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A COMMENT ON FREE GROUP FACTORS

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Abstract. Let M be a finite von Neumann algebra acting on the standard Hilbert space $L^2(M)$. We look at the space of those bounded operators on $L^2(M)$ that are compact as operators from M into $L^2(M)$. The case where M is the free group factor is particularly interesting.

1. Introduction. In the paper [Oz1], it is proved that the free group factor $\mathcal{L}\mathbb{F}_r$ is solid, i.e., the relative commutant $B' \cap \mathcal{L}\mathbb{F}_r$ of any diffuse subalgebra B is amenable. The proof relies on C*-algebra techniques. (See [Pe, Po2] for purely von Neumann algebraic proofs of this fact.) In particular, the crucial ingredient in [Oz1] is Akemann and Ostrand's theorem ([AO]) stating that the $*$-homomorphism

$$\mu\colon C^*_\lambda\mathbb{F}_r \otimes_{\mathrm{alg}} C^*_\rho\mathbb{F}_r \ni \sum a_k \otimes x_k \mapsto \sum a_k x_k + \mathbb{K}(\ell^2\mathbb{F}_r) \in \mathbb{B}(\ell^2\mathbb{F}_r)/\mathbb{K}(\ell^2\mathbb{F}_r)$$

is continuous w.r.t. the minimal tensor norm. It would be interesting to know how much of the proof in [Oz1] can be carried out at the level of von Neumann algebras. In this paper, we will prove a version of Akemann and Ostrand's theorem in the von Neumann setting. For this purpose, we consider the set of those operators in $\mathbb{B}(L^2(M))$ that are compact as operators from M into $L^2(M)$, where $M = \mathcal{L}\mathbb{F}_r$ or any finite von Neumann algebra.

2. Compact operators. Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathbb{B}(\mathcal{H})$ (resp. $\mathbb{K}(\mathcal{H})$) the C*-algebra of all bounded (resp. compact) linear operators on $\mathcal{H}$. Let $\Omega \subset \mathcal{H}$ be a closed balanced bounded convex subset. (Recall that Ω is said to be balanced if $\alpha\Omega \subset \Omega$ for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$.) We define the closed left ideal $\mathbb{K}^L_\Omega$ of $\mathbb{B}(\mathcal{H})$ by

$$\mathbb{K}^L_\Omega = \{x \in \mathbb{B}(\mathcal{H}) : x\Omega \text{ is norm compact in } \mathcal{H}\}.$$

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We define a seminorm $\|\cdot\|_\Omega$ on $\mathbb{B}(\mathcal{H})$ by

$$\|x\|_\Omega = \sup\{\|x\xi\| : \xi \in \Omega\}.$$

We will use the following trivial proposition without quoting it.

PROPOSITION. *Let $\Omega \subset \mathcal{H}$ be as above. Then, for $x \in \mathbb{B}(\mathcal{H})$, the following are equivalent.*

1. $x \in \mathbb{K}_\Omega^L$.
2. *x is weak-norm continuous on Ω.*
3. *For every weakly null sequence (ξ_n) in Ω, one has $\|x\xi_n\| \to 0$.*
4. *For every sequence of (finite-rank) projections (Q_n) strongly converging to 1 on $\mathcal{H}$, one has $\|x - Q_n x\|_\Omega \to 0$.*
5. *There exists a sequence (x_n) in $\mathbb{K}(\mathcal{H})$ such that $\|x - x_n\|_\Omega \to 0$.*

DEFINITION. We denote by $\mathbb{K}_\Omega$ the hereditary C*-subalgebra of $\mathbb{B}(\mathcal{H})$ associated with the left ideal $\mathbb{K}_\Omega^L$:

$$\mathbb{K}_\Omega = (\mathbb{K}_\Omega^L)^* \cap \mathbb{K}_\Omega^L = (\mathbb{K}_\Omega^L)^* \cdot \mathbb{K}_\Omega^L.$$

Let $x \in \mathbb{K}_\Omega$. For finite-rank projections $Q_n \nearrow 1$ on $\mathcal{H}$, we define $x_n = aQ_n b$, where $b = |x|^{1/2}$ and $a = xb^{-1}$ are in $\mathbb{K}_\Omega$. Then, $x_n \in \mathbb{K}(\mathcal{H})$ satisfies $\|x_n\| \le \|x\|$ and $\|x - x_n\|_\Omega + \|x^* - x_n^*\|_\Omega \to 0$.

3. Finite von Neumann algebras. Let M be a finite von Neumann algebra with a distinguished faithful normal trace τ, and $L^2(M)$ be the GNS-Hilbert space associated with (M, τ). We will write $\hat{a}$ for $a \in M$ when viewed as a vector in $L^2(M)$, and $\|a\|_2 = \|\hat{a}\| = \tau(a^*a)^{1/2}$. From now on, we set

$$\Omega = \{\hat{a} : a \in M,\ \|a\| \le 1\} \subset L^2(M)$$

and write $\mathbb{K}_M$ instead of $\mathbb{K}_\Omega$. It is clear that both M and M' are in the multiplier of $\mathbb{K}_M$. The C*-algebra $\mathbb{K}_M$ is much larger than $\mathbb{K}(L^2(M))$. Indeed, if p_n are mutually orthogonal projections in M (or in M') and x_n are compact contractive operators such that $x_n = p_n x_n p_n$, then $\sum x_n \in \mathbb{K}_M$. The following is useful in understanding the nature of the norm $\|\cdot\|_\Omega$.

LEMMA. *For every $x \in \mathbb{B}(\mathcal{H})$, one has*

$$\|x\|_\Omega \le \inf\{\|y\|\|b\|_2 + \|z\|\|c'\|_2\} \le 4\|x\|_\Omega,$$

where the infimum is taken over all possible decompositions $x = yb + zc'$ with $y, z \in \mathbb{B}(\mathcal{H})$, $b \in M$ and $c' \in M'$.

Proof. Since

$$\|yb\|_\Omega = \sup_{a \in (M)_1} \|yb\hat{a}\| \le \|y\| \sup_{a \in (M)_1} \|\widehat{ba}\| \le \|y\| \sup_{a \in (M)_1} \|\hat{b}\|\|a\| = \|y\|\|b\|_2$$

and $\|zc'\|_\Omega \le \|z\|\|c'\|_2$ similarly, one has $\|yb + zc'\|_\Omega \le \|y\|\|b\|_2 + \|z\|\|c'\|_2$.

To prove the other inequality, let $x \in \mathbb{B}(\mathcal{H})$ be given such that $\|x\|_\Omega = 1$. We observe that $\|x\|_\Omega$ is nothing but the norm as an operator from M into $L^2(M)$. It follows from the noncommutative little Grothendieck inequality (Theorem 9.4 in [Pi]) that there are unit vectors $\zeta, \eta \in L^2(M)$ such that $\|x\hat{a}\|^2 \le \|a\zeta\|^2 + \|\eta a\|^2$ for all $a \in M$. We view ζ and

η as square integrable operators affiliated with M (see Appendix F in [BO] or Chapter IX in [Ta]), and let $q = \chi_{(\|x\|^2,\infty)}(\zeta\zeta^*)$, $p = \chi_{(\|x\|^2,\infty)}(\eta^*\eta)$. It follows that

$$\begin{aligned}\|x\hat{a}\|^2 &\le 2\big(\|x(p^\perp \hat{a} q^\perp)\|^2 + \|x(p\hat{a}q^\perp)\|^2 + \|x(\hat{a}q)\|^2\big)\\ &\le 2\big(\|aq^\perp\zeta\|_2^2 + \|\eta p^\perp a\|_2^2 + \|x\|^2\|pa\|_2^2 + \|x\|^2\|aq\|_2^2\big)\\ &= 2\big(\|b\hat{a}\|_2^2 + \|c'\hat{a}\|_2^2\big),\end{aligned}$$

where b is the left multiplication operator by $(p^\perp\eta^*\eta p^\perp + \|x\|^2 p)^{1/2}$ and c' is the right multiplication operator by $(q^\perp\zeta\zeta^* q^\perp + \|x\|^2 q)^{1/2}$. Note that one has $b \in M$, $\|b\| \le \|x\|$ and $\|b\|_2 \le \|\zeta\|_2 = 1$; and likewise for $c' \in M'$. It follows that there are operators $y, z \in \mathbb{B}(\mathcal{H})$ with $yy^* + zz^* \le 2$ such that $x = yb + zc'$. ■

The "cb-version" of the norm $\|\cdot\|_\Omega$ is defined to be

$$\begin{aligned}\|x\|_\tau &= \sup\Big\{\Big(\sum \|x\hat{a}_n\|^2\Big)^{1/2} : (a_n)_{n=1}^\infty \in M \text{ such that } \sum a_n a_n^* \le 1\Big\}\\ &= \|x\colon M' \ni a' \mapsto xa'\hat{1} \in L^2(M)_{\mathrm{col}}\|_{\mathrm{cb}}\\ &= \inf\{\|y\|\|b\|_2 : y \in \mathbb{B}(\mathcal{H}) \text{ and } b \in M \text{ with } x = yb\}.\end{aligned}$$

We do not elaborate on this norm here. See [Ma] for more information about the topology associated with this norm.

4. Free group factors. We write λ and ρ respectively for the left and the right regular representation of a countable discrete group Γ on $\ell^2\Gamma$. Recall that the group Γ is in the class $\mathcal{S}$ if it is exact and the $*$-homomorphism

$$\mu\colon C^*_\lambda\Gamma \otimes_{\mathrm{alg}} C^*_\rho\Gamma \ni \sum a_k \otimes x_k \mapsto \sum a_k x_k + \mathbb{K}(\ell^2\Gamma) \in \mathbb{B}(\ell^2\Gamma)/\mathbb{K}(\ell^2\Gamma)$$

is continuous w.r.t. the minimal tensor norm. Free groups as well as hyperbolic groups are in the class $\mathcal{S}$. (See [Oz2].) Let Γ be an ICC group so that the group von Neumann algebra $\mathcal{L}\Gamma = \lambda(\Gamma)'' \subset \mathbb{B}(\ell^2\Gamma)$ is a factor. We note that $L^2(\mathcal{L}\Gamma)$ is canonically isomorphic to $\ell^2\Gamma$. We denote $\mathcal{R}\Gamma = (\rho(\Gamma))'' = (\mathcal{L}\Gamma)'$ and consider the $*$-homomorphism

$$\pi\colon C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma) \ni \sum a_k x_k \mapsto \sum a_k \otimes x_k \in \mathcal{L}\Gamma \otimes_{\min} \mathcal{R}\Gamma \subset \mathbb{B}(\ell^2\Gamma \otimes \ell^2\Gamma),$$

which is well-defined by Takesaki's theorem on the minimal tensor norm. The following theorem extends Akemann and Ostrand's theorem ([AO]).

THEOREM. *Let Γ be an ICC group which is in the class $\mathcal{S}$. Then*

$$\ker \pi = \mathbb{K}_{\mathcal{L}\Gamma} \cap C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma).$$

Proof. Take any sequence (ξ_n) of unit vectors in $\Omega = \{\hat{a} : a \in \mathcal{L}\Gamma,\ \|a\| \le 1\}$, which weakly converges to 0. We define a state ω on $C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma)$ by the Banach limit $\omega(x) = \mathrm{Lim}\langle x\xi_n, \xi_n\rangle$. Let $y \in \mathrm{alg}(\mathcal{L}\Gamma, \mathcal{R}\Gamma)$ be arbitrary. Since $y\Omega \subset K_y\Omega$ for some constant $K_y > 0$, one has $\omega(y^* \cdot y) \le K_y^2\tau(\cdot)$ both on $\mathcal{L}\Gamma$ and on $\mathcal{R}\Gamma$. Therefore, the GNS representation π_ω of ω is binormal on $C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma)$. Moreover, since $\mathbb{K}_{\mathcal{L}\Gamma} \cap C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma) \subset \ker \pi_\omega$, the $*$-homomorphism from $C^*_\lambda\Gamma \otimes_{\mathrm{alg}} C^*_\rho\Gamma$ into $\pi_\omega(C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma))$ is continuous w.r.t. the minimal tensor norm. It follows from Lemma 9.2.9 in [BO] that the $*$-homomorphism from $\mathcal{L}\Gamma \otimes_{\mathrm{alg}} \mathcal{R}\Gamma$ into $\pi_\omega(C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma))$ is continuous w.r.t. the minimal tensor norm, too. Because of simplicity of $\mathcal{L}\Gamma$, this means that $\pi_\omega(C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma)) = \mathcal{L}\Gamma \otimes_{\min} \mathcal{R}\Gamma$, or equivalently that $\ker\pi_\omega = \ker\pi$. Therefore, $\mathbb{K}_{\mathcal{L}\Gamma} \cap C^*(\mathcal{L}\Gamma, \mathcal{R}\Gamma) \subset \ker\pi$. On the other hand, if $x \geq 0$ and $x \notin \mathbb{K}_{\mathcal{L}\Gamma}$, then there is a normalized weakly null sequence (ξ_n) in Ω such that $\omega(x^2) = \operatorname{Lim}\|x\xi_n\|^2 > 0$ and *a fortiori* $x \notin \ker\pi_\omega$. ∎

It follows that $\mathbb{K}(\ell^2\Gamma) \subset \ker\pi \subset \mathbb{K}_{\mathcal{L}\Gamma}$. The first inclusion is strict. Indeed, it is not hard to show that $\ker\pi$ is non-separable. It is likely that the second is strict as well.

Recall that a finite von Neumann algebra N has the property (Γ) if there is a sequence (u_n) of unitary elements in N such that $u_n \to 0$ ultraweakly and $[u_n, a] \to 0$ ultrastrongly for every $a \in N$. We observe the following: Let $M \subset \mathbb{B}(L^2(M))$ be a finite von Neumann algebra and $N \subset M$ be a von Neumann subalgebra with the property (Γ). Then, one has $\mathbb{K}_M \cap C^*(N, M') = \{0\}$. Indeed, if (u_n) is as above, then on the one hand $u_n^* x u_n \to x$ for every $x \in C^*(N, M')$, but on the other hand $u_n^* x u_n \to 0$ for every $x \in \mathbb{K}_M$. This observation, combined with the above theorem, implies the main theorem of [Oz1]: A von Neumann subalgebra of $\mathcal{L}\Gamma$ which has the property (Γ) is necessarily amenable.

5. Boundary of free group factors. Let $\mathbb{F}_r$ be the free group of rank $r \in \mathbb{N}$. For each $t \in \mathbb{F}_r$, we define $\chi_t \in \ell^\infty\mathbb{F}_r$ to be the characteristic function of the set of those elements in $\mathbb{F}_r$ whose last segments in the reduced forms are t. Let

$$A = C^*(\{\chi_t : t \in \mathbb{F}_r\}) \subset \ell^\infty\mathbb{F}_r$$

and observe that $[A, C^*_\lambda\mathbb{F}_r] \subset \mathbb{K}(\ell^2\mathbb{F}_r)$. Indeed, $\lambda(s)\chi_t\lambda(s)^*\delta_x = \chi_t\delta_x$ if $|x| \geq |s| + |t|$, and hence $[\chi_t, \lambda(s)]$ has finite rank. It is well-known that $B := C^*(A, \rho(\mathbb{F}_r)) \cong A \rtimes \mathbb{F}_r$ is nuclear. Akemann and Ostrand's theorem stating that $\mathbb{F}_r$ is in the class $\mathcal{S}$ follows from this and

$$[C^*_\lambda\mathbb{F}_r, B] \subset \text{norm-cl}\big(C^*_\lambda\mathbb{F}_r \cdot [C^*_\lambda\mathbb{F}_r, A]\big) \subset \mathbb{K}(\ell^2\mathbb{F}_r).$$

It would be interesting to know whether a similar fact holds true at the level of von Neumann algebras. Namely,

Problem. Is it true that $[A, \mathcal{L}\mathbb{F}_r] \subset \mathbb{K}_{\mathcal{L}\mathbb{F}_r}$?

We recall Popa's theorem ([Po1]) stating that every derivation from a von Neumann algebra $M \subset \mathbb{B}(\mathcal{H})$ into $\mathbb{K}(\mathcal{H})$ is inner. In particular, $[x, M] \subset \mathbb{K}(\mathcal{H})$ only if $x \in \mathbb{K}(\mathcal{H}) + M'$. Nevertheless, the above problem has a positive answer if $r = 1$, i.e., if $\mathbb{F}_r = \mathbb{Z}$. Indeed, let $\chi = \chi_{n\geq 0}$ for simplicity. Then, the projection χ is the Riesz projection which is bounded on $L^4(\widehat{\mathbb{Z}})$ (or on any L^q with $1 < q < \infty$, see [Ga]). It follows from the Hölder inequality that for every $a \in \mathcal{L}(\mathbb{Z}) \cong L^\infty(\widehat{\mathbb{Z}})$, one has

$$\|[\chi, a]\|_\Omega \leq \|\chi\|_{2,4}\|a\|_{4,\infty} + \|a\|_{2,4}\|\chi\|_{4,\infty} \leq C\|a\|_{L^4(\widehat{\mathbb{Z}})},$$

where $\|\cdot\|_{p,q}$ stands for the operator norm from $L^q(\widehat{\mathbb{Z}})$ into $L^p(\widehat{\mathbb{Z}})$. Since $C^*_\lambda\mathbb{Z}$ is dense in $\mathcal{L}\mathbb{Z}$ w.r.t. the L^4-norm and $[\chi, C^*_\lambda\mathbb{Z}] \in \mathbb{K}(\ell^2\mathbb{Z})$, one obtains that $[\chi, \mathcal{L}\mathbb{Z}] \in \mathbb{K}_{\mathcal{L}\mathbb{Z}}$. The author is unable to extend this argument to $\mathbb{F}_r$ with $r \geq 2$, because he does not know whether the 'Riesz projection' on $\mathbb{F}_r$ is a bounded operator from $\mathcal{L}\mathbb{F}_r$ into $L^4(\mathcal{L}\mathbb{F}_r)$.

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References

[AO] C. A. Akemann and P. A. Ostrand, *On a tensor product C^*-algebra associated with the free group on two generators*, J. Math. Soc. Japan 27 (1975), 589–599.

[BO] N. P. Brown and N. Ozawa, *C^*-algebras and Finite-Dimensional Approximations*, Grad. Stud. Math. 88, Amer. Math. Soc., Providence, RI, 2008.

[Ga] J. Garnett, *Bounded Analytic Functions*, revised first edition, Graduate Texts in Mathematics 236, Springer, New York, 2007.

[Ma] B. Magajna, *A topology for operator modules over W^*-algebras*, J. Funct. Anal. 154 (1998), 17–41.

[Oz1] N. Ozawa, *Solid von Neumann algebras*, Acta Math. 192 (2004), 111–117.

[Oz2] N. Ozawa, *A Kurosh-type theorem for type* II_1 *factors*, Int. Math. Res. Not. 2006, Art. ID 97560, 21 pp.

[Pe] J. Peterson, *L^2-rigidity in von Neumann algebras*, Invent. Math. 175 (2009), 417–433.

[Pi] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Series in Math. 60, Amer. Math. Soc., Providence, RI, 1986.

[Po1] S. Popa, *The commutant modulo the set of compact operators of a von Neumann algebra*, J. Funct. Anal. 71 (1987), 393–408.

[Po2] S. Popa, *On Ozawa's property for free group factors*, Int. Math. Res. Not. 2007, Art. ID rnm036, 10 pp.

[Ta] M. Takesaki, *Theory of Operator Algebras. II*, Encyclopaedia of Mathematical Sciences 125, Springer, Berlin, 2003.