$

Then $\mathcal{A} \otimes \mathcal{A}$ has an algebra structure with multiplication

$$m_\Psi = (m_\mathcal{A} \otimes m_\mathcal{A}) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id}).$$

The comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined on the generator by the formula

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

and the counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ is given by $\varepsilon(x^n) = \delta_{n,0}$, both being algebra homomorphisms (for further details, see [11] and references given there).

Then, $\mathcal{A} \otimes \mathcal{A}$ can be considered as the algebra with generators $a = 1 \otimes x$ and $b = x \otimes 1$, satisfying the relation $ab = qba$.

1.3. Notation. Unless otherwise stated, we assume that $q > 0$ and that sequences appearing in the paper are indexed by non-negative integers. To avoid ambiguity while talking about two relations describing q-deformed commutativity, we shall use the name *q-commutativity* to refer to the relation ($q1$) and the name *q-commutation relation* in case of the relation ($q2$). We shall also use the standard notation of q-calculus (see for example [11]):

- q-integer: $[n]_q := \dfrac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}$,
- q-factorial(s): $[n]_q! := [n-1]_q! \cdot [n]_q, \quad (q, q)_n := \prod_{k=1}^{n} (1 - q^k)$,
- q-exponents: $e_q(z) := \sum_{n=0}^{\infty} \dfrac{z^n}{(q, q)_n}, \quad E_q(z) := \sum_{n=0}^{\infty} \dfrac{q^{n(n-1)/2} z^n}{(q, q)_n}$,
- q-integral (Jackson integral): $\int_{-\infty}^{\infty} f(t) d_q(t) := (1 - q) \sum_{k=-\infty}^{\infty} \sum_{\varepsilon=\pm 1} q^k f(\varepsilon q^k x)$.

2. Convolutions for q-commutation relations. There were several attempts to define a q-deformed convolution, which would correspond to the q-CCR scheme and interpolate between the classical convolution (for $q = 1$) and its free analogue ($q = 0$).

In 1995, A. Nica [18] defined the *R_q-transform*, $q \in [0, 1]$, which is a bijection between the space of moment sequences and the space of formal power series vanishing at 0. It associates to a moment sequence $(\mu_n)_n$ the series $\sum_{n=0}^{\infty} \alpha_n z^n$, coefficients of which (called *q-cumulants*) can be calculated via the generalized moment-cumulant formula

$$\mu_n = \sum_{\pi \in \mathcal{P}_n} q^{c_0(\pi)} \prod_{j=1}^{k} [|B_j| - 1]_q! \alpha_{|B_j|}. \tag{1}$$

Here $\mathcal{P}_n$ stands for the set of all partitions of $\{1, \ldots, n\}$, $B_1, \ldots, B_k$ are the blocks of partition π, $|B_j|$ is the cardinality of the j-th block. The number $c_0(\pi)$ is the left-reduced number of crossings of π. By a crossing we mean a 4-tuple (m_1, m_2, m_3, m_4) such that $1 \leq m_1 < m_2 < m_3 < m_4 \leq n$ and for which there exists i, j such that $m_1, m_3 \in B_i$ and $m_2, m_4 \in B_j$. The left-reduced number of crossings is the number of crossings (m_1, m_2, m_3, m_4) in which m_1, m_2 are minimal in the corresponding blocks. For example, for the partition $\pi = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, this number is three, since the 4-tuples $(1, 2, 3, 4)$, $(1, 2, 3, 6)$ and $(1, 2, 5, 6)$ satisfy the definition.

Nica showed also that the R_q-transform can be described in two other ways, one of them using the weighted shifts S_q, satisfying the q-CCR.

Now, the q-convolution $\boxplus_q$ is, by definition (see [18], Definition 4.1), the operation on the space of moment sequences, which is linearized by the R_q-transform, that is if $\mu = (\mu_n)_n$ and $\nu = (\nu_n)_n$ are two moment sequences, then

$$\mu \boxplus_q \nu = R_q^{-1}(R_q(\mu) + R_q(\nu)).$$

The R_q-transform is, on one hand, a q-analogue of the logarithm of a Fourier transform (up to a linear bijection), and on the other hand, a generalization of the R-transform, described by Voiculescu [23]. It is a useful tool in investigations of limit distributions and, in particular, the role of central limit is played by $R_q^{-1}(z^2)$, which is a measure associated to q-continuous Hermite polynomials.

The definition of q-convolution was rather algebraic and it was not clear whether the resulting object is again a measure. It was 10 years later, in 2005, when F. Oravecz [19] gave a negative answer to this question. He proved that for $0 < q < 1$, the q-convolution does not preserve positivity, since the Poisson type limit is not a measure. More precisley, if we consider $\mu_N = (1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_1$ (where δ_x is a Dirac measure at x, $N \in \mathbb{N}$, $\lambda > 0$), then the Poisson limit is

$$\lim_{N \to +\infty} \underbrace{\mu_N \boxplus_q \ldots \boxplus_q \mu_N}_{N\text{-fold}}.$$

Thanks to the R_q-transform and its linearity property, the q-cumulants (coefficients α_n) of $\mu_N \boxplus_q \ldots \boxplus_q \mu_N$ are easily computed:

$$\alpha_n^{(N)} := \alpha_n(\mu_N \boxplus_q \ldots \boxplus_q \mu_N) = N \cdot \alpha_n(\mu_N), \quad n \in \mathbb{N}.$$

The q-cumulants of the Poisson limit are the limits $\hat{\alpha}_n = \lim_{N \to +\infty} \alpha_n^{(N)}$. But, as was shown by Oravecz, the sequence $(\hat{\mu}_n)_n$ corresponding to $(\hat{\alpha}_n)_n$ via moment-cumulant formula (1) is not positive definite and thus is not a moment sequence.

In 1996, H. van Leeuwen and H. Maassen [14] were the next to look for a good q-convolution, interpolating between the free ($q = 0$) and the classical ($q = 1$) one, and found an important obstruction. They constructed operators X_0 and X_1 on the q-Fock space, that are q-independent, and show that there exists a function $\gamma : \mathcal{R} \to \mathcal{R}$ such that X_0 and $\gamma(X_0)$ have the same distribution, but the distributions of $\gamma(X_0) + X_1$ and $X_0 + X_1$ are different. This means that the distribution of the sum of two (independent) random variables is no longer determined by the distributions of the summands. Thus, they concluded, no q-deformed convolution can exist.

The authors knew already the q-convolution of A. Nica and they noted that it did not contradict their result. The clue lies in the notion of q-independence. Two random variables X and Y in q-Fock space are called q-independent if they are of the form $X = a(f) + a(f)^*$ and $Y = a(g) + a(g)^*$ with $f \perp g$ (they are q-Gaussians corresponding to orthogonal vectors). It turned out that the structure on the q-Fock space and the Nica's law for convolution give different result when applied to functions on q-Gaussians.

Finally, let us mention the paper by M. Anshelevich [1] in 2001. He defines q-cumulants in a different way than Nica and a transform LH_q linearizing them. The q-cumulants r_n corresponding to the moment sequence $(\mu_n)_n$ (or to a measure μ uniquely determined by the moments) are to be calculated recursively from the formula

$$\mu_n = r_n + \sum_{\pi \in \mathcal{P}_n \setminus \{\hat{1}\}} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} r_{|B|},$$

where $\hat{1}$ denotes the one-block partition $((1, 2, \ldots, n))$ and $\mathrm{rc}(\pi)$ is the number of restricted crossings of the partition π, defined by Ph. Biane [3]. This is the number of crossings (m_1, m_2, m_3, m_4), where m_1 follows (in the block) m_3 and m_2 follows m_4. This number can be described graphically: if the set $\{1, \ldots, n\}$ is represented on the x axis in the plane and we join by a semicircle (above the x axis) any two points belonging to the same block such that no other element from the same block is between them, then the restricted crossing counts the number of intersections of the semicircles.

The difference between the number of restricted crossings $\mathrm{rc}(\pi)$ and the number of left-reduced crossings $c_0(\pi)$ is that in the first case we restrict ourselves to crossings with successive elements of blocks, whereas in the second case, we consider only those 4-tuples where m_1 and m_2 are the minimal elements in blocks. In both cases, we disregard some other crossings. For example, in the partition $\pi = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, the crossing $(1, 2, 3, 6)$ counts into $c_0(\pi)$, but not into $\mathrm{rc}(\pi)$ (1 and 2 are minimal, but 6 does not follow 2), while the crossing $(2, 3, 4, 5)$ is a converse case (3 is not minimal in the block). This is the reason why $\mathrm{rc}(\pi)$ can (but need not!) differ from $c_0(\pi)$. More precisely, the number of restricted crossings can be smaller or bigger than, or equal to the number of left-reduced crossings, as shown in the following examples:

- for $\pi = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, $\mathrm{rc}(\pi) = 3 = c_o(\pi)$,
- for $\pi = \{\{1, 3, 6\}, \{2, 4\}, \{5, 7\}\}$, $\mathrm{rc}(\pi) = 3 > 2 = c_o(\pi)$,
- for $\pi = \{\{1, 3, 4, 5\}, \{2, 6\}\}$, $\mathrm{rc}(\pi) = 2 < 3 = c_o(\pi)$.

All this implies that for $q \notin \{0, 1\}$, the q-cumulants defined by Anshelevich and the ones by Nica are different.

The Lévy-Khinchin transform LH_q acts on a subclass of determinate moment sequences (the set of the so-called q-infinitely divisible measures, denoted $\mathcal{ID}_c(q)$) in the following way: for a positive Borel measure τ on $\mathcal{R}$ with all moments $(\tau_n)_n$ finite and for $\lambda \in \mathcal{R}$, we define $\mathrm{LH}_q^{-1}(\lambda, \tau)$ to be a moment sequence, or equivalently a probability measure, (uniquely) determined by the cumulants $r_1 = \lambda$, $r_n = \tau_{n-2}$, $n \geq 2$. Then $\mathcal{ID}_c(q)$ is defined as the image of LH_q^{-1} and LH_q is the inverse of LH_q^{-1}. The q-convolution of two moment sequences of q-infinitely divisible measures μ and ν is defined by the rule

$$\mathrm{LH}_q(\mu *_q \nu) = \mathrm{LH}_q(\mu) + \mathrm{LH}_q(\nu).$$

Then $(\mathcal{ID}_c(q), *_q)$ is an abelian semigroup.

3. q-convolution. The q-deformation of the convolution corresponding to the q-commutativity ($ab = qba$) was studied in 2001 by G. Carnovale and T. H. Koornwinder in the paper [5]. They focused on the following definition:

Definition 1. Let us consider $0 < q < 1$. Let f be a function on $\mathbb{R}$ such that all its "weighted" moments

$$m_n(f) := q^{\frac{n(n+1)}{2}} \int t^n f(t) d_q(t)$$

are finite and let g be a function on some subset of $\mathbb{C}$. Then the q-convolution $f \star_q g$ is defined by

$$(f \star_q g)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n m_n(f)}{[n]_q!} (\partial_q^n g)(x)$$

(for $x \in \mathbb{C}$ such that the above definition makes sense). Here $\partial_q^n g$ denotes the n^{th} q-derivative of a function g, where ∂_q is defined as $(\partial_q g)(x) = \frac{g(x)-g(qx)}{(1-q)x}$.

Three main motivations for the definition are mentioned in the paper:

1. The 1-convolution coincides with the classical convolution of sequences, so the q-convolution can be regarded as a q-deformation of the classical convolution.
2. The q-convolution is the operation that transforms into multiplication by the q-Fourier transform

$$(\mathcal{F}_q f)(y) := \int_{-\infty}^{+\infty} E_q(iqxy) f(x) d_q x,$$

defined by T. H. Koornwinder [11]. This means that

$$\mathcal{F}_q(\mu \star_q \nu)(x) = \mathcal{F}_q(\mu)(x) \cdot \mathcal{F}_q(\nu)(x). \tag{2}$$

3. The q-convolution is a modification of the convolution defined by Kempf and Majid [8] on the braided covector algebras which is bosonic and invariant under translation. The q-convolution is adapted to the braided line, where the Jackson integral is translation invariant, but is not bosonic. Moreover, a slight modification is needed to ensure associativity.

The authors showed that if f and g are "good", then the convolution is well-defined, associative and commutative. "Good function" meant there that we have some analyticity and moments growth conditions (see [5] for details). Anyway, for "good functions" the q-Fourier transform can be rewritten as the power series

$$(\mathcal{F}_q f)(y) = \sum_{k=0}^{\infty} \frac{(iy)^k}{(q;q)_k} m_k(f)$$

and what is more, we have a nice formula for the moments of the q-convolution

$$m_n(f \star_q g) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q m_k(f) m_{n-k}(g). \tag{3}$$

In both formulas $m_k(f)$ denotes the "weighted" moment from Definition 1, but the formula (3) remains unchanged if, instead of "weighted" moments, we take

$$\mu_k(f) = q^{k(k-1)/2} \int t^k f(t) d_q(t),$$

which are the k^{th} q-moments of the measure $d\mu(t) = f(t)d_q(t)$ (see [12] for more on the notion of q-moments).

Now, the idea is to take (3) as a general definition and study the q-convolution as an operation on sequences. This was the approach presented in [13].

Definition 2. Let $q > 0$ and let $(\mu_n)_n$, $(\nu_n)_n$ be two sequences. Their q-convolution is the sequence

$$(\mu \star_q \nu)_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mu_k \nu_{n-k}, \quad n \in \mathbb{N}.$$

What can we say about such an operation? Observe that the 1-convolution coincides with the classical convolution of the sequences, so it can be called a q-deformation of the classical convolution.

By a direct calculation one shows that the q-convolution of two sequences is associative and commutative.

As far as the positivity preserving property is concerned, one can show that if μ and ν are measures on $\mathbb{R}$ with all moments finite and if $(\mu_n)_n$ and $(\nu_n)_n$ are their moment or q-moment sequences, then the resulting sequence need neither be a moment sequence, nor a q-moment sequence. If we restrict to q-moment sequences of measures on $[0, +\infty)$, then the positivity is preserved only if $q < 1$ (see [13] for details).

Finally, the (formal) power series

$$(\mathcal{F}_q \mu)(y) = \sum_{k=0}^{\infty} \frac{(iy)^k}{(q;q)_k} \mu_k$$

defines the q-Fourier transform which satisfies (2).

4. (p, q)**-convolution.** Another convolution corresponding to the (generalized) q-commutativity was defined recently in our paper with E. Ricard [13].

DEFINITION 3. Let $p, q > 0$ and let $(\mu_n)_n$, $(\nu_n)_n$ be two sequences. We shall call the (p,q)-*convolution* of this sequences, and denote by $\{(\mu \star_{p,q} \nu)_n\}_n$, the sequence given by the formula

$$(\mu \star_{p,q} \nu)_n = \sum_{k=0}^{n} \left(\frac{q}{p}\right)^{k(n-k)} \left[\begin{array}{c} n \\ k \end{array}\right]_p^2 \mu_k \nu_{n-k}. \tag{4}$$

The motivation for such a definition comes from the following algebraic interpretation which we just sketch here (see [13] for further details).

Let us fix two parameters $p, q > 0$ and let $(\mu_n)_n$, $(\nu_n)_n$ be two sequences. We say that a sequence $(\mu_n)_n$ is $q\mathrm{PD}^+$ if for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \dots, \alpha_n \in \mathbb{C}$

$$\sum_{i,j=0}^{n} q^{-ij} \alpha_i \bar{\alpha}_j \mu_{i+j} \geq 0 \quad \text{and} \quad \sum_{i,j=0}^{n} q^{-ij} \alpha_i \bar{\alpha}_j \mu_{i+j+1} \geq 0.$$

This is equivalent to saying that the sequence $(\mu_n)_n$ is the q-moment sequence of a measure on the positive real half-line, that is, there exists a measure μ on $[0, +\infty)$ such that

$$\mu_n = q^{n(n-1)/2} \int_0^{+\infty} t^n d\mu(t), \quad n \in N,$$

(see [12] for details).

Consider a unital $*$-algebra $\mathcal{A}$ generated by a and b that are q-normal and (p,q)-commute, that is they satisfy

$$aa^* = qa^*a, \quad bb^* = qb^*b, \quad ab = pba, \quad ab^* = qb^*a. \tag{5}$$

Moreover, let $\mathcal{A}_a$ and $\mathcal{A}_b$ be two unital $*$-subalgebras, generated by the q-normal a and the q-normal b, respectively, *i.e.*

$$\mathcal{A}_a = \mathbb{C}[a, a^*]/(aa^* - qa^*a), \quad \mathcal{A}_b = \mathbb{C}[b, b^*]/(bb^* - qb^*b).$$

Given the sequences $(\mu_n)_n$ and $(\nu_n)_n$, we define the functionals

$$\begin{aligned} &\mu : \mathcal{A}_a \to \mathbb{C}, \mu(a^m a^{*n}) := \delta_{m,n} \mu_m, \\ &\nu : \mathcal{A}_b \to \mathbb{C}, \nu(a^m a^{*n}) := \delta_{m,n} \nu_m, \\ &\Phi : \mathcal{A} \to \mathbb{C}, \ \Phi[a^k a^{*l} b^m b^{*n}] := \delta_{k,l} \delta_{m,n} \mu_k \nu_m. \end{aligned}$$

THEOREM 4.1. *If the sequences $(\mu_n)_n$, $(\nu_n)_n$ are $q\mathrm{PD}^+$, then the functionals μ and ν are positive* (*and thus states*) *on $\mathcal{A}_a$ and $\mathcal{A}_b$, respectively, and the mapping Φ is a state on $\mathcal{A}$.*

Now, we follow the idea that convolution is related to sums of random variables. We consider the unital $*$-algebra C genarated by $a + b$. This is a subalgebra of $\mathcal{A}$ and the generator $a + b$ is q-normal. Thanks to the (p,q)-commutation relations (5), we can compute the mixed moments of Φ on $\mathbb{C}$. What we get is the following

$$\Phi[(a+b)^m (a^* + b^*)^n] = \delta_{m,n} \sum_{k=0}^{n} \left(\frac{q}{p}\right)^{k(n-k)} \left[\begin{array}{c} n \\ k \end{array}\right]_p^2 \mu_k \nu_{n-k}.$$

Moreover, we know that the sequence $(\Phi_{n,n})_n$ is $q\mathrm{PD}^+$, since it comes from the state on an algebra generated by a q-normal element. We conclude:

Corollary 4.2. *The mapping*

$$\big((\mu_n)_n, (\nu_n)_n\big) \mapsto (\Phi_n)_n, \quad \textit{where} \quad \Phi_n := \sum_{k=0}^{n} \left(\frac{q}{p}\right)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \mu_k \nu_{n-k},$$

is an operation on q-moments of measures on $\mathbb{R}_+$.

This operation is exactly the (p, q)-convolution from Definition 3. We want to study its properties. Note first that this is no longer a deformation of the classical convolution (the case $q = 1$ does not lead to the classical convolution).

Thanks to the algebraic interpretation presented above, we get (for free!) the fact that the (p, q)-convolution preserves measures on $\mathbb{R}_+$. Unfortunately, no hope for the stronger positivity preserving conditions – the (p, q)-convolution of two measures on $\mathbb{R}$ need not be a measure on $\mathbb{R}$ (see for example the Dirac measures $\mu = \delta_1, \nu = \delta_{-1}$).

A straightforward calculation shows that the (p, q)-convolution is associative and commutative. What is more, we can see from the formula

$$(\mu \star_{p,q} \nu)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_{p^{-1}} q^{k(n-k)} \mu_k \nu_{n-k}$$

that the (p, q)-convolution is symmetric w.r.t. $p \leftrightarrow p^{-1}$.

By modifying the definition of the q-Fourier transform, we can define the (formal) power series

$$\mathcal{F}_{p,q}(\mu)(x) = \sum_{k=0}^{\infty} \frac{q^{-k(k-1)/2} x^k}{(p,p)_k (p^{-1}, p^{-1})_k} \mu_k^{(q)},$$

where $\mu_k^{(q)}$ is the k-th q-moment of measure μ. This new transform is quite difficult to study, but one can show that it satisfies the following property:

$$\mathcal{F}_{p,q}(\mu \star_{p,q} \nu)(x) = \mathcal{F}_{p,q}(\mu)(x) \cdot \mathcal{F}_{p,q}(\nu)(x),$$

which means that it transforms the convolution of q-moments sequences into multiplication of (formal) power series. So it plays the role of the Fourier transform for the (p, q)-convolution.

5. Relations between q-convolution and (p, q)-convolution. Although the motivation and the constructions of both q- and (p, q)-convolutions were different, they have a similar formula and share some properties. A useful way to compare them is the table opposite.

In the special case when $p = q$ we have a nice formula joining both convolutions. Recall that, by the Schur's Lemma, the pointwise (Schur-Hadamard) product of two moment sequences is again a moment sequence. The corresponding measure transformation is the (classical) multiplicative convolution $\boxtimes$.

Let us denote by N the sequence $\{[n]_q!\}_n$ which is positive definite or, equivalently, is a moment sequence (see [13]) and let us adopt the notation $\{(N \boxtimes \mu)_n\}_n$ for the sequence with the coefficients $(N \boxtimes \mu)_n = [n]_q! \cdot \mu_n$ (for a given sequence $\mu = \{\mu_n\}_n$).

q-**convolution**	(p,q)-**convolution**
FORMULA	
$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_k \nu_{n-k}$	$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \left(\frac{q}{p}\right)^{k(n-k)} \mu_k \nu_{n-k}$
PROPERTIES	
associative, commutative, **not** symmetric $q \leftrightarrow q^{-1}$, does **not** preserve measures on $\mathbb{R}$, preserves measures on $\mathbb{R}_+$ **only if** $q < 1$.	associative, commutative, symmetric $p \leftrightarrow p^{-1}$, does **not** preserve measures on $\mathbb{R}$, preserves measures on $\mathbb{R}_+$ **for all** $p, q > 0$.
FOURIER TRANSFORM	
$\mathcal{F}_q(\mu)(x) = \sum_{k=0}^{\infty} \frac{\mu_k^{(q)}}{(q,q)_k} (ix)^k,$	$\mathcal{F}_{p,q}(\mu)(x) = \sum_{k=0}^{\infty} \frac{q^{-k(k-1)/2} \mu_k^{(q)}}{(p,p)_k (p^{-1},p^{-1})_k} x^k,$

THEOREM 5.1. *For any sequences $(\mu_n)_n$ and $(\nu_n)_n$ we have*

$$(N \boxtimes \mu) \star_{q,q} (N \boxtimes \nu) = N \boxtimes (\mu \star_q \nu).$$

Proof. Indeed, we have

$$\begin{aligned}\big((N \boxtimes \mu) \star_{q,q} (N \boxtimes \nu)\big)_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 (N \boxtimes \mu)_k (N \boxtimes \nu)_{n-k} \\ &= \sum_{k=0}^{n} \frac{[n]_q!^2}{[k]_q!^2 [n-k]_q!^2} [k]_q! \mu_k [n-k]_q! \nu_{n-k} \\ &= [n]_q! \sum_{k=0}^{n} \frac{[n]_q!}{[k]_q! [n-k]_q!} \mu_k \nu_{n-k} = \big(N \boxtimes (\mu \star_q \nu)\big)_n. \ \blacksquare\end{aligned}$$

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