NONCOMMUTATIVE HARMONIC ANALYSIS
WITH APPLICATIONS TO PROBABILITY II
BANACH CENTER PUBLICATIONS, VOLUME 89
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

TWO-LEVEL t-DEFORMATION

ŁUKASZ JAN WOJAKOWSKI

Mathematical Institute, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: Lukasz.Wojakowski@math.uni.wroc.pl

Abstract. In the present paper we define and study the properties of a deformation of measures and convolutions that works in a similar way to the U_t deformation of Bożejko and Wysoczański, but in its definition operates on two levels of Jacobi coefficients of a measure, rather than on one.

1. Deformation of measures. In the papers [BW1, BW2] Bożejko and Wysoczański introduced the concept of t-deformation of probability measures U_t. This topic was further studied in [W] and extended in many ways in a number of papers [BKW, KW1, KW2, KY2, Or1, Or2], see [W] for a more complete list of references. The original t-deformation of measures, as well as the other studied ones, can also be used to define deformations either of equivalently both the free and conditionally free convolutions, when the deformation is invertible, or of the conditionally free convolution alone, if it is not the case.

The original U_t-deformation of any probability measure μ with compact support can be interpreted as a multiplication of the two Jacobi coefficients α_0 and λ_0 in the continued fraction notation of the Cauchy transform $G_\mu(z)$. For

$$G_\mu(z) = \cfrac{1}{z-\alpha_0-\cfrac{\lambda_0}{z-\alpha_1-\cfrac{\lambda_1}{z-\alpha_2-\cfrac{\lambda_2}{z-\alpha_3-\ddots}}}},$$

2000 *Mathematics Subject Classification*: Primary 46L53; Secondary 46L54.

Key words and phrases: free convolution, deformation, continued fractions, limit theorems.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc89-0-21

the Cauchy transform of the deformed measure $U_t(\mu)$ is given by

$$G_{U_t(\mu)}(z) = \cfrac{1}{z - t\,\alpha_0 - \cfrac{t\,\lambda_0}{z - \alpha_1 - \cfrac{\lambda_1}{z - \alpha_2 - \cfrac{\lambda_2}{z - \alpha_3 - \ddots}}}}.$$

This concept can be generalized in several ways. Here we consider a natural counterpart V_t, consisting in multiplying by a constant t not only the first level in the continued fraction, but the second as well. For a compactly supported measure μ we will denote such deformation by $V_t(\mu)$, thus getting

$$G_{V_t(\mu)}(z) = \cfrac{1}{z - t\,\alpha_0 - \cfrac{t\,\lambda_0}{z - t\,\alpha_1 - \cfrac{t\,\lambda_1}{z - \alpha_2 - \cfrac{\lambda_2}{z - \alpha_3 - \ddots}}}}.$$

For the ease of calculations let us first consider *symmetric* measures with moments, i.e. having all coefficients α_i equal to zero:

$$G_\mu(z) = \cfrac{1}{z - \cfrac{\lambda_0}{z - \cfrac{\lambda_1}{z - \cfrac{\lambda_2}{z - \ddots}}}} \qquad G_{V_t(\mu)}(z) = \cfrac{1}{z - \cfrac{t\,\lambda_0}{z - \cfrac{t\,\lambda_1}{z - \cfrac{\lambda_2}{z \ddots}}}}.$$

We look for a conversion formula

$$z - \frac{1}{G_\mu(z)} = \cfrac{\lambda_0}{z - \cfrac{\lambda_1}{z - \cfrac{\lambda_2}{z - \cfrac{\lambda_3}{z - \ddots}}}}$$

$$\frac{1}{\lambda_0}\left(z - \frac{1}{G_\mu(z)}\right) = \cfrac{1}{z - \cfrac{\lambda_1}{z - \cfrac{\lambda_2}{z - \cfrac{\lambda_3}{z - \ddots}}}}$$

$$z - \frac{1}{\frac{1}{\lambda_0}\left(z - \frac{1}{G_\mu(z)}\right)} = \cfrac{\lambda_1}{z - \cfrac{\lambda_2}{z - \cfrac{\lambda_3}{z - \cfrac{\lambda_4}{z - \ddots}}}}$$

$$z(1-t) + \frac{t}{\frac{1}{\lambda_0}\left(z - \frac{1}{G_\mu(z)}\right)} = z - \cfrac{t\,\lambda_1}{z - \cfrac{\lambda_2}{z - \cfrac{\lambda_3}{z - \cfrac{\lambda_4}{z - \ddots}}}}$$

$$\frac{t\,\lambda_0}{z(1-t)+\frac{t}{\frac{1}{\lambda_0}\left(z-\frac{1}{G_\mu(z)}\right)}} = \cfrac{t\,\lambda_0}{z-\cfrac{t\,\lambda_1}{z-\cfrac{\lambda_2}{z-\cfrac{\lambda_3}{z-\ddots}}}}$$

$$\frac{1}{z-\frac{t\,\lambda_0}{z(1-t)+\frac{t}{\frac{1}{\lambda_0}\left(z-\frac{1}{G_\mu(z)}\right)}}} = \cfrac{1}{z-\cfrac{t\,\lambda_0}{z-\cfrac{t\,\lambda_1}{z-\cfrac{\lambda_2}{z-\ddots}}}} = G_{V_t\mu}(z).$$

Hence

$$G_{V_t\mu}(z) = \frac{z-t\,z-G_\mu(z)\,z^2+G_\mu(z)\,t\,z^2-G_\mu(z)\,t\,\lambda_0}{z^2-t\,z^2-G_\mu(z)\,z^3+G_\mu(z)\,t\,z^3-t\,\lambda_0}. \tag{1}$$

By a similar calculation in the case of general, not necessarily symmetric, measures we get

$$G_{V_t\mu}(z) = \cfrac{1}{z-t\,\alpha_0-\cfrac{t\,\lambda_0}{z(1-t)+\cfrac{t\,\lambda_0}{z-\alpha_0-\cfrac{1}{G_\mu(z)}}}},$$

$$\begin{aligned} G_{V_t\mu}(z) = {} & \big(z-t\,z-z^2\,G_\mu(z)+t\,z^2\,G_\mu(z)+z\,\alpha_0\,G_\mu(z)-t\,z\,\alpha_0\,G_\mu(z)-t\,\lambda_0\,G_\mu(z)\big)/ \\ & \big[z^2-t\,z^2-t\,z\,\alpha_0+t^2\,z\,\alpha_0-t\,\lambda_0-z^3\,G_\mu(z)+t\,z^3\,G_\mu(z)+z^2\,\alpha_0\,G_\mu(z)+ \\ & -t^2\,z^2\,\alpha_0\,G_\mu(z)-t\,z\,{\alpha_0}^2\,G_\mu(z)+t^2\,z\,{\alpha_0}^2\,G_\mu(z)-t\,\alpha_0\,\lambda_0\,G_\mu(z)+t^2\,\alpha_0\,\lambda_0\,G_\mu(z)\big]. \end{aligned} \tag{2}$$

We can generalize the deformation V_t to measures with finite second moments with the use of the Nevanlinna representations. Let us recall a lemma by Maassen ([Ma], Proposition 2.2, see also [Ak]) that characterizes the reciprocals of the Cauchy transforms of measures with finite variance:

LEMMA 1. *A holomorphic function $F:\mathbb{C}^+\to\mathbb{C}^+$ is the reciprocal of the Cauchy transform of a measure μ with finite second moment if and only if there exists a positive finite measure ρ on $\mathbb{R}$ such that*

$$F(z) = z+\alpha_0+\int_{-\infty}^{\infty}\frac{d\rho_0(x)}{x-z},$$

where $\alpha_0\in\mathbb{R}$ is the first moment of the measure μ.

The crucial observation now is that the measure ρ_0 can be normalized by factoring its total weight $\lambda_0<\infty$ before the integral, and thus turned into a probability measure $\tilde\rho_0$:

$$F(z) = z+\alpha_0-\lambda_0\int_{-\infty}^{\infty}\frac{d\tilde\rho_0(x)}{z-x}.$$

The integral in the above equation is now nothing else but the Cauchy transform of the probability measure $\tilde\rho_0$, moreover, we may use the standard Nevanlinna integral representation theorem (see for instance Chapter 1 of [W]) for its reciprocal:

$$G_{\tilde\rho_0}(z)=\int_{-\infty}^{\infty}\frac{d\tilde\rho_0(x)}{z-x}=\frac{1}{F_{\tilde\rho_0}(z)}=\frac{1}{z+\alpha_1-\int_{-\infty}^{\infty}\frac{1+x\,z}{z-x}d\rho_1(x)}.$$

where $\alpha_1\in\mathbb{R}$ and ρ_1 is a finite positive measure on $\mathbb{R}$. We have thus proved the following theorem.

THEOREM 2. *A holomorphic function $F:\mathbb{C}^+\to\mathbb{C}^+$ is the reciprocal of the Cauchy transform of a measure μ with finite second moment if and only if there exist $\alpha_0,\alpha_1\in\mathbb{R}$, $\lambda_0\geq 0$ and a positive finite measure ρ_1 on $\mathbb{R}$ such that*

$$F_\mu(z)=z+\alpha_0-\frac{\lambda_0}{z+\alpha_1-\int_{-\infty}^{\infty}\frac{1+x\,z}{z-x}d\rho_1(x)}. \tag{3}$$

When $\lambda_0=0$ the measure μ is the Dirac mass at α_0.

DEFINITION 3. For a probability measure μ with finite second moment and $F_\mu(z)$ given by equation (3) we define its V_t deformation for $t>0$ as the probability measure $V_t(\mu)$ whose reciprocal of the Cauchy transform is given by

$$F_{V_t(\mu)}(z)=z+t\,\alpha_0-\frac{t\,\lambda_0}{z+t\,\alpha_1-\int_{-\infty}^{\infty}\frac{1+x\,z}{z-x}d(t\,\rho_1)(x)}. \tag{4}$$

2. Recurrence formula for moments of symmetric measures. Bożejko and Wysoczański in [BW2] have calculated a recurrence relation for the moments of the deformed measure; we can give a similar result for the deformation V_t. Since we have the relations between the moment generating function $M_\mu(z)$ and the Cauchy transform $G_\mu(z)$:

$$G_\mu(z)=\frac{1}{z}\,M_\mu\left(\frac{1}{z}\right),\qquad M_\mu(z)=\frac{1}{z}\,G_\mu\left(\frac{1}{z}\right),$$

we get from (1) the following:

$$\frac{t\,\lambda_0}{t-1}\big(z^2\,M_\mu(z)-z^2\,M_{V_t(\mu)}(z)\big)+M_\mu(z)\,M_{V_t(\mu)}(z)-M_\mu(z)-M_{V_t(\mu)}(z)+1=0.$$

We now differentiate the above equation $k>0$ times with the use of the chain rule:

$$\frac{t\,\lambda_0}{t-1}\sum_{j=0}^{k}\binom{k}{j}D^{(j)}(z^2)\big(M_\mu^{(k-j)}(z)-M_{V_t(\mu)}^{(k-j)}(z)\big)+\sum_{j=0}^{k}\binom{k}{j}M_\mu^{(j)}(z)\,M_{V_t(\mu)}^{(k-j)}(z)$$
$$-M_\mu^{(k)}(z)-M_{V_t(\mu)}^{(k)}(z)=0,$$

and notice that when we now evaluate the differentials at zero, only one term in the first sum survives, thus giving

$$\frac{t\,\lambda_0\,k!}{t-1}\left(\frac{M_\mu^{(k-2)}(z)}{(k-2)!}-\frac{M_{V_t(\mu)}^{(k-2)}(z)}{(k-2)!}\right)+k!\sum_{j=0}^{k}\frac{M_\mu^{(j)}(z)}{j!}\,\frac{M_{V_t(\mu)}^{(k-j)}(z)}{(k-j)!}-M_\mu^{(k)}(z)-M_{V_t(\mu)}^{(k)}(z)=0,$$

hence

$$\frac{t\,\lambda_0}{t-1}\big(m_\mu(k-2)-m_{V_t(\mu)}(k-2)\big)+\sum_{j=0}^{k}m_\mu(j)\,m_{V_t(\mu)}(k-j)-m_\mu(k)-m_{V_t(\mu)}(k)=0.$$

Since our deformation maps probability measures into probability measures, we know that $m_{V_t(\mu)}(0) = m_\mu(0) = 1$, we know also that odd moments of both measures are zero, by the fact that they are symmetric. Eventually, for $k = 2n$ we get the recurrence

$$\frac{t\,\lambda_0}{t-1}\big(m_\mu(2n-2) - m_{V_t(\mu)}(2n-2)\big) + \sum_{j=1}^{n-1} m_\mu(2j)\, m_{V_t(\mu)}(2n-2j) = 0.$$

By the Accardi–Bożejko formula, the second moment of measures with zero expectation is the symmetric Jacobi coefficient with index 0, we thus get $\lambda_0 = m_\mu(2)$ and $m_{V_t(\mu)}(2) = t\, m_\mu(2)$. This allows for further simplification for the case $n > 2$. We thus have the following

PROPOSITION 4. *The moments of the V_t deformation of a symmetric probability measure μ satisfy the following recurrence relation:*

$$m_{V_t(\mu)}(2) = t\, m_\mu(2),$$
$$m_{V_t(\mu)}(2n-2) = t^2\, m_\mu(2n-2) + \frac{t-1}{\lambda_0}\sum_{j=2}^{n-2} m_\mu(2j)\, m_{V_t(\mu)}(2n-2j) \quad \textit{for} \quad n > 2.$$

3. Convolution and limit theorems. Similarly as in the deformation given by U_t we define the V_t-free convolution for two probability measures with moments μ and ν by

$$\mu \boxplus_{V_t} \nu = V_{1/t}\,(V_t\,\mu \boxplus V_t\,\nu)\,.$$

This convolution is quite clearly associative and the deformation commutes with dilations, thus we have a central limit theorem.

THEOREM 5. *Let μ be a probability measure with mean 0 and second moment 1. Then the sequence of measures $D_{\frac{1}{\sqrt{n}}}(\mu) \boxplus_{V_t} \cdots \boxplus_{V_t} D_{\frac{1}{\sqrt{n}}}(\mu)$ tends in the $*$-weak topology to the measure $\mu_t = V_{1/t}(\omega_t)$ where ω_t is the semicircle law with mean 0 and second moment t.*

Proof. In the previous section we saw that for a measure μ with zero mean its second moment $m_{V_t\mu}(2) = t\, m_\mu(2)$. We have by definition

$$D_{\frac{1}{\sqrt{n}}}(\mu) \boxplus_{V_t} \cdots \boxplus_{V_t} D_{\frac{1}{\sqrt{n}}}(\mu) = V_{1/t}\big(D_{\frac{1}{\sqrt{n}}}\, V_t(\mu) \boxplus \cdots \boxplus D_{\frac{1}{\sqrt{n}}}\, V_t(\mu)\big)$$

and $D_{\frac{1}{\sqrt{n}}}\, V_t(\mu) \boxplus \cdots \boxplus D_{\frac{1}{\sqrt{n}}}\, V_t(\mu)$ obviously tends to the measure ω_t. ∎

We shall now compute the measure μ_t explicitly, by finding its Cauchy transform and then using Stieltjes inversion formula. We know the Cauchy transform of the semicircle law ω_t (see for instance [HP]):

$$G_{\omega_t}(z) = \frac{z - \sqrt{z^2 - 4t}}{2t} = \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{z - \cfrac{t}{z - \ddots}}}},$$

where the branch of the square root is chosen so that for negative x we have

$$\sqrt{x + i\,\epsilon} \longrightarrow_{\epsilon\to 0} = i\,\sqrt{-x}$$

whereas for positive x we have for positive ϵ the limit

$$\sqrt{x+i\,\epsilon}\longrightarrow_{\epsilon\to 0^+} = \sqrt{x}$$

and for negative ϵ the limit

$$\sqrt{x+i\,\epsilon}\longrightarrow_{\epsilon\to 0^-} = -\sqrt{x}.$$

Now

$$G_{\mu_t}(z) = \cfrac{1}{z-\cfrac{1}{z-\cfrac{1}{z-\cfrac{t}{z-\ddots}}}} = \cfrac{1}{z-\cfrac{1}{z-G_{\omega_t}(z)}} = \cfrac{1}{z-\cfrac{1}{z-\frac{z-\sqrt{z^2-4t}}{2t}}},$$

hence

$$G_{\mu_t}(z) = \frac{z(2t-1)+\sqrt{z^2-4t}}{z^2(2t-1)+z\sqrt{z^2-4t}-2t} = \frac{2z^3(t-1)+z(3-2t)-\sqrt{z^2-4t}}{2z^4(t-1)-4z^2(t-1)+2t}.$$

The density $f_{\tilde{\mu}_t}(x)$, $x\in\mathbb{R}$ of the absolutely continuous part of the measure μ_t can now be calculated by the Stieltjes formula

$$\begin{aligned} f_{\tilde{\mu}_t}(x) &= -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}G(x+i\,\epsilon)\\ &= \frac{2t\sqrt{4t-x^2}}{\pi\left(\,(x^2(2t-1)-2t)^2-x^2(x^2-4t)\right)}, \quad \text{with } x\in[-2\sqrt{t},2\sqrt{t}], \end{aligned}$$

whereas $\hat{\mu}_t$, the discrete part of the measure μ_t, will appear for $0<t<\frac{3}{4}$ and will have support in the poles of the Cauchy transform, with weights equal to the corresponding residues:

$$\hat{\mu}_t = \frac{1}{8}\left(2-\frac{1}{\sqrt{1-t}}+\sqrt{\frac{5-4\,t-4\sqrt{1-t}}{1-t}}\right)\left(\delta_{-\sqrt{\frac{1+\sqrt{1-t}}{\sqrt{1-t}}}}+\delta_{\sqrt{\frac{1+\sqrt{1-t}}{\sqrt{1-t}}}}\right).$$

One can check that the measure μ_t has no singular part, since the absolutely continuous and discrete parts integrate to 1. A diagram of this measure is presented in Figure 1.

It is also natural to consider the Poisson limit theorem:

THEOREM 6. *Let π_λ denote the V_t-free Poisson measure of parameter λ, defined as the weak limit*

$$\pi_\lambda = \lim_{N\to\infty}\underbrace{\mu_N \boxplus_{V_t}\mu_N\boxplus_{V_t}\cdots\boxplus_{V_t}\mu_N}_{N},$$

where

$$\mu_N = \left(1-\frac{\lambda}{N}\right)\delta_0+\frac{\lambda}{N}\,\delta_1.$$

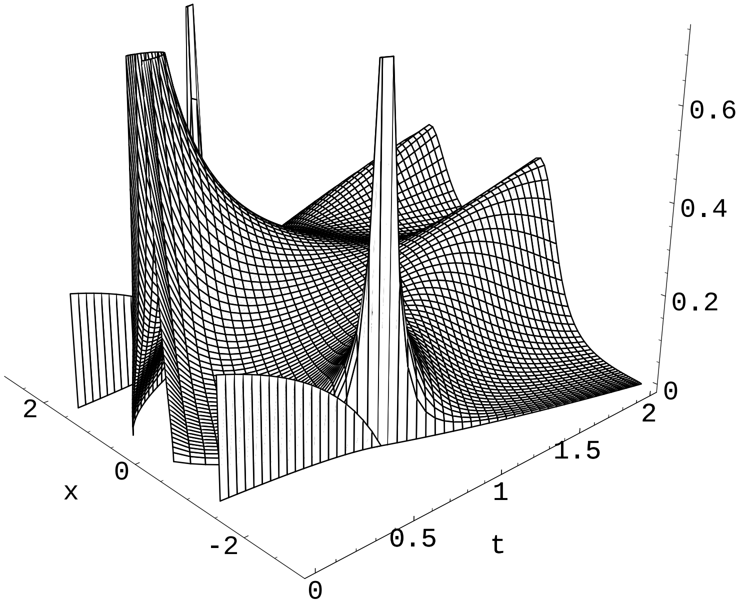

Fig. 1. Density of the measure μ_t for $t \in (0, 2]$

Then

$$G_{\pi_\lambda}(z) = \cfrac{1}{z - \lambda - \cfrac{\lambda}{z - (1+\lambda) - \cfrac{\lambda}{z - t\,(1+\lambda) - \cfrac{t\,\lambda}{\ddots}}}}.$$

Proof. Since we know that for z in the upper half plane we have

$$G_{\mu_N}(z) = \frac{1 - \frac{\lambda}{N}}{z} + \frac{\frac{\lambda}{N}}{z-1} = \frac{z - (1 - \frac{\lambda}{N})}{z\,(z-1)},$$

by the definition we obtain that

$$G_{V_t(\mu_N)}(z) = \frac{z^2 - t\,z - \frac{\lambda}{N}\,t + \left(\frac{\lambda}{N}\right)^2\,t + \frac{\lambda}{N}\,t^2 - \left(\frac{\lambda}{N}\right)^2\,t^2}{z - t + \frac{\lambda}{N}\,t}.$$

The Voiculescu's $R^{\boxplus}$-transform of $V_t(\mu_N)$ should satisfy the relation

$$\frac{1}{G_{V_t(\mu_N)}(z)} = z - R^{\boxplus}_{V_t(\mu_N)}\left(G_{V_t(\mu_N)}(z)\right)$$

which can be solved as

$$\begin{aligned} R^{\boxplus}_{V_t(\mu_N)}(z) &= \frac{-1 + t\,z - \sqrt{(-1+t\,z)^2 + \frac{4\,t\,z\,(1+z-t\,z)\,\lambda}{N} + \frac{4\,(-1+t)\,t\,z^2\,\lambda^2}{N^2}}}{2\,z} \\ &= \frac{t\,(-1 + (-1+t)\,z)\,\lambda}{N\,(-1+t\,z)} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

Since we know that

$$R^{\boxplus}_{\left(V_t(\mu_N)\right)^{\boxplus N}}(z) = N \cdot R^{\boxplus}_{V_t(\mu_N)}(z),$$

taking the limit as $N \to \infty$, we obtain

$$R^{\boxplus}_{V_t(\pi_\lambda)}(z) = \frac{t\,(-1+(-1+t)\,z)\,\lambda}{(-1+t\,z)},$$

which implies that the free cumulants of V_t-transformed measure $V_t(\pi_\lambda)$ are given by

$$\begin{cases} R^{\boxplus}_{V_t(\pi_\lambda)}(1) = t\,\lambda, \\ R^{\boxplus}_{V_t(\pi_\lambda)}(n) = t^{n-1}\,\lambda \quad \text{for} \quad n \geq 2. \end{cases}$$

Now let us determine the measure of the V_t-deformed free Poisson law exactly. The Cauchy transform of π_λ should satisfy the following relation for z in some neighbourhood of ∞:

$$\frac{1}{G_{V_t(\pi_\lambda)}(z)} = z - R^{\boxplus}_{V_t(\pi_\lambda)}\big(G_{V_t(\pi_\lambda)}(z)\big) = z - \frac{t\,\big(-1+(-1+t)\,G_{V_t(\pi_\lambda)}(z)\big)\,\lambda}{\big(-1+t\,G_{V_t(\pi_\lambda)}(z)\big)},$$

that is,

$$\begin{aligned} G_{V_t(\pi_\lambda)}(z) &= \frac{z-t\,(-1+\lambda)-\sqrt{z^2+t^2\,(1+\lambda)^2-2\,t\,(z+2\,\lambda+z\,\lambda)}}{2\,t\,(z+\lambda-t\,\lambda)} \\ &= \frac{1}{\frac{z-t\,\lambda+t}{2}+\frac{\sqrt{(z-t\,(1+\lambda))^2-4\,t\,\lambda}}{2}}, \end{aligned}$$

which implies that

$$G_{V_t(\pi_\lambda)}(z) = \cfrac{1}{z-t\,\lambda-\cfrac{t\,\lambda}{z-t\,(1+\lambda)-\cfrac{t\,\lambda}{z-t\,(1+\lambda)-\cfrac{t\,\lambda}{\ddots}}}}.$$

Thus we have

$$\begin{aligned} G_{\pi_\lambda}(z) &= \cfrac{1}{z-\lambda-\cfrac{\lambda}{z-(1+\lambda)-\cfrac{\lambda}{z-t\,(1+\lambda)-\cfrac{t\,\lambda}{\ddots}}}} \\ &= \Big[2\,(t-1)\,z^3-2\,(t-1)\,z^2\,(1+2\,\lambda)+z\,\lambda\,(1+2\,(t-1)\,\lambda) \\ &\quad +\lambda\Big(t-2\,\lambda+t\,\lambda-\sqrt{(t-z+t\,\lambda)^2-4\,t\,\lambda}\Big)\Big] \\ &/\big[2\,\big((t-1)\,(z-1)\,z^3+z\,(t-3\,(t-1)\,z^2)\,\lambda+(t-1)\,z\,(2+3\,z)\,\lambda^2-(t-1)\,(1+z)\,\lambda^3\big)\big]. \end{aligned}$$

The appropriate choice of the branch of a square root gives in limit for real $z = x$

$$\sqrt{(t-z+t\lambda)^2-4t\lambda} = \begin{cases} \sqrt{(t-x+t\lambda)^2-4t\lambda} & \text{for } x \geq t+t\lambda+2\sqrt{t\lambda}, \\ -\sqrt{(t-x+t\lambda)^2-4t\lambda} & \text{for } x \leq t+t\lambda-2\sqrt{t\lambda}, \\ i\sqrt{4t\lambda-(t-x+t\lambda)^2} & \text{for } t+t\lambda-2\sqrt{t\lambda} < x < t+t\lambda+2\sqrt{t\lambda}. \end{cases} \blacksquare$$

We can now determine the probability measure π_λ by using the Stieltjes inversion formula.

The density $f_{\pi_\lambda}(x),\ x \in \mathbb{R}$ of the absolutely continuous part of the measure π_λ is equal to

$$\begin{aligned} f_{\pi_\lambda}(x) &= -\frac{1}{\pi}\lim_{\epsilon\to 0^+} \mathrm{Im} G_{\pi_\lambda}(x+i\epsilon) \\ &= \frac{1}{2\pi}\frac{\lambda\sqrt{4t\lambda-(t-x+t\lambda)^2}}{(t-1)(x-1)x^3+x(t-3(t-1)x^2)\lambda+(t-1)x(2+3x)\lambda^2-(t-1)(1+z)\lambda^3}, \end{aligned}$$

with $x \in \left[t+t\lambda-2\sqrt{t\lambda}, t+t\lambda+2\sqrt{t\lambda}\right]$.

The denominator of the Cauchy transform

$$\left((t-1)(z-1)z^3+z(t-3(t-1)z^2)\lambda+(t-1)z(2+3z)\lambda^2-(t-1)(1+z)\lambda^3\right)$$

can have 4 roots, which means that the measure can have 4 atoms.

Acknowledgments. This research was partially supported by KBN grant no 1P03A 01330 and the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability", MTKD-CT-2004-013389.

References

[Ak] N. I. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, Edinburgh and London, 1965.

[BKW] M. Bożejko, A. D. Krystek and Ł. J. Wojakowski, *Remarks on the r- and Delta-convolutions*, Math. Z. 253 (2006), 177–196.

[BLS] M. Bożejko, M. Leinert and R. Speicher, *Convolution and limit theorems for conditionally free random variables*, Pac. J. Math. 175, (1996), 357–388.

[BS1] M. Bożejko and R. Speicher, *Ψ-independent and symmetrized white noises*, in: L. Accardi (ed.), Quantum Probability and Related Topics VI, World Scientific, Singapore, 1991, 219–236.

[BW1] M. Bożejko and J. Wysoczański, *New examples of convolutions and non-commutative central limit theorems*, Banach Cent. Publ. 43 (1998), 95–103.

[BW2] M. Bożejko and J. Wysoczański, *Remarks on t-transformations of measures and convolutions*, Ann. Inst. Henri Poincaré Probab. Stat. 37 (2001), 737–761.

[HP] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy*, American Mathematical Society, 2000.

[KW1] A. D. Krystek and Ł. J. Wojakowski, *Associative convolutions arising from conditionally free convolution*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 8 (2005), 515–545.

[KW2] A. D. Krystek and Ł. J. Wojakowski, *Convolution associated with the free* cosh-*law*, in: QP-PQ: Quantum Probability and White Noise Analysis 20, L. Accardi et al. (eds.), World Scientific, 2007, 295–301.

[KY2] A. D. Krystek and H. Yoshida, *Generalized t-transformations of probability measures and deformed convolution*, Probability and Mathematical Statistics 24 (2004), 97–119.

[Ma] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. 106 (1992), 409–438.

[Or1] F. Oravecz, *The number of pure convolutions arising from conditionally free convolution*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 8 (2005), 327–355.

[Or2] F. Oravecz, *Pure convolutions arising from conditionally free convolution*, preprint, 2004.

[W] Ł. J. Wojakowski, *Probability interpolating between free and boolean*, Dissertationes Mathematicae 446 (2007).