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POISSON BOUNDARIES OF DISCRETE QUANTUM GROUPS

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Abstract. This is a survey article about a theory of a Poisson boundary associated with a discrete quantum group. The main problem of the theory, that is, the identification problem is explained and solved for some examples.

1. Introduction. In [7], M. Izumi initiates the theory of a Poisson boundary for a random walk on a discrete quantum group. The main problem is the identification, that is, to find a more concrete realization as a von Neumann algebra. For example, the $SU_q(n)$ case is studied in [7, 8], and their main result shows the identification of the Poisson boundary with the quantum flag manifold $\mathbb{T}\backslash SU_q(n)$. This result is generalized to a co-amenable compact quantum group with commutative fusion rules by the author in [18]. Starting from basics of compact or discrete quantum groups, we explain the idea of its proof.

In this article, another new problem is also presented in §4.2. More precisely, our conjecture is the following:

CONJECTURE 1. *Let $\mathbb{G}$ be a compact quantum group and μ a generating probability measure. Then the following equality holds:*

$$H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\mathrm{class}} = Z(H^\infty(\widehat{\mathbb{G}}, P_\mu)).$$

Here $H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\mathrm{class}}$ means the classical Poisson boundary on the center of the discrete quantum group, and $Z(H^\infty(\widehat{\mathbb{G}}, P_\mu))$ is the center of the Poisson boundary. In the final

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part of this article, we will verify this conjecture for $SU_q(2)$. We do not know if that holds for other q-deformations of classical compact Lie groups, but it does for some non-amenable examples such as $A_o(F)$ and $A_u(F)$ [19, 20].

2. Quantum groups. Throughout this survey, we will mainly treat quantum groups of compact or discrete type. Our standard references are [5, 11, 22]. We denote by $\otimes$ minimal tensor products or spatial tensor products for C^*-algebras or von Neumann algebras, respectively.

2.1. Compact quantum groups. The following definition of a compact quantum group has been introduced by S. L. Woronowicz [22]:

DEFINITION 2.1 (Woronowicz). A *compact quantum group* (c.q.g.) $\mathbb{G}$ is a pair $(C(\mathbb{G}), \delta)$ that satisfies the following conditions:

1. $C(\mathbb{G})$ is a separable unital C^*-algebra;
2. (Coproduct) The map $\delta\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a coproduct, i.e. it is a faithful unital $*$-homomorphism satisfying the co-associativity condition,
$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta;$$
3. (Cancellation property) The vector spaces $\delta(C(\mathbb{G}))(\mathbb{C} \otimes C(\mathbb{G}))$ and $\delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes \mathbb{C})$ are dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

EXAMPLE 2.2. A compact group $\mathbb{G}$ is regarded as a compact quantum group. Indeed, via the identification $C(\mathbb{G}) \otimes C(\mathbb{G}) = C(\mathbb{G} \times \mathbb{G})$, a coproduct δ is defined by
$$\delta(x)(r, s) := x(rs) \text{ for all } x \in C(\mathbb{G}), r, s \in \mathbb{G}.$$
The cancellation property means $rs = rt$ or $sr = tr$ imply $s = t$ for $r, s, t \in \mathbb{G}$. Note that a compact semigroup with cancellation property is a compact group.

Let Γ be a discrete group. Then the full group C^*-algebra $\mathrm{C}^*\Gamma$ and the reduced group C^*-algebra $\mathrm{C}^*_{\mathrm{red}}\Gamma$ are compact quantum groups. The coproducts are given by $\delta(r) = r \otimes r$ for $r \in \Gamma$.

The following state called *Haar state* plays an important role in the study of quantum groups.

THEOREM 2.3 (Woronowicz). *There exists a unique state $h \in C(\mathbb{G})^*$ such that*
$$(\mathrm{id} \otimes h)(\delta(a)) = h(a)1 = (h \otimes \mathrm{id})(\delta(a)) \textit{ for all } a \in C(\mathbb{G}).$$

EXAMPLE 2.4. The Haar state of $\mathrm{C}^*_{\mathrm{red}}\Gamma$ is given by the canonical tracial state $(\cdot\, \delta_e, \delta_e)$. Composing the state and the surjection $\mathrm{C}^*\Gamma \to \mathrm{C}^*_{\mathrm{red}}\Gamma$, we obtain the Haar state on $\mathrm{C}^*\Gamma$, which is not faithful when Γ is non-amenable.

2.2. Reduced quantum groups. Let $N_h := \{a \in C(\mathbb{G}) \mid h(a^*a) = 0\}$. Then it is known that N_h is in fact an ideal of $C(\mathbb{G})$, and we can consider the *reduced compact quantum group* $C(\mathbb{G}_{\mathrm{red}}) := C(\mathbb{G})/N_h$ with a natural coproduct. By definition, h is faithful on $C(\mathbb{G}_{\mathrm{red}})$.

Let $(L^2(\mathbb{G}), \pi_h, \Omega_h)$ be the GNS representation associated with the Haar state h, that is,

- $L^2(\mathbb{G})$ is a Hilbert space;
- $\pi_h \colon C(\mathbb{G}) \to B(L^2(\mathbb{G}))$ is a $*$-homomorphism;
- $\Omega_h \in L^2(\mathbb{G})$ is the GNS cyclic vector, i.e. we have $L^2(\mathbb{G}) = \overline{\pi_h(C(\mathbb{G}))\Omega_h}$ and $h(a) = (\pi_h(a)\Omega_h, \Omega_h)$.

Note that N_h is precisely equal to $\ker \pi_h$. Hence we can regard $C(\mathbb{G}_{\mathrm{red}}) = \pi_h(C(\mathbb{G}))$. We often omit π_h.

2.3. Multiplicative unitaries. From the bi-invariance of the state h, the following theorem follows:

THEOREM 2.5. *There exists a unitary* $V \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ *satisfying*

$$V(a\Omega_h \otimes \xi) = \delta(a)(\Omega_h \otimes \xi) \text{ for all } a \in C(\mathbb{G}), \xi \in L^2(\mathbb{G});$$

Then V satisfies the following notable *pentagon equation*:

$$V_{12}V_{13}V_{23} = V_{23}V_{12}. \tag{2.1}$$

So, V is called the *multiplicative unitary* [5]. By definition, we have the following implementation formula:

$$V(a \otimes 1)V^* = \delta(a) \quad \text{for all } a \in C(\mathbb{G}_{\mathrm{red}}). \tag{2.2}$$

2.4. Von Neumann algebraic quantum groups. We denote by $L^\infty(\mathbb{G})$ the weak closure of $C(\mathbb{G}_{\mathrm{red}})$ in $B(L^2(\mathbb{G}))$. The coproduct δ extends to the normal morphism from $L^\infty(\mathbb{G})$ into $L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ through (2.2). Then the pair $(L^\infty(\mathbb{G}), \delta)$ is called the *von Neumann algebraic compact quantum group* [11]VT. There exists a modular automorphism for h on $C(\mathbb{G}_{\mathrm{red}})$, and the Haar state $h(\cdot) = (\cdot\Omega_h, \Omega_h)$ is faithful on $L^\infty(\mathbb{G})$ [22].

2.5. Kac type quantum groups

DEFINITION 2.6. A compact quantum group is said to be of *Kac type* when the Haar state is tracial, i.e. $h(ab) = h(ba)$ for all $a, b \in C(\mathbb{G})$.

A compact group or a C^*-group algebra of a discrete group are typical examples of Kac type quantum groups. They are commutative or co-commutative. Woronowicz's twisted quantum group $SU_{-1}(n)$ is also of Kac type, which is neither commutative nor co-commutative [21]. Readers should note the first such example discovered by G. I. Kac and V. G. Paljutkin [9], which is 8-dimensional C^*-algebra (8 is the smallest dimension allowing a non-trivial Kac algebra).

2.6. Representation theory

DEFINITION 2.7. Let H be a Hilbert space. A unitary $v \in B(H) \otimes L^\infty(\mathbb{G})$ is called a *(right unitary) representation* if it satisfies

$$(\mathrm{id} \otimes \delta)(v) = v_{12}v_{13}. \tag{2.3}$$

We have the following natural operations:

- (**direct sum**)

$$v_1 \oplus v_2 := \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \in B(H_1 \oplus H_2) \otimes L^\infty(\mathbb{G}).$$

- (**tensor product**)
$$v_1 \otimes v_2 := (v_1)_{13}(v_2)_{23} \in B(H_1 \otimes H_2) \otimes L^\infty(\mathbb{G}).$$
- (**conjugation**) Let $v = (v_{ij})_{i,j\in I}$ be a matrix form of a representation. Consider
$$v^c := (v_{ij}^*)_{i,j\in I}$$
which may not be a unitary, but this still satisfies (2.3). In fact, if the dimension is finite, then it is unitarizable, i.e. there exists a positive invertible operator Q such that
$$\overline{v} := (Q^{1/2} \otimes 1)v^c(Q^{-1/2} \otimes 1)$$
is a unitary matrix (see [22]).

We introduce the intertwiner space between unitary representations $v_i \in B(H_i) \otimes L^\infty(\mathbb{G})$, $i = 1, 2$,
$$\mathrm{Mor}(v_1, v_2) := \{T \in B(H_1, H_2) \mid (T \otimes 1)v_1 = v_2(T \otimes 1)\}.$$

Definition 2.8. Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a unitary representation.

- A unitary representation is said to be *irreducible* when $\mathrm{Mor}(v, v) = \mathbb{C}1_H$.
- Let $w \in B(K) \otimes L^\infty(\mathbb{G})$ be a unitary representation. We say that v and w are *equivalent* if $\mathrm{Mor}(v, w)$ contains a unitary.

Theorem 2.9 (Woronowicz). *For any compact quantum group $\mathbb{G}$, the following hold:*

1. *An irreducible representation is finite dimensional;*
2. *A finite dimensional representation is the direct sum of irreducibles;*
3. *Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a finite dimensional representation. Then $v \in B(H) \otimes C(\mathbb{G})$.*

We denote by $\mathrm{Irr}(\mathbb{G})$ the set of equivalence classes of irreducible representations of $\mathbb{G}$. For each $\pi \in \mathrm{Irr}(\mathbb{G})$, we choose a corresponding representation $v_\pi \in B(H_\pi) \otimes L^\infty(\mathbb{G})$. Note that $\dim(H_\pi) < \infty$ from the previous theorem.

Definition 2.10. We say that $\mathbb{G}$ has *commutative fusion rules* when $v \otimes w$ is equivalent to $w \otimes v$ for any unitary representations v, w.

Example 2.11. If $\mathbb{G}$ is a compact group, then a usual flip map intertwines $v \otimes w$ and $w \otimes v$. For a q-deformation, an R-matrix takes place of that.

3. Discrete quantum groups. Let $\mathbb{G}$ be a c.q.g. In this section, we study basic properties of the dual $\widehat{\mathbb{G}}$.

3.1. Right group algebras. Recall the multiplicative unitaries V, which are right and left representations of $\mathbb{G}$ on $L^2(\mathbb{G})$. We introduce the following subspace:
$$R(\mathbb{G}) := \overline{\mathrm{span}}^{\mathrm{w}}\{(\mathrm{id} \otimes \omega)(V) \mid \omega \in L^\infty(\mathbb{G})_*\},$$
which we call the *right group algebra*.

Define the map $\beta \colon B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G})) \otimes B(L^2(\mathbb{G}))$,
$$\beta(x) := V^*(1 \otimes x)V \text{ for } x \in B(L^2(\mathbb{G})).$$

THEOREM 3.1. *The following hold:*

- $R(\mathbb{G})$ *is a von Neumann algebra;*
- *The restriction* $\Delta := \beta|_{R(\mathbb{G})}$ *defines the coproduct, i.e.*

$$\Delta(R(\mathbb{G})) \subset R(\mathbb{G}) \otimes R(\mathbb{G}), \quad (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

So, the pair $(R(\mathbb{G}), \Delta)$ is a bialgebra. In fact, it is known that there exist weights φ and ψ on $R(\mathbb{G})$ such that

- $\varphi((\omega \otimes \mathrm{id})(\Delta(x))) = \omega(1)\psi(x)$ for all $\omega \in R(\mathbb{G})_*^+$, $x \in R(\mathbb{G})_+$.
- $\psi((\mathrm{id} \otimes \omega)(\Delta(x))) = \omega(1)\varphi(x)$ for all $\omega \in R(\mathbb{G})_*^+$, $x \in R(\mathbb{G})_+$;

Therefore, $\widehat{\mathbb{G}} := (R(\mathbb{G}), \Delta)$ is a quantum group in the sense of [11]. Using the usual Peter-Weyl type isomorphism,

$$L^2(\mathbb{G}) \cong \bigoplus_{\pi \in \mathrm{Irr}(\mathbb{G})} H_{\overline{\pi}} \otimes H_\pi,$$

we obtain the isomorphism,

$$R(\mathbb{G}) \to \bigoplus_{\pi \in \mathrm{Irr}(\mathbb{G})} B(H_\pi).$$

Hence $\widehat{\mathbb{G}}$ is also called a *discrete quantum group*.

3.2. Actions of quantum groups

DEFINITION 3.2. Let $\mathbb{G} = (L^\infty(\mathbb{G}), \delta)$ be a locally compact quantum group and M a von Neumann algebra. A map $\alpha\colon M \to M \otimes L^\infty(\mathbb{G})$ is called a *(right) action* when it satisfies the following:

- α is a unital faithful normal $*$-homomorphism;
- $(\mathrm{id} \otimes \delta) \circ \alpha = (\alpha \otimes \mathrm{id}) \circ \alpha$.

A left action is similarly defined.

EXAMPLE 3.3. The map $\beta\colon B(L^2(\mathbb{G})) \to R(\mathbb{G}) \otimes B(L^2(\mathbb{G}))$ is a left action of $\widehat{\mathbb{G}}$. Similarly $\alpha\colon B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G})) \otimes L^\infty(\mathbb{G})$ defined by $\alpha(x) = V(x \otimes 1)V^*$ is a right action of $\mathbb{G}$.

3.3. Quantum subgroups and left (right) coideals. There are several ways to define a quantum subgroup of a compact quantum group, but we adopt the following definition in this article.

DEFINITION 3.4. Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. We say that $\mathbb{H}$ is a *quantum subgroup* of $\mathbb{G}$ if there exists a unital $*$-homomorphism $r_{\mathbb{H}}\colon A(\mathbb{G}) \to A(\mathbb{H})$ such that

- $r_{\mathbb{H}}$ is surjective;
- $\delta_{\mathbb{H}} \circ r_{\mathbb{H}} = (r_{\mathbb{H}} \otimes r_{\mathbb{H}}) \circ \delta_{\mathbb{G}}$.

This definition is weaker than the usual C^*-version, which requires $r_{\mathbb{H}}$ is a C^*-homomorphism from $C(\mathbb{G})$ onto $C(\mathbb{H})$. Readers should notice that the map $r_{\mathbb{H}}$, called the *restriction*, need not be unique. So, rigorously we have to say the pair $\{\mathbb{H}, r_{\mathbb{H}}\}$ is a quantum subgroup, but we simply say $\mathbb{H}$ is a quantum subgroup. Note that $\mathbb{H}$ acts on $A(\mathbb{G})$ from both sides as $\mathbb{H} \overset{\gamma^\ell}{\curvearrowright} A(\mathbb{G}) \overset{\gamma^r}{\curvearrowleft} \mathbb{H}$ defined by

$$\gamma^\ell := (r_{\mathbb{H}} \otimes \mathrm{id}) \circ \delta_{\mathbb{G}}, \quad \gamma^r := (\mathrm{id} \otimes r_{\mathbb{H}}) \circ \delta_{\mathbb{G}}.$$

Then we define the *non-commutative quotient spaces* by the following fixed point algebras:

$$A(\mathbb{H}\backslash\mathbb{G}) := \{a \in A(\mathbb{G}) \mid \gamma^\ell(a) = 1 \otimes a\},$$

$$A(\mathbb{G}/\mathbb{H}) := \{a \in A(\mathbb{G}) \mid \gamma^r(a) = a \otimes 1\}.$$

The weak closures in $B(L^2(\mathbb{G}))$ are denoted by $L^\infty(\mathbb{H}\backslash\mathbb{G})$ and $L^\infty(\mathbb{G}/\mathbb{H})$, respectively.

Note that the left $\mathbb{H}$-action γ^ℓ and the right $\mathbb{G}$-action $\delta_{\mathbb{G}}$ are commuting, i.e. $(\mathrm{id} \otimes \delta_{\mathbb{G}}) \circ \gamma^\ell = (\gamma^\ell \otimes \mathrm{id}) \circ \delta_{\mathbb{G}}$. Hence $\mathbb{G}$ is also acting on $A(\mathbb{H}\backslash\mathbb{G})$ by δ. Similarly the coproduct $\delta_{\mathbb{G}}$ defines a left action on $A(\mathbb{G}/\mathbb{H})$. Since these actions preserve the Haar state, they extend to the quotient spaces $L^\infty(\mathbb{H}\backslash\mathbb{G})$ or $L^\infty(\mathbb{G}/\mathbb{H})$, respectively. They are typical examples of right or left coideals.

Definition 3.5. Let $B \subset L^\infty(\mathbb{G})$ be a von Neumann subalgebra. Then we say that

- B is a left coideal if $\delta(B) \subset L^\infty(\mathbb{G}) \otimes B$;
- B is a right coideal if $\delta(B) \subset B \otimes L^\infty(\mathbb{G})$;
- a left (right) coideal B is of *quotient type* if $B = L^\infty(\mathbb{G}/\mathbb{H})$ (resp. $L^\infty(\mathbb{H}\backslash\mathbb{G})$) for some quantum subgroup $\mathbb{H}$.

Thanks to Gelfand theorem, every left coideal is of quotient type when $\mathbb{G}$ is a compact group [1]. However, this is not true in general [13, 14, 17]. Indeed, we have the following characterization [18].

Theorem 3.6 (Tomatsu). *Let $B \subset L^\infty(\mathbb{G})$ be a right coideal. Then the following are equivalent:*

- *B is of quotient type;*
- *There exists an expectation $E_B\colon L^\infty(\mathbb{G}) \to B$ preserving the Haar state, and moreover $\widehat{\mathbb{G}}$ acts on B, i.e. $\beta(B) \subset R(\mathbb{G}) \otimes B$.*

This theorem has been proved for co-amenable quantum groups [18], but the same proof works because we have changed the definition of quantum subgroups.

3.4. Amenability and co-amenability. For details of the theory of amenability for quantum groups, readers are referred to [2, 3, 4, 16] and references therein.

Definition 3.7. We say that $\widehat{\mathbb{G}}$ is *amenable* when there exists an *invariant mean* m on $R(\mathbb{G})$, that is, $m \in R(\mathbb{G})^*$ is a state such that

$$m((\mathrm{id} \otimes \omega)(\Delta(x))) = \omega(1)m(x) = m((\omega \otimes \mathrm{id})(\Delta(x))).$$

In this case, $\mathbb{G}$ is said to be *co-amenable.*

Theorem 3.8 (Bedos-Murphy-Tuset, Tomatsu). *The following are equivalent:*

- *$\mathbb{G}$ is co-amenable;*
- *$C(\mathbb{G}_{\mathrm{red}})$ has a bounded counit ε that is a $*$-homomorphism $\varepsilon\colon C(\mathbb{G}_{\mathrm{red}}) \to \mathbb{C}$ such that $(\varepsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \delta$.*

4. Poisson boundaries. We briefly recall the notion of the Poisson boundary for a discrete quantum group. We refer to [7, 8] for definitions of terminology.

4.1. Identification problems. Let $\phi_\pi \in B(H_\pi)_*$ be a right $\mathbb{G}$-invariant state. Define a transition operator P_π on $R(\mathbb{G})$ by $P_\pi(x) = (\mathrm{id} \otimes \phi_\pi)(\Delta_R(x))$ for $x \in R(\mathbb{G})$. When $\widehat{\mathbb{G}}$ is a discrete group, P_g, $g \in \widehat{\mathbb{G}}$, is nothing but the right translation of functions by $g \in \widehat{\mathbb{G}}$, which is an automorphism. However, the map P_π is not an automorphism but a faithful normal unital completely positive (u.c.p.) map in general.

For a probability measure μ on $\mathrm{Irr}(\mathbb{G})$, we set the *non-commutative Markov operator*,

$$P_\mu := \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} \mu(\pi) P_\pi.$$

We assume μ is generating, that is, $\mathrm{supp}(\mu)$ generates $\mathrm{Irr}(\mathbb{G})$ as a semigroup in the following sense: for any $\pi \in \mathrm{Irr}(\mathbb{G})$, there exist $\rho_1, \dots, \rho_n \in \mathrm{supp}(\mu)$ such that the representation π is contained in the tensor product representation $\rho_1 \otimes \cdots \otimes \rho_n$.

Then we define an operator system,

$$H^\infty(\widehat{\mathbb{G}}, P_\mu) := \{x \in R(\mathbb{G}) \mid P_\mu(x) = x\}.$$

We often regard $\mathrm{id} - P_\mu$ as a Laplace operator on $\widehat{\mathbb{G}}$, and we say that each element of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is P_μ-*harmonic*. That operator system has the von Neumann algebra structure defined by

$$x \cdot y = \lim_{n \to \infty} P_\mu^n(xy) \quad \text{for } x, y \in H^\infty(\widehat{\mathbb{G}}, P_\mu), \tag{4.1}$$

where the limit is taken in the strong topology [7, Theorem 3.6]. The von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is called the (non-commutative) *Poisson boundary* of $\{R(\mathbb{G}), P_\mu\}$. The following theorem is probably well-known to specialists.

THEOREM 4.1. *The von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is amenable.*

Proof. We know that $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is isomorphic to $(\mathcal{R}^\alpha)' \cap \mathcal{R}$, where $\mathcal{R} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ is an ITP action [7]. Let $E_\alpha := (\mathrm{id} \otimes h) \circ \alpha$ be the averaging expectation. Take a faithful state $\omega \in (\mathcal{R}^\alpha)_*$, and set $\psi := \omega \circ E_\alpha$. Since $\sigma_t^\psi|_{\mathcal{R}^\alpha} = \sigma_t^\omega$, $\sigma_t^\psi((\mathcal{R}^\alpha)' \cap \mathcal{R}) = (\mathcal{R}^\alpha)' \cap \mathcal{R}$. Thanks to Takesaki theorem [15], we see that there exists an expectation from $\mathcal{R}$ onto $(\mathcal{R}^{\mathbb{G}})' \cap \mathcal{R}$, and $(\mathcal{R}^{\mathbb{G}})' \cap \mathcal{R}$ is amenable. Hence so is $H^\infty(\widehat{\mathbb{G}}, P_\mu)$. ∎

Now we recall the actions $\widehat{\mathbb{G}} \overset{\beta}{\curvearrowright} B(L^2(\mathbb{G})) \overset{\alpha}{\curvearrowleft} \mathbb{G}$ defined by

$$\beta(x) := V^*(1 \otimes x)V, \quad \alpha(x) = V(x \otimes 1)V^*.$$

Since we can prove P_μ and α or β are commuting on $R(\mathbb{G})$, the Poisson boundary is a $\widehat{\mathbb{G}}$-$\mathbb{G}$-von Neumann algebra [7]. We should note that if $\widehat{\mathbb{G}}$ is a discrete group, then α is trivial. Hence non-triviality of α on $R(\mathbb{G})$ is a purely quantum phenomenon.

In Poisson boundary theory, one of the most important problems is the following:

PROBLEM 4.2 (Identification problem). *Realize $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ as a more concrete $\widehat{\mathbb{G}}$-$\mathbb{G}$-von Neumann algebra.*

In [7], it is shown that $H^\infty(SU_q(2), P_\mu) \cong L^\infty(\mathbb{T} \backslash SU_q(2))$. This result is generalized to $SU_q(n)$ by Izumi-Neshveyev-Tuset [8]. Their key ingredient is the *Poisson integral*

$\Theta\colon L^\infty(\mathbb{G}) \to R(\mathbb{G})$ defined by

$$\Theta(a) := (\mathrm{id} \otimes h)(\beta(a)) \quad \text{for } a \in L^\infty(\mathbb{G}).$$

Then the faithful normal u.c.p. map Θ is $\widehat{\mathbb{G}}$-$\mathbb{G}$-equivariant, i.e.

$$\Delta \circ \Theta = (\mathrm{id} \otimes \Theta) \circ \beta, \quad \alpha \circ \Theta = (\Theta \otimes \mathrm{id}) \circ \delta.$$

It is known that the range space $\operatorname{Im}\Theta$ is contained in $H^\infty(\widehat{\mathbb{G}}, P_\mu)$. Here, we should note that $\operatorname{Im}\Theta$ does not depend on μ. Indeed, if we define the operator system

$$H^\infty(\widehat{\mathbb{G}}) := \{x \in R(\mathbb{G}) \mid P_\pi(x) = x \text{ for all } \pi \in \operatorname{Irr}(\mathbb{G})\},$$

then $\operatorname{Im}\Theta \subset H^\infty(\widehat{\mathbb{G}})$.

From now, we focus on $\mathbb{G}$ with commutative fusion rules. Then $H^\infty(\widehat{\mathbb{G}}, P_\mu) = H^\infty(\widehat{\mathbb{G}})$ for any generating probability measure μ [8].

Our main theorem of this section is the following [18]:

THEOREM 4.3. *Let $\mathbb{G}$ be a co-amenable compact quantum group. Assume that its fusion algebra is commutative. Then the following statements hold:*

1. *There exists a unique maximal quantum subgroup of Kac type $\mathbb{H}$, that is, if $\mathbb{K}$ is another Kac quantum subgroup of $\mathbb{G}$, then $L^\infty(\mathbb{H}\backslash\mathbb{G}) \subset L^\infty(\mathbb{K}\backslash\mathbb{G})$;*
2. *The Poisson integral $\Theta\colon L^\infty(\mathbb{H} \setminus \mathbb{G}) \to H^\infty(\widehat{\mathbb{G}})$ is an isomorphism.*

Proof. We present a sketch of a proof. What we want to construct first is the inverse map $\Lambda\colon H^\infty(\widehat{\mathbb{G}}) \to L^\infty(\mathbb{G})$. To do this, we recall $\Theta = (\mathrm{id} \otimes h) \circ \beta$. Then we would use a "Haar state" on $R(\mathbb{G})$, intuitively. If $\widehat{\mathbb{G}}$ was finite dimensional, it would be sufficient to replace β with α. A problem is that we do not have such a functional in general. However, since we have assumed amenability of $\widehat{\mathbb{G}}$, we have an invariant mean m on $R(\mathbb{G})$, which may be singular. Then we introduce $\Lambda\colon R(\mathbb{G}) \to L^\infty(\mathbb{G})$ given by

$$\Lambda(x) := (m \otimes \mathrm{id})(\alpha(x)) \text{ for } x \in R(\mathbb{G}),$$

where the u.c.p. map $m \otimes \mathrm{id}\colon R(\mathbb{G}) \otimes L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is well-defined.

Next we compute $\Theta \circ \Lambda$, and we obtain $\Theta \circ \Lambda = \mathrm{id}$ on $H^\infty(\widehat{\mathbb{G}})$ because of the ergodicity of $H^\infty(\widehat{\mathbb{G}}) \overset{\alpha}{\curvearrowleft} \mathbb{G}$, which comes from the commutativity of fusion rules. In particular, we have shown $\operatorname{Im}\Theta = H^\infty(\widehat{\mathbb{G}})$. Moreover, we can see that Λ is multiplicative on $H^\infty(\widehat{\mathbb{G}})$: for any $x \in H^\infty(\widehat{\mathbb{G}})$, we have

$$x^* \cdot x = \Theta(\Lambda(x))^* \cdot \Theta(\Lambda(x)) \le \Theta(\Lambda(x)^*\Lambda(x)) \le \Theta(\Lambda(x^* \cdot x)) = x^* \cdot x.$$

Hence we obtain $\Theta(\Lambda(x)^*\Lambda(x)) = \Theta(\Lambda(x^* \cdot x))$. Since $\Lambda(x)^*\Lambda(x) \le \Lambda(x^* \cdot x)$ and Θ is faithful, $\Lambda(x)^*\Lambda(x) = \Lambda(x^* \cdot x)$ holds.

Next we study the image of Λ, and put $B := \operatorname{Im}\Lambda$, which is a von Neumann subalgebra in $L^\infty(\mathbb{G})$. We consider $E := \Lambda \circ \Theta\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$. It is easy to see that E is a Haar state preserving expectation onto B. Since Λ is $\widehat{\mathbb{G}}$-$\mathbb{G}$-equivariant, $\widehat{\mathbb{G}}$ and $\mathbb{G}$ are acting on B. Hence B is a right coideal with nice properties, and B must be of quotient type by Theorem 3.6.

So, B is of the form $L^\infty(\mathbb{H}\backslash\mathbb{G})$ for some quantum subgroup $\mathbb{H}$. By definition, E is also $\widehat{\mathbb{G}}$-equivariant, and we see that $\mathbb{H}$ is of Kac type. With a little effort, we can see that $\mathbb{H}$ is the maximal quantum subgroup of Kac type. ∎

For the q-deformation of a classical compact Lie group, we can show that the maximal Kac quantum subgroup is exactly equal to the maximal torus [18]. Hence we have the following result which generalizes the main result of [8]:

COROLLARY 4.4. *Let $\mathbb{G}_q$ be the q-deformation of a classical compact Lie group $\mathbb{G}$. Then the Poisson integral $\Theta\colon L^\infty(\mathbb{T}\setminus\mathbb{G}_q)\to H^\infty(\widehat{\mathbb{G}_q})$ is an isomorphism.*

4.2. Description of centers of Poisson boundaries. Here, we will propose a new problem on a Poisson boundary. Recall that P_μ and α are commuting, and P_μ acts on the fixed point algebra $R(\mathbb{G})^\alpha$, which is nothing but the center $Z(R(\mathbb{G}))=\ell_\infty(\mathrm{Irr}(\mathbb{G}))$. Hence we introduce the classical part of a Poisson boundary,

$$H^\infty(\widehat{\mathbb{G}},P_\mu)_{\mathrm{class}}:=H^\infty(\widehat{\mathbb{G}},P_\mu)\cap Z(R(\mathbb{G})).$$

Let us denote the center of $H^\infty(\widehat{\mathbb{G}},P_\mu)$ by $Z(H^\infty(\widehat{\mathbb{G}},P_\mu))$, where we should again remember the product structure (4.1). It is trivial by (4.1) that the classical part $H^\infty(\widehat{\mathbb{G}},P_\mu)_{\mathrm{class}}$ is contained in $Z(H^\infty(\widehat{\mathbb{G}},P_\mu))$. Now we present following our problem:

CONJECTURE 4.5. *Let $\mathbb{G}$ be a compact quantum group and μ a generating probability measure. Then the following equality holds:*

$$H^\infty(\widehat{\mathbb{G}},P_\mu)_{\mathrm{class}}=Z(H^\infty(\widehat{\mathbb{G}},P_\mu)).$$

When $\mathbb{G}$ has commutative fusion rules, then the classical part is trivial [6, 7]. So, the conjecture means the factoriality of $H^\infty(\widehat{\mathbb{G}},P_\mu)$.

There are some positive observations about this conjecture. For $SU_q(2)$ case, it is true because the quantum flag manifold (or a Podleś sphere) $L^\infty(\mathbb{T}\backslash SU_q(2))$ is a type I_∞ factor. However, that is unknown for other q-deformations. The conjecture holds even for non-amenable examples such as $A_o(F)$ and $A_u(F)$ [19, 20]. It has seemed to be affirmative so far.

4.3. A criterion on a factoriality of $L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$. From now on, $\mathbb{G}$ is the q-deformation of a classical compact Lie group [10]. It is proved that the maximal Kac quantum subgroup of $\mathbb{G}$ is the maximal torus $\mathbb{T}_{\max}$ [18]. Then Theorem 4.3 implies the isomorphism,

$$\Theta\colon L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})\to H^\infty(\widehat{\mathbb{G}}).$$

Recall the left action $\mathbb{T}_{\max}\overset{\gamma^\ell}{\curvearrowright}L^\infty(\mathbb{G})$ introduced in §3.3.

LEMMA 4.6. *Let Z be the center of $L^\infty(\mathbb{G})$. Then the left action $\mathbb{T}_{\max}\overset{\gamma^\ell}{\curvearrowright}L^\infty(\mathbb{G})$ is centrally ergodic, i.e. $Z^{\gamma^\ell}=\mathbb{C}$*

Proof. Recall the left $\widehat{\mathbb{G}}$-action $\beta(x)=V^*(x\otimes 1)V$, $x\in L^\infty(\mathbb{G})$. Then $L^\infty(\mathbb{G})^\beta=Z$. Since $Z^{\gamma^\ell}=L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})^\beta$, $\Theta(Z^{\gamma^\ell})=H^\infty(\widehat{\mathbb{G}})^\beta$, this is equal to $\mathbb{C}$ because $R(\mathbb{G})^\beta=\mathbb{C}$. Hence $Z^{\gamma^\ell}=\Theta^{-1}(\mathbb{C})=\mathbb{C}$. ∎

Hence there exists a subgroup $\Gamma\subset\widehat{\mathbb{T}_{\max}}$ such that the following decomposition holds:

$$Z=\bigoplus_{\chi\in\Gamma}Z_\chi,$$

where $Z_\chi:=\{x\in Z\mid \gamma^\ell_z(x)=\chi(z)x \text{ for } z\in\mathbb{T}_{\max}\}$. From the previous lemma, $Z_0=\mathbb{C}$, and each Z_χ is one-dimensional subspace spanned by a unitary.

THEOREM 4.7. *If the action* $\mathbb{T}_{\max} \overset{\gamma^\ell}{\curvearrowright} Z$ *is faithful, then the following hold:*

1. $L^\infty(\mathbb{G}) = Z \vee L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$*;*
2. $L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$ *is the type* I_∞ *factor;*
3. $H^\infty(\widehat{\mathbb{G}})$ *is the type* I_∞ *factor.*

Proof. By our assumption, Γ coincides with $\widehat{\mathbb{T}_{\max}}$. Hence for any $\chi \in \widehat{\mathbb{T}_{\max}}$, there exists a unitary $u_\chi \in Z_\chi$. Then $u_\chi^* L^\infty(\mathbb{G})_\chi = L^\infty(\mathbb{G})^{\gamma^\ell} = L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$. Hence $L^\infty(\mathbb{G})_\chi = u_\chi L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$, and (1) holds.

It is known that $C(\mathbb{G})$ is a type I C^*-algebra [10], and so are $L^\infty(\mathbb{G})$ and $L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$, which is trivially infinite dimensional. By (1), the center of $L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$ is contained in Z. The central ergodicity of γ^ℓ implies the factoriality of $L^\infty(\mathbb{T}_{\max}\backslash\mathbb{G})$.

(3) is trivial from Theorem 4.3. ∎

From the next subsection, we present a complete proof for the $SU_q(2)$ case.

4.4. Quantum group $SU_q(2)$**.** The definition of $SU_q(2)$ is as follows. Our notations are same as those of [12]. The continuous function algebra $C(SU_q(2))$ is the universal C^*-algebra generated by four elements x, u, v and y with the following relations:

$$ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy, \quad uv = vu,$$
$$xy - q^{-1}uv = 1 = yx - quv,$$
$$x^* = y, \quad u^* = -q^{-1}v.$$

The coproduct δ is given by

$$\begin{pmatrix} \delta(x) & \delta(u) \\ \delta(v) & \delta(y) \end{pmatrix} := \begin{pmatrix} x \otimes 1 & u \otimes 1 \\ v \otimes 1 & y \otimes 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \otimes x & 1 \otimes u \\ 1 \otimes v & 1 \otimes y \end{pmatrix},$$

which means the following matrix is an (irreducible) representation:

$$w(1/2) := \begin{pmatrix} x & u \\ v & y \end{pmatrix}.$$

Each element of $\mathrm{Irr}(SU_q(2))$ is determined by the highest weight, which is an element of $(1/2)\mathbb{Z}_+ = \{0, 1/2, \cdots\}$ in this case. Each $\nu \in (1/2)\mathbb{Z}_+$ is called the *spin* and the dimension of $w(\nu)$ is $2\nu + 1$. The quantum dimension of $w(\nu)$ is given by the *q-integer* $(2\nu+1)_q := (q^{-2\nu-1} - q^{2\nu+1})/(q^{-1} - q)$ [12].

On tensor products, we have the same formula (Clebsch-Gordan rule) as that of $SU(2)$,

$$w(\mu) \oplus w(\nu) = w(|\mu - \nu|) \oplus w(|\mu - \nu| + 1) \oplus \cdots \oplus w(\mu + \nu - 1) \oplus w(\mu + \nu). \tag{4.2}$$

For $\ell \in (1/2)\mathbb{Z}_+$, we set $I_\ell := \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$. Set the positive operator $\zeta := u^*u = -q^{-1}uv$.

THEOREM 4.8. *For each* $\ell \in (1/2)\mathbb{Z}_+$ *and* $i, j \in I_\ell$, *the matrix elements* $w(\ell)_{i,j}$ *are expressed in terms of the little q-Jacobi polynomials in* ζ *as follows:*

1. *Case* $i + j \leq 0$, $i \geq j$*:*

$$q^{(\ell+j)(j-i)} \begin{bmatrix} \ell + i \\ i - j \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \ell - j \\ i - j \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{-i-j} v^{i-j} P_{\ell+j}^{(i-j,-i-j)}(\zeta; q^2);$$

2. *Case $i+j \leq 0$, $i \leq j$:*

$$q^{(\ell+i)(i-j)} \begin{bmatrix} \ell-i \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \ell+j \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{-i-j} u^{j-i} P_{\ell+i}^{(j-i,-i-j)}(\zeta; q^2);$$

3. *Case $i+j \geq 0$, $i \leq j$:*

$$q^{(j-i)(j-\ell)} \begin{bmatrix} \ell-i \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \ell+j \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} P_{\ell-j}^{(j-i,i+j)}(\zeta; q^2) u^{j-i} y^{i+j};$$

4. *Case $i+j \geq 0$, $i \geq j$:*

$$q^{(i-j)(i-\ell)} \begin{bmatrix} \ell+i \\ i-j \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \ell-j \\ i-j \end{bmatrix}_{q^2}^{\frac{1}{2}} P_{\ell-i}^{(i-j,i+j)}(\zeta; q^2) v^{i-j} y^{i+j},$$

where we have used the q-binomial coefficients and the little q-Jacobi polynomials:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}, \quad (t;q)_m = \prod_{s=0}^{m-1} (1 - tq^s),$$

$$P_n^{(\alpha,\beta)}(\zeta; q^2) = \sum_{r \geq 0} \frac{(q^{-2n}; q^2)_r (q^{2\alpha+2\beta+2n+2}; q^2)_r}{(q^2; q^2)_r (q^{2\alpha+2}; q^2)_r} (q^2\zeta)^r.$$

Corollary 4.9. *For each $\ell \in (1/2)\mathbb{Z}_+$ and $i, j \in I_\ell$, the following equality holds:*

$$w(\ell)_{i,j}^* = (-q)^{i-j} w(\ell)_{-i,-j}.$$

Let σ^h be the modular automorphism group for h. Then we have the following formula [12, 21, 22]:

$$\sigma_t^h(w(\ell)_{r,s}) = q^{-2(r+s)it} w(\ell)_{r,s} \quad \text{for all } t \in \mathbb{R}. \tag{4.3}$$

4.5. Description of $L^\infty(\mathbb{T} \backslash SU_q(2))$. Our aim is to prove the faithfulness of $\mathbb{T} \overset{\gamma^\ell}{\curvearrowright} Z(L^\infty(SU_q(2)))$, which implies the factoriality of the standard Podleś sphere $L^\infty(\mathbb{T} \backslash SU_q(2))$ by Theorem 4.7. We let $Z := Z(L^\infty(SU_q(2)))$.

It is known that the spectrum of $\zeta = u^*u$ is quantized. More precisely, we obtain $\mathrm{Sp}(\zeta) = \{q^{2k}\}_{k=0}^\infty$ [12, 21]. Hence there exist orthogonal projections $\{p_k\}_{k=0}^\infty$ in $C(SU_q(2))$ such that

$$\zeta = \sum_{k=0}^\infty q^{2k} p_k.$$

Lemma 4.10. *The set $\{p_k\}_{k=0}^\infty$ is a partition of unity in $L^\infty(SU_q(2))$.*

Proof. Let $p := \sum_{k=0}^\infty p_k$ and $p' := 1 - p$. Using $\zeta x = q^2 x \zeta$, we see that

$$x p_k = p_{k+1} x. \tag{4.4}$$

Summing up both sides over $k \geq 0$, we have $xp = (p - p_0)x$. Since $xx^* + \zeta = 1$, we have $p_0 x x^* p_0 = 0$, and $xp = px$. Then

$$\begin{aligned} h(p') &= h((xx^* + uu^*)p') = h(xx^*p') \\ &= h(x^*p'\sigma_{-i}^h(x)) = q^{-2}h(x^*p'x) = q^{-2}h(p'x^*x) \\ &= q^{-2}h(p'(x^*x + u^*u)) = q^{-2}h(p'), \end{aligned}$$

and $h(p') = 0$. Since h is faithful, $p' = 0$. ∎

Remark 4.11. We can compute $h(p_k)$ and directly verify the above result as follows. Recall the formula $h(\zeta^n) = (1-q^2)/(1-q^{2n+2})$ [12]. Since ζ^n converges to p_0 as $n \to \infty$ (in the norm topology), we have $h(p_0) = 1-q^2$. Next using (4.4), we have

$$h(xp_kx^*) = h(p_{k+1}xx^*) = h(p_{k+1}(1-\zeta)) = (1-q^{2k+2})h(p_{k+1}).$$

Thanks to $\sigma_t^h(x) = q^{2it}x$, we also obtain

$$h(xp_kx^*) = h(p_kx^*\sigma_{-i}^h(x)) = q^2h(p_kx^*x) = q^2h(p_k(1-q^2\zeta)) = q^2(1-q^{2k+2})h(p_k).$$

Therefore, $h(p_{k+1}) = q^2h(p_k)$, and $h(p_k) = q^{2k}(1-q^2)$. The summation of $\{h(p_k)\}_{k=0}^\infty$ is indeed equal to 1.

Theorem 4.12. *The following hold:*

1. *γ^ℓ is a faithful action on Z;*
2. *$L^\infty(SU_q(2)) = Z \vee L^\infty(\mathbb{T}\backslash SU_q(2))$;*
3. *Z is $\mathbb{T}$-equivariantly isomorphic to $L^\infty(\mathbb{T})$;*
4. *$L^\infty(\mathbb{T}\backslash SU_q(2))$ is the type I_∞ factor.*

Proof. Let $u = a|u| = a\zeta^{1/2}$ be the polar decomposition of u. By Lemma 4.10, $aa^* = a^*a = 1$. We show that $a \in Z$. Indeed, we have

$$qx|u|^2 = uxu^* = a|u|x|u|a^* = qax|u|^2a^* = qaxa^*|u|^2.$$

Again by Lemma 4.10, we have $x = axa^*$. The $*$-algebra $L^\infty(SU_q(2))$ is generated by x and u, so $a \in Z$. Since $\gamma_z^\ell(u) = zu$, we have $\gamma_z^\ell(a) = za$ for all $z \in \mathbb{T}$. The other statements are trivial from Theorem 4.7. ∎

Hence Conjecture 4.5 holds for $SU_q(2)$. Although we have discussed and got the previous result using somewhat general results, we can directly obtain that from Woronowicz's classification of irreducible representations of the C^*-algebra $C(SU_q(2))$ as follows.

Let us consider the tensor product von Neumann algebra $N := L^\infty(\mathbb{T}) \otimes B(\ell_2)$. Define the four operators x', v', u' and y' in this algebra by

$$x' = z \otimes \sum_{k=0}^\infty \sqrt{1-q^{2k+2}}e_{k+1,k}, \quad u' = z \otimes \sum_{k=0}^\infty -q^k e_{k,k}$$

$$v' = \overline{z} \otimes \sum_{k=0}^\infty q^{k+1}e_{k,k}, \quad y' = \overline{z} \otimes \sum_{k=0}^\infty \sqrt{1-q^{2k+2}}e_{k,k+1},$$

where $z \in L^\infty(\mathbb{T})$ is a canonical generating unitary. Then they satisfy the same relations as those of $C(SU_q(2))$. Hence there exists a surjection $\pi\colon C(SU_q(2)) \to C^*(x', u', v', y')$ by universality.

Consider the tensor product state $h_\mathbb{T} \otimes \mathrm{Tr}_\rho$ on N, where $h_\mathbb{T}$ is the Haar state on $L^\infty(\mathbb{T})$ and the density is $\rho := \sum_{k=0}^\infty (1-q^2)q^{2k}e_{kk}$. We can verify $h = (h_\mathbb{T} \otimes \mathrm{Tr}_\rho) \circ \pi$ on the $*$-algebra generated by x, u, v, y from Theorem 4.8. Hence the surjection π extends to the isomorphism between $L^\infty(SU_q(2))$ and $C^*(x', u', v', y')'' = N$. It is trivial that π intertwines the torus actions γ^ℓ and $\beta \otimes \mathrm{id}$, where β is the rotation on $L^\infty(\mathbb{Z})$. Hence $L^\infty(\mathbb{T}\backslash SU_q(2))$ is isomorphic to $B(\ell_2)$.

4.6. More complete description of Z. We close this article by giving generators of Z for $SU_q(2)$ and studying their relations. Recall the torus actions γ and γ^ℓ on $L^\infty(SU_q(2))$,

$$\begin{pmatrix} \gamma_z^\ell(x) & \gamma_z^\ell(u) \\ \gamma_z^\ell(v) & \gamma_z^\ell(y) \end{pmatrix} = \begin{pmatrix} zx & zu \\ \bar{z}v & \bar{z}y \end{pmatrix}, \quad \begin{pmatrix} \gamma_z(x) & \gamma_z(u) \\ \gamma_z(v) & \gamma_z(y) \end{pmatrix} = \begin{pmatrix} zx & \bar{z}u \\ zv & \bar{z}y \end{pmatrix} \quad \text{for } z \in \mathbb{T}.$$

The following formulae are immediate from Theorem 4.8:

$$\gamma_z^\ell(w(\nu)_{i,j}) = z^{-2i} w(\nu)_{i,j}, \quad \gamma_z(w(\nu)_{i,j}) = z^{-2j} w(\nu)_{i,j}. \tag{4.5}$$

LEMMA 4.13. *We have the following equalities:*

1. $xw(\nu)_{r,-r} = \alpha_{\nu,r} w(\nu-1/2)_{r-1/2,-r-1/2} + \beta_{\nu,r} w(\nu+1/2)_{r-1/2,-r-1/2}$
2. $w(\nu)_{r,-r} x = \alpha'_{\nu,r} w(\nu-1/2)_{r-1/2,-r-1/2} + \beta'_{\nu,r} w(\nu+1/2)_{r-1/2,-r-1/2}$
3. $yw(\nu)_{r,-r} = \gamma_{\nu,r} w(\nu-1/2)_{r+1/2,-r+1/2} + \delta_{\nu,r} w(\nu+1/2)_{r+1/2,-r+1/2}$
4. $w(\nu)_{r,-r} y = \gamma'_{\nu,r} w(\nu-1/2)_{r+1/2,-r+1/2} + \delta'_{\nu,r} w(\nu+1/2)_{r+1/2,-r+1/2}$,

where the constants are given by

$$\alpha_{\nu,r} = \gamma'_{\nu,r} = q^{\nu+1} \frac{\sqrt{(\nu+r)_q(\nu-r)_q}}{(2\nu+1)_q}, \quad \beta_{\nu,r} = \delta'_{\nu,r} = q^{-\nu} \frac{\sqrt{(\nu+r+1)_q(\nu-r+1)_q}}{(2\nu+1)_q},$$

$$\alpha'_{\nu,r} = \gamma_{\nu,r} = q^{-\nu-1} \frac{\sqrt{(\nu+r)_q(\nu-r)_q}}{(2\nu+1)_q}, \quad \beta'_{\nu,r} = \delta_{\nu,r} = q^{\nu} \frac{\sqrt{(\nu+r+1)_q(\nu-r+1)_q}}{(2\nu+1)_q},$$

Proof. The element $xw(\nu)_{r,-r}$ is spanned by $w(\nu\pm1/2)_{i,j}$. Using (4.5), we see that there exist some complex numbers α and β such that

$$xw(\nu)_{r,-r} = \alpha w(\nu-1/2)_{r-1/2,-r-1/2} + \beta w(\nu+1/2)_{r-1/2,-r-1/2}.$$

From Theorem 4.8, we have

$$\begin{aligned} w(\nu)_{r,-r} &= q^{-2r(\nu-r)} \begin{bmatrix} \nu+r \\ 2r \end{bmatrix}_{q^2} v^{2r} P_{\nu-r}^{2r,0}(\zeta, q^2) \\ w(\nu-1/2)_{r-1/2,-r-1/2} &= q^{-2r(\nu-r-1)} \begin{bmatrix} \nu+r-1 \\ 2r \end{bmatrix}_{q^2}^{1/2} \begin{bmatrix} \nu+r \\ 2r \end{bmatrix}_{q^2}^{1/2} xv^{2r} P_{\nu-r-1}^{2r,1}(\zeta, q^2) \\ w(\nu+1/2)_{r-1/2,-r-1/2} &= q^{-2r(\nu-r)} \begin{bmatrix} \nu+r \\ 2r \end{bmatrix}_{q^2}^{1/2} \begin{bmatrix} \nu+r+1 \\ 2r \end{bmatrix}_{q^2}^{1/2} xv^{2r} P_{\nu-r}^{2r,1}(\zeta, q^2). \end{aligned}$$

By comparing the constant term and the degree $(\nu-r)$ term of ζ, we can obtain $\beta = q^{-\nu}(2\nu+1)_q^{-1}\sqrt{(\nu+r+1)_q(\nu-r+1)_q}$ and $\alpha = q^{\nu+1}(2\nu+1)_q^{-1}\sqrt{(\nu+r)_q(\nu-r)_q}$. Similarly we get the other equalities. ∎

Let $r \in (1/2)\mathbb{Z}_+$ and Z_{-2r} the spectral subspace for the action γ^ℓ for the eigenvalue $-2r$. Since $Z \subset L^\infty(SU_q(2))^{\sigma^h}$, an element $a \in Z_{-2r}$ has the following expansion in $L^2(SU_q(2))$:

$$a = \sum_{\nu=r}^{\infty} \lambda_{\nu,r} w(\nu)_{r,-r},$$

where $\lambda_{\nu,r} \in \mathbb{C}$ and the summation is taken for $\nu = r, r+1, \ldots$. From the previous lemma, the commutativity of a and x implies

$$\sum_{\nu\geq r}\lambda_{\nu,r}(\alpha_{\nu,r}w(\nu-1/2)_{r-1/2,-r-1/2}+\beta_{\nu,r}w(\nu+1/2)_{r-1/2,-r-1/2})$$
$$=\sum_{\nu\geq r}\lambda_{\nu,r}(\alpha'_{\nu,r}w(\nu-1/2)_{r-1/2,-r-1/2}+\beta'_{\nu,r}w(\nu+1/2)_{r-1/2,-r-1/2}).$$

From this we get the recurrence formula $\lambda_{\nu+1,r}(\alpha_{\nu+1,r}-\alpha'_{\nu+1,r})=\lambda_{\nu,r}(\beta'_{\nu,r}-\beta_{\nu,r})$ whose solution is

$$\lambda_{\nu,r}=\frac{(2\nu+1)_q}{(\nu)_q(\nu+1)_q}\frac{(r)_q(r+1)_q}{(2r+1)_q}\lambda_{r,r},$$

where $(t)_q:=(q^{-t}-q^t)/(q^{-1}-q)$ for any $t\in\mathbb{R}$.

LEMMA 4.14. *For $r\in(1/2)\mathbb{Z}_+$, the following element a_r is a well-defined unitary in Z_{-2r} and commutes with x, u, v and y:*

$$a_r=q^{-r}(r)_q\sum_{\nu=r}^{\infty}\frac{(2\nu+1)_q}{(\nu)_q(\nu+1)_q}w(\nu)_{r,-r}.$$

Proof. Since γ^ℓ is faithful on Z, the above elements have to be well-defined. Using the formulae $h(w(\nu)^*_{r,-r}w(\nu)_{r,-r})=(2\nu+1)^{-1}q^{2r}$ and $(2\nu+1)_q=(\nu+1)_q^2-(\nu)_q^2$, we have

$$h(a_r^*a_r)=q^{-2r}(r)_q^2\sum_{\nu=r}^{\infty}\frac{(2\nu+1)_q^2}{(\nu)_q^2(\nu+1)_q^2}\cdot(2\nu+1)_q^{-1}q^{2r}=(r)_q^2\sum_{\nu=r}^{\infty}\frac{(2\nu+1)_q}{(\nu)_q^2(\nu+1)_q^2}$$
$$=(r)_q^2\sum_{\nu=r}^{\infty}\frac{(\nu+1)_q^2-(\nu)_q^2}{(\nu)_q^2(\nu+1)_q^2}=(r)_q^2\sum_{\nu=r}^{\infty}\left(\frac{1}{(\nu)_q^2}-\frac{1}{(\nu+1)_q^2}\right)=1.$$

Hence a_r is a unitary. ∎

Next we compare $a_{1/2}$ with b that is a unitary of the polar decomposition of v, i.e. $v=b|v|=qb\zeta^{1/2}$. From this equality, we get $v^*b=q\zeta^{1/2}$. Using $h(p_k)=(1-q^2)q^{2k}$, we obtain

$$h(v^*b)=qh(ze^{1/2})=q\sum_{k=0}^{\infty}q^k(1-q^2)q^{2k}=\frac{q(1-q^2)}{1-q^3}.$$

By definition, $a_{1/2}$ is equal to

$$q^{-1/2}(1/2)_q\sum_{\nu=1/2}^{\infty}\frac{(2\nu+1)_q}{(\nu)_q(\nu+1)_q}w(\nu)_{1/2,-1/2}=q^{-1/2}(1/2)_q\frac{(2)_q}{(1/2)_q(3/2)_q}v+\cdots.$$

Using $h(v^*v)=q^2h(\zeta)=q^2(1+q^2)^{-1}$, we have

$$h(v^*a_{1/2})=q^{-1/2}(1/2)_q\cdot\frac{(2)_q}{(1/2)_q(3/2)_q}\cdot\frac{q^2}{1+q^2}=\frac{q(1-q^2)}{1-q^3}.$$

Since $\dim Z_{-1}=1$, $a_{1/2}$ is a scalar multiple of b, but we have shown $a_{1/2}=b$ by the above calculation.

Therefore $v=a_{1/2}|v|$ is the polar decomposition. Next we compute $a^k_{1/2}$, $k\geq1$, which has to be a scalar multiple of $a_{k/2}$. In a similar way to the previous calculation, we obtain

$$h((v^*)^ka_{1/2}^k)=q^k(1-q^2)(1-q^3)^{-1}=h((v^*)^ka_{k/2}).$$

Summarizing our discussions, we have the following:

THEOREM 4.15. *Let a_r be as above. Then the following hold:*

1. $Z = \{a_{1/2}\}''$;
2. $v = a_{1/2}|v|$ *is the polar decomposition;*
3. $a_{1/2}^k = a_{k/2}$ *for all integer* $k \geq 1$.

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