NONCOMMUTATIVE HARMONIC ANALYSIS
WITH APPLICATIONS TO PROBABILITY II
BANACH CENTER PUBLICATIONS, VOLUME 89
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

JUCYS-MURPHY ELEMENTS AND THE UNITARY WEINGARTEN FUNCTION

JONATHAN I. NOVAK

Combinatorics & Optimization, University of Waterloo
200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1
E-mail: j2novak@math.uwaterloo.ca

Abstract. We describe an approach to the unitary Weingarten function based on the JM elements of symmetric group algebras. When combined with previously known properties of the Weingarten function, this gives a surprising connection with the Moebius function of the lattice of noncrossing partitions.

1. Introduction

1.1. Moments of the unitary group. Let $\mathbb{U}_d = \{U \in GL_d(\mathbb{C}) : U^* = U^{-1}\}$ be the compact group of $d \times d$ complex unitary matrices equipped with (normalized) Haar measure dU. In the context of random matrix theory, one often wishes to compute integrals (expectations) of the form

$$I_d(i,j,i',j') = \int_{\mathbb{U}_d} u_{i(1)j(1)} \dots u_{i(n)j(n)} \overline{u}_{i'(1)j'(1)} \dots \overline{u}_{i'(n)j'(n)} dU, \tag{1}$$

where n is a positive integer and $i,j,i',j' : [n] \to [d]$ are functions.

Integrals of this form, which describe the joint moments of the entries of a Haar-distributed random unitary matrix from $\mathbb{U}_d$, were studied extensively by Collins [C] and Collins-Śniady [CS], where they were called the *moments* of the unitary group $\mathbb{U}_d$. The computation of these moments is ostensibly a probabilistic problem; however, the fact that $\mathbb{U}_d$ is a compact algebraic group allows for an essentially algebraic solution.

1.2. The Weingarten function. We assume henceforth that $d \geq n$. This assumption simplifies the presentation of some results concerning unitary group integrals. It

2000 *Mathematics Subject Classification*: Primary 05E05, 05E10; Secondary 15B52.
Key words and phrases: Jucys-Murphy elements, Weingarten function, symmetric functions, Haar unitary random matrices.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc89-0-14

was shown in [CS] that, in most cases, this assumption can be lifted without too much difficulty.

The main result of Collins in [C] is that there exists a function $\mathrm{Wg}_d : S(n) \to \mathbb{C}$, where $S(n)$ is the symmetric group on $[n] := \{1, \dots, n\}$, with the following remarkable property:

$$I_d(i,j,i',j') = \sum_{\sigma,\tau \in S(n)} \prod_{k=1}^{n} \delta_{i(k),i'(\sigma(k))}\delta_{j(k),j'(\sigma(k))} \mathrm{Wg}_d(\tau\sigma^{-1}), \tag{2}$$

where δ is the Kronecker delta. The special function Wg_d is called the *Weingarten function*, and formula (2) above is the *Weingarten convolution formula.* The existence of the Weingarten function was deduced by Collins from the classical Schur-Weyl duality between representations of $\mathbb{U}_d$ and $S(n)$ in the tensors

$$\underbrace{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{n}. \tag{3}$$

The problem of computing the integrals (1) is thus reduced, via the Weingarten convolution formula (2), to the problem of finding a tractable expression for the Weingarten function.

1.3. Main result. Since Wg is a function on permutations, we view it as an element of the symmetric group algebra $\mathbb{C}[S(n)]$, i.e. as a formal sum

$$\mathrm{Wg}_d = \sum_{\sigma \in S(n)} \mathrm{Wg}_d(\sigma)\sigma. \tag{4}$$

In this note we show that a remarkably simple formula for Wg_d can be given in terms of the Jucys-Murphy elements of $\mathbb{C}[S(n)]$:

$$\begin{aligned} J_1 &= 0, \\ J_2 &= (1,2), \\ J_2 &= (1,3) + (2,3), \\ &\vdots \\ J_n &= (1,n) + (2,n) + \cdots + (n-1,n). \end{aligned}$$

Our main theorem is the following:

THEOREM 1.1. *Let $d \geq n$ be positive integers. Then $d+J_1, d+J_2, \dots, d+J_n \in \mathbb{C}[S(n)]$ are invertible, and*

$$\mathrm{Wg}_d = (d+J_1)^{-1}(d+J_2)^{-1} \dots (d+J_n)^{-1}. \tag{5}$$

2. Proofs

2.1. Background on JM elements. The JM elements $J_1, \dots, J_n \in \mathbb{C}[S(n)]$ were introduced independently by Jucys [J] and Murphy [M]. These seemingly simple transposition sums have many remarkable properties, and are a key object in the modern approach to the representation theory of the symmetric groups initiated by Okounkov and Vershik [OV]. In particular, the JM elements generate a maximal commutative subalgebra of $\mathbb{C}[S(n)]$.

Let $Z(n)$ denote the center of $\mathbb{C}[S(n)]$. Let $\mathbb{Y}(n)$ be the set of Young diagrams with n cells, and recall the canonical surjection $T : S(n) \to \mathbb{Y}(n)$ which maps each permutation to its cycle type. Given $\mu \in \mathbb{Y}(n)$, let $\mathcal{C}_\mu$ denote the fibre of T over μ, i.e. the conjugacy class of permutations of cycle type μ, which we identify with the sum

$$\mathcal{C}_\mu = \sum_{T(\sigma)=\mu} \sigma \in Z(n). \tag{6}$$

We also denote by $\ell(\mu)$ the number of rows in a Young diagram μ. In particular, $\ell(T(\sigma))$ is the number of factors in the decomposition of σ into a product of disjoint cycles.

We will benefit greatly from the following result of Jucys:

THEOREM 2.1 ([J]). *For $1 \leq r \leq n$, let*

$$e_r(J_1, J_2, \ldots, J_n) = \sum_{1 \leq i_i < i_2 < \cdots < i_r \leq n} J_{i_1} J_{i_2} \ldots J_{i_r} \tag{7}$$

be the elementary symmetric polynomials in JM elements. Then

$$e_r(J_1, J_2, \ldots, J_m) = \sum_{\substack{\mu \in \mathbb{Y}(n) \\ \ell(\mu)=n-r}} \mathcal{C}_\mu. \tag{8}$$

In other words, $e_r(J_1, J_2, \ldots, J_n)$ is the indicator function of permutations with exactly $n - r$ cycles.

For example,

$$e_1(J_1, J_2, \ldots, J_n) = J_1 + J_2 + \cdots + J_n \tag{9}$$

is the formal sum of all permutations with exactly $n - 1$ cycles (i.e. the conjugacy class of transpositions).

2.2. Proof of Theorem 1.1. Our starting point is the following fundamental theorem of Collins [C]: for $d \geq n$, the inverse of Wg_d is

$$\mathrm{Wg}_d^{-1} = \sum_{\mu \in \mathbb{Y}(n)} d^{\ell(\mu)} \mathcal{C}_\mu. \tag{10}$$

Now we simply compute using Jucys' Theorem (2.1)

$$\begin{aligned} (d + J_1)(d + J_2) \ldots (d + J_n) &= \sum_{k=0}^{n} d^{n-k} e_k(J_1, J_2, \ldots, J_n) &(11)\\ &= \sum_{k=0}^{n} \sum_{\substack{\mu \in \mathbb{Y}(n) \\ \ell(\mu)=n-k}} d^{n-k} \mathcal{C}_\mu &(12)\\ &= \sum_{\mu \in \mathbb{Y}(n)} d^{\ell(\mu)} \mathcal{C}_\mu. &(13) \end{aligned}$$

It follows immediately from the assumption $d \geq n$ that $d + J_k$ is invertible for each $1 \leq k \leq n$. Thus

$$\mathrm{Wg}_d = (d + J_1)^{-1}(d + J_2)^{-1} \ldots (d + J_n)^{-1}, \tag{14}$$

as desired.

3. Application: the Laurent expansion. Consider a fixed permutation $\sigma \in S(n)$ with cycle type $\tau = (\tau_1, \tau_2, \ldots, \tau_\ell)$. $\mathrm{Wg}_d(\sigma)$ is then a rational function of d with denominator of degree n [C]. Thus there exists a Laurent expansion

$$\mathrm{Wg}_d(\sigma) = \sum_{r \geq 0} \frac{a_r(\sigma)}{d^{n+r}}. \tag{15}$$

The structure of the Laurent coefficients remains somewhat mysterious. Collins [C] has shown that

$$\min\{r : a_r(\sigma) \neq 0\} = n - \ell, \tag{16}$$

and moreover

$$a_{n-\ell}(\sigma) = \mathrm{Moeb}(\sigma). \tag{17}$$

Recall that Moeb is the *Moebius function* defined by

$$\mathrm{Moeb}(\sigma) = \prod_{k=1}^{\ell} (-1)^{\tau_k - 1} \mathrm{Cat}_{\tau_{k-1}}, \tag{18}$$

where

$$\mathrm{Cat}_m = \frac{1}{m+1}\binom{2m}{m} \tag{19}$$

is a Catalan number. Moeb coincides with the Moebius function of the lattice of non-crossing partitions, see [NS].

The existence of the Laurent expansion (15) means that there exists a sequence of functions $a_0, a_1, \ldots, a_r, \cdots \in \mathbb{C}[S(n)]$ such that, if we define

$$H_N = \sum_{r=0}^{N} \frac{a_r}{d^{n+r}}, \tag{20}$$

then

$$\mathrm{Wg}_d(\sigma) = \lim_{N \to \infty} H_N(\sigma) \tag{21}$$

for all $\sigma \in S(n)$ (where we are assuming $d \geq n$).

THEOREM 3.1. *For each $r \geq 0$,*

$$a_r = (-1)^r h_r(J_1, J_2, \ldots, J_n), \tag{22}$$

where

$$h_r(J_1, J_2, \ldots, J_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} J_{i_1} J_{i_2} \ldots J_{i_n} \tag{23}$$

is the rth complete homogeneous symmetric polynomial in the JM elements (i.e. simply the sum of all distinct monomials of degree r).

Proof. If $x_1, \dots, x_n, t$ are formal indeterminates, one has the generating function

$$\sum_{r\geq 0} h_r(x_1,\dots,x_n)t^r = (1-x_1t)^{-1}(1-x_2t)^{-1}\dots(1-x_nt)^{-1}. \tag{24}$$

Simply specialize $x_1 = J_1, x_2 = J_2, \dots, x_n = J_n$ and $t = -\frac{1}{d}$ and apply Theorem 1.1 to obtain the stated result. ■

One immediate consequence of this expression of a_r in terms of the JM elements is the following:

$$a_0(\sigma) = \begin{cases} 1, & \text{if } \sigma = e, \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

This is simply because $h_0(J_1, J_2, \dots, J_n)$ is the unit element of the group algebra $\mathbb{C}[S(n)]$, i.e. the indicator function of the identity element.

Let us point out how remarkably this h-function result interacts with the work of Collins. Recall that the number $|\sigma| := n - \ell(T(\sigma))$ is the minimal length of a factorization of σ into transpositions. Our remarks above amount to the following striking result:

THEOREM 3.2. *For any permutation $\sigma \in S(n)$,*

$$\mathrm{Moeb}(\sigma) = (-1)^{|\sigma|} h_{|\sigma|}(J_1,\dots,J_n)(\sigma). \tag{26}$$

It should be possible to prove this remarkable fact using only intrinsic properties of JM elements, and this will be the subject of future work.

References

[C] B. Collins, *Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability*, International Mathematics Research Notices 17 (2003), 953–982.

[CS] B. Collins and P. Śniady, *Integration with respect to the Haar measure on unitary, orthogonal, and symplectic group*, Communications in Mathematical Physics 264 (2006), 773–795.

[J] A. Jucys, *Symmetric polynomials and the center of the symmetric group ring*, Reports on Mathematical Physics 5 (1974), 107–112.

[M] G. Murphy, *A new construction of Young's seminormal representation of the symmetric group*, Journal of Algebra 69 (1981), 287–291.

[NS] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, London Mathematical Society Lecture Note Series 335, Cambridge University Press, 2006

[OV] A. Okounkov and A. Vershik, *A new approach to the representation theory of symmetric groups*, Selecta Mathematica 4 (1996), 581–605.