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ON A TRANSLATION PROPERTY OF POSITIVE DEFINITE FUNCTIONS

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Abstract. If G is a locally compact group with a compact invariant neighbourhood of the identity e, the following property $(*)$ holds: For every continuous positive definite function $h \geq 0$ with compact support there is a constant $C_h > 0$ such that $\int L_x h \cdot g \leq C_h \int hg$ for every continuous positive definite $g \geq 0$, where L_x is left translation by x. In [L], property $(*)$ was stated, but the above inequality was proved for special h only. That "for one h" implies "for all h" seemed obvious, but turned out not to be obvious at all. We fill this gap by means of a new structure theorem for IN-groups.

For $p \in \mathbb{N}$ even, property $(*)$ easily implies the following property $(*)_p$: For every relatively compact invariant neighbourhood U of e, there is a constant $C_U > 0$ such that $\|\chi_{xU} \cdot g\|_p \leq C_U \|\chi_U \cdot g\|_p$ for every continuous positive definite function g. For all other $p \in (1, \infty)$, property $(*)_p$ fails (see [L]). In the special case of the unit circle, the $\| \, \|_p$-norm results are essentially due to N. Wiener, S. Wainger, and H. S. Shapiro. For compact abelian groups they are due to M. Rains, and for locally compact abelian groups to J. Fournier.

G denotes a locally compact group with identity element e. We assume G to be an IN-group (IN = "invariant neighbourhood"), i.e. G has a compact neighbourhood of e which is invariant under inner automorphisms of G. Integrals are taken with respect

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to left Haar measure on G, and subsets of G which we consider are assumed to be Haar measurable. By $P(G)$ resp. $P_c(G)$ we denote the set of positive definite continuous functions on G resp. the subset of those f in $P(G)$ whose support $\mathrm{supp}(f)$ is compact. The characteristic function of a set $A \subset G$ is denoted χ_A. Left translation of a function f by $a \in G$ is defined by $L_a f(x) = f(a^{-1}x)$ for $x \in G$. As usual, convolution and involution of functions are written $f * g$ and f^* respectively. Since IN-groups are unimodular, the involution is given by $f^*(x) = \overline{f(x^{-1})}$ for $x \in G$. The Dirac measure in the point $a \in G$ is δ_a. If π is a unitary representation of G, we denote its integrated version on the algebra $L^1(G)$ of integrable functions by π again.

DEFINITION. We say that G has property $(*)$ if for every non-negative $h \in P_c(G)$ there is some $C_h > 0$ such that $\int L_x h \cdot g \le C_h \int hg$ for all $x \in G$ and all non-negative $g \in P(G)$.

LEMMA 1. *Let G be locally compact. Let $x \in G$ and U be a relatively compact neighbourhood of e such that $\delta_x * \chi_U = \chi_U * \delta_x$. Then, letting $h = \chi_U^* * \chi_U$, we have*

$$\text{(i)} \quad \int L_x h \cdot g \le \int h \cdot g \quad \text{for all non-negative } g \in P(G).$$

In particular, (i) holds if x is in the center of G or U is an invariant neighbourhood of e.

Proof. We may write g in the form $g(x) = \langle \pi(x)\xi, \xi\rangle$ where π is a unitary representation of G and ξ is a cyclic vector of π. Now

$$\int L_x h \cdot g = \langle \pi(L_x h)\xi, \xi\rangle = \langle \pi(x)\pi(\chi_U^*)\pi(\chi_U)\xi, \xi\rangle = \langle \pi(x)\pi(\chi_U)\xi, \pi(\chi_U)\xi\rangle$$
$$\le \|\pi(\chi_U)\xi\|^2 = \langle \pi(\chi_U^* * \chi_U)\xi, \xi\rangle = \int hg. \quad \blacksquare$$

REMARK. Let $f, h \in P_c(G)$ be non-negative and suppose there is $C_h > 0$ such that $\int L_x h \cdot g \le C_h \int hg$ for all $x \in G$ and non-negative $g \in P(G)$. If $f > 0$ on $\mathrm{supp}\ h$, a corresponding inequality $\int L_x f \cdot g \le C_f \int fg$ holds. This follows from the fact that there are constants $c_0, \ldots, c_n > 0$ and elements $x_1, \ldots, x_n \in G$ such that $h \le c_0 f$ and $f \le \sum_{i=1}^n c_i L_{x_i} h$.

LEMMA 2. *Let G be an IN-group and H an open normal subgroup of G. If H has property $(*)$, so has G.*

Proof. By the Remark, it suffices to prove inequality (i) for functions $h = \chi_V^* * \chi_V$ where V is a small compact neighbourhood of e in G. If U is an invariant compact neighbourhood of e in G, so is $U \cap H$. Let $f = \chi_{U\cap H}^* * \chi_{U\cap H}$. Let $h = \chi_V^* * \chi_V$ with $V \subset U \cap H$. For non-negative $g \in P(G)$ and any $x \in G$ we have $\int L_x h \cdot g \le \int L_x f \cdot g \le \int fg$ by Lemma 1. Since f and h have support in H, there are $y_1, \ldots, y_n \in H$ and constants $c_1, \ldots, c_n > 0$ such that $f \le \sum_{i=1}^n c_i L_{y_i} h$. Hence $\int fg \le \sum_{i=1}^n c_i \int L_{y_i} h \cdot g \le \sum_{i=1}^n c_i C_h^H \int hg$, since these integrals are in H and H has property $(*)$. Altogether we have $\int L_x h \cdot g \le C_h^H \big(\sum_{i=1}^n c_i\big) \int hg$, which proves property $(*)$ for G. $\blacksquare$

LEMMA 3. *Let G be an IN group, V an open neighbourhood of e, $\mathcal{F}$ the system of all compact invariant neighbourhoods of e, and $N = \bigcap_{U\in\mathcal{F}} U$. Then there are a compact invariant neighbourhood M of e and elements $n_1, \ldots, n_k \in N$ such that $M \subset \bigcup_{j=1}^k n_j V$.*

Proof. Since N is compact, there are $n_1, \ldots, n_k \in N$ with $N \subset \bigcup_{j=1}^k n_j V$. This means $\bigcap_{U\in\mathcal{F}}(U\setminus\bigcup_{j=1}^k n_j V) = \emptyset$. This being an intersection of compact sets, the finite intersection property implies that there is a finite subsystem $\mathcal{F}' \subset \mathcal{F}$ with $\bigcap_{U\in\mathcal{F}'}(U\setminus\bigcup_{j=1}^k n_j V) = \emptyset$, that is, $\bigcap_{U\in\mathcal{F}'} \subset \bigcup_{j=1}^k n_j V$. So take $M = \bigcap_{U\in\mathcal{F}'} U$. ∎

THEOREM 1. *Every IN-group G contains an open normal subgroup H such that the intersection of all compact H-invariant neighbourhoods of the identity of H is contained in the center of H.*

Proof. Let G be an arbitrary IN-group with N_G the intersection of all its compact invariant neighbourhoods of the identity. The group G/N_G is SIN and therefore—according to theorem 2.13 in [GM]—an extension: $\{1\} \to V \times K \to G/N_G \to D \to \{1\}$, in which D is discrete, V a vector subgroup and K compact. The inverse image of $V \times K$ under the canonical group homomorphism $G \to G/N_G$ is an open normal IN-subgroup H of G with $G/H \simeq D$.

In particular $N_H \subset N_G \trianglelefteq H$ and

$$H/N_G = V \times K.$$

Let L be the inverse image of K under the canonical group homomorphism $H \to H/N_G$. Then L is a compact normal subgroup of H with $H/L \simeq {}^{H/N_G}\!/_{L/N_G} = {}^{V\times K}\!/_K = V$.

The component of the identity of locally compact groups is fully characteristic (Lemma 7 in [H]). Thus

$$\begin{aligned} V = V_0 = (H/L)_0 = H_0L/L \\ \Rightarrow H/H_0L \simeq {}^{H/L}\!/_{H_0L/L} = V/V_0 = \{e\} \Rightarrow H = H_0L \\ \Rightarrow H/H_0 = H_0L/H_0 \simeq L/L\cap H_0 \quad \text{which is compact.} \end{aligned}$$

The group H therefore meets the requirements of Theorem XII in [H] which implies the following:

- H contains a compact group C and a solvable connected group S such that:

$$H/S\cap C = S/S\cap C \ltimes C/S\cap C.$$

  $S/S\cap C$ is a normal vector subgroup of H/N_H.
- $S \cap C$ is the intersection of all compact invariant neighbourhoods of e in H, i.e. $S \cap C = N_H$.
- H contains a subgroup H_1 of finite index. H_1 contains the component of the identity H_0 and H_1/N_H is the centralizer of S/N_H (proof in [H] page 53). Moreover N_H is in the center of H_1.

Theorem 2.13 in [GM] says that H is $[FC]^-$. Thus $H/N_H \in [FC]^-$ (being the continuous homomorphic image of an $[FC]^-$ group). Application of 3.4 in [GM] yields that the normal vector subgroup S/N_H has to be central in H/N_H. In particular its centralizer H_1/N_H is the whole group H/N_H. So H_1 is a subgroup of H containing N_H and intersecting every N_H orbit. Therefore $H_1 = H$ and N_H is contained in the center of $H_1 = H$. ∎

REMARK. In the special case that G is connected (then one has $H = G$), the theorem is due to Iwasawa [I].

Theorem 2. *Every IN-group has property* $(*)$.

Proof Let G be an IN-group and let H be as in Theorem 1. By Lemma 2, it suffices to prove property $(*)$ for H. If W is a fixed compact invariant neighbourhood of e in G, $W' = W \cap H$ is a compact invariant neighbourhood of e in H, so H is an IN-group, too. Let N be the intersection of all compact (H-) invariant neighbourhoods of e in H. If V is an open relatively compact neighbourhood of e in H, then, by Lemma 3, there are a compact invariant neighbourhood U of e in H and elements $n_1, \dots, n_k \in N$ such that $U \subset \bigcup_{i=1}^k n_i V$. On the other hand, there are $y_1, \dots, y_\ell \in H$ such that $V \subset \bigcup_{i=1}^\ell y_i U$. Taking into account that the n_i and χ_U are central in H and $L^1(H)$, respectively, we obtain

$$\text{(i)}\quad \chi_U^* * \chi_U \le \Big(\sum_{i=1}^k L_{n_i}\chi_V\Big)^* * \Big(\sum_{i=1}^k L_{n_i}\chi_V\Big) = \sum_{i,j=1}^k L_{n_i^{-1}n_j}\chi_V^* * \chi_V,$$

$$\text{(ii)}\quad \chi_V^* * \chi_V \le \Big(\sum_{i=1}^\ell L_{y_i}\chi_U\Big)^* * \Big(\sum_{i=1}^\ell L_{y_i}\chi_U\Big) = \sum_{i,j=1}^\ell L_{y_i^{-1}y_j}\chi_U^* * \chi_U.$$

For $y \in H$ and non-negative $g \in P(H)$ we obtain from (ii) and (i) by Lemma 1

$$\text{(iii)}\quad \int L_y\chi_V^* * \chi_V \cdot g \le \int \sum_{i,j=1}^\ell L_{yy_i^{-1}y_j}\chi_U^* * \chi_U \cdot g \le \ell^2 \int \chi_U^* * \chi_U \cdot g,$$

$$\text{(iv)}\quad \int \chi_U^* * \chi_U \cdot g \le \int \sum_{i,j=1}^k L_{n_i^{-1}n_j}\chi_V^* * \chi_V \cdot g \le k^2 \int \chi_V^* * \chi_V \cdot g.$$

So $\int L_y\chi_V^* * \chi_V \cdot g \le k^2\ell^2 \int \chi_V^* * \chi_V \cdot g$. Considering small V and using the Remark after Lemma 1 one sees that H has property $(*)$. ■

Finally, the statement of Theorem 1.6 in [L] is correct:

Corollary. *If G is an IN-group, $(*)_p$ holds for all even natural numbers p and fails for all other $p \in (1, \infty)$.*

The proof for even p is an easy consequence of Theorem 2, using the fact that $|g|^p = g^{p/2} \cdot \bar{g}^{p/2} \in P(G)$ for $g \in P(G)$ and that every χ_U can be sandwiched between two nonzero non-negative functions h, $h' \in P_c(G)$. For the proof that property $(*)_p$ fails for all non-even $p \in (1, \infty)$, see [L].

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