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THE STOCHASTIC LIMIT IN THE ANALYSIS OF SOME MODIFIED OPEN BCS MODELS

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Abstract. We use the so called *stochastic limit approach* as a tool to discuss the open BCS model of low temperature superconductivity. We also briefly discuss the role of a second reservoir in the analysis of the transition from a normal to a superconducting phase.

1. Introduction. In a recent paper, [3], we have analyzed the open BCS model as given in [9, 10] using the techniques of the *stochastic limit* (SL), [1]. Among other results, we have shown that the same values of the critical temperature and of the order parameters can be found using the SL, in a significantly simpler way. This procedure suggested using this approach in order to generalize the original model in the attempt to obtain some extra control on the value of the critical temperature T_c. This has been done in [4], where we have started our analysis on the role of a second reservoir in the definition of the model and its consequences on the value of T_c.

In this paper we review the results of these two papers: in particular, we devote the next section to summarizing our results concerning the original model, [3], while in Section 3 we consider the case where more reservoirs are considered, [4]. Section 4 contains our conclusions. We also add an Appendix to review some useful results concerning the stochastic limit approach, which are used in the main body of the paper.

2. The original model. The model discussed in [3] consists of the *system*, which is described by means of spin variables, and the *reservoir*, which is given in terms of bosonic operators. It is contained in a box of volume $V = L^3$, with N lattice sites. We define,

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following [9, 10],

$$H_N^{(sys)} = \widetilde{\epsilon}\sum_{j=1}^N \sigma_j^0 - \frac{g}{N}\sum_{i,j=1}^N \sigma_i^+\sigma_j^- . \tag{1}$$

The algebra of the Pauli matrices is given by $[\sigma_i^+,\sigma_j^-] = \delta_{ij}\sigma_i^0$, $[\sigma_i^\pm,\sigma_j^0] = \mp 2\delta_{ij}\sigma_i^\pm$. Then, introducing the operators $S_N^\alpha = \frac{1}{N}\sum_{i=1}^N \sigma_i^\alpha$ and $R_N = S_N^+ S_N^- = R_N^\dagger$, $H_N^{(sys)}$ can be simply written as $H_N^{(sys)} = N(\widetilde{\epsilon}S_N^0 - gR_N)$ and we have:

$$[S_N^0, R_N] = [H_N^{(sys)}, R_N] = [H_N^{(sys)}, S_N^0] = 0,$$

for any given $N > 0$. These S_N^α are all bounded by 1 in the operator norm, and the commutators $[S_N^\alpha, \sigma_j^\beta]$ go to zero in norm as $\frac{1}{N}$ when $N \to \infty$, for all j, α and β.

As for the reservoir, we introduce N bosonic modes $a_{\vec{p},j}$: $j = 1, 2, \dots, N$, one for each lattice site. $\vec{p}$ is the value of the momentum of the j-th boson which, if we impose periodic boundary conditions on the wave functions, has necessarily the form $\vec{p} = \frac{2\pi}{L}\vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ and $n_j \in \mathbb{Z}$. These operators satisfy the following CCR,

$$[a_{\vec{p},i}, a_{\vec{q},j}] = [a_{\vec{p},i}^\dagger, a_{\vec{q},j}^\dagger] = 0, \qquad [a_{\vec{p},i}, a_{\vec{q},j}^\dagger] = \delta_{ij}\delta_{\vec{p}\vec{q}} \tag{2}$$

and their free dynamics is given by

$$H_N^{(res)} = \sum_{j=1}^N \sum_{\vec{p}\in\Lambda_N} \epsilon_{\vec{p}}\, a_{\vec{p},j}^\dagger a_{\vec{p},j}. \tag{3}$$

Here $\Lambda_N = \{\vec{p} = \frac{2\pi}{L}\vec{n},\ \vec{n} \in \mathbb{Z}^3\}$. It may be useful to notice that the energy of the different bosons is independent of the lattice site: $\epsilon_{\vec{p}} = \frac{\vec{p}^2}{2m} = \frac{4\pi^2(n_1^2+n_2^2+n_3^2)}{2mL^2}$.

The interaction between reservoir and system is

$$H_N^{(I)} = \sum_{j=1}^N (\sigma_j^+ a_j(f) + h.c.), \tag{4}$$

where $a_j(f) = \sum_{\vec{p}\in\Lambda_N} a_{\vec{p},j} f(\vec{p})$, f being a given test function which will be later asked to satisfy some regularity conditions. Notice that, in order to keep the notation reasonably simple, we are not using the tensor product symbol here and along this paper, whenever the meaning of the formulas is clear.

The finite volume open system is now described by the following hamiltonian,

$$H_N = H_N^0 + \lambda H_N^{(I)}, \text{ where } H_N^0 = H_N^{(sys)} + H_N^{(res)} \tag{5}$$

and λ is the coupling constant.

We have seen in [3] that the free evolution of the interaction hamiltonian, $H_N^{(I)}(t) = e^{iH_N^0 t} H_N^{(I)} e^{-iH_N^0 t}$, can be written as

$$H_N^{(I)}(t) = \sum_{j=1}^N \sum_{\alpha=0,\pm} \left(\rho_\alpha^j a_j(fe^{it\nu_\alpha}) + h.c\right), \tag{6}$$

where

$$\left\{\begin{array}{l}\rho_0^j = \frac{g^2S^+}{\omega^2}(2S^-\sigma_j^+ + S^0\sigma_j^0 + 2S^+\sigma_j^-),\\ \rho_+^j = \frac{gS^+}{\omega^2}\left(gS^-\frac{\omega-gS^0}{\omega+gS^0}\sigma_j^+ + \frac{\omega-gS^0}{2}\sigma_j^0 - gS^+\sigma_j^-\right)\\ \rho_-^j = \frac{gS^+}{\omega^2}\left(gS^-\frac{\omega+gS^0}{\omega-gS^0}\sigma_j^+ - \frac{\omega+gS^0}{2}\sigma_j^0 - gS^+\sigma_j^-\right),\end{array}\right. \tag{7}$$

with

$$\omega = g\sqrt{(S^0)^2 + 4S^+S^-},\ \nu = 2\widetilde{\epsilon} + gS^0 \tag{8}$$

and

$$\nu_\alpha(\vec{p}) = \nu - \epsilon_{\vec{p}} + \alpha\omega. \tag{9}$$

Here α takes the values 0, + and − and $S^\alpha = \mathcal{F}\text{-strong}\lim_{N\to\infty} S_N^\alpha$. $\mathcal{F}$-strong indicates the strong topology restricted to a certain family $\mathcal{F}$ of *relevant* states, see [7] and references therein for the details. The introduction of $\mathcal{F}$ is needed because the sequence S_N^α does not converge in the uniform, strong or even in the weak topology, [7].

As we briefly sketch in the Appendix, the stochastic limit procedure is strongly linked to the result of the following limit, [1],

$$I(t) = \lim_{\lambda\to0} I_\lambda(t) = \lim_{\lambda\to0}\left(-\frac{i}{\lambda}\right)^2\int_0^t dt_1\int_0^{t_1} dt_2\,\omega_{tot}\left(H_N^{(I)}\left(\frac{t_1}{\lambda^2}\right)H_N^{(I)}\left(\frac{t_2}{\lambda^2}\right)\right),$$

which turns out to be, [3],

$$I(t) = -t\sum_{j=1}^N\sum_{\alpha=0,\pm}\{\omega_{sys}(\rho_\alpha^j{\rho_\alpha^j}^\dagger)\Gamma_\alpha^{(a)} + \omega_{sys}({\rho_\alpha^j}^\dagger\rho_\alpha^j)\Gamma_\alpha^{(b)}\}. \tag{10}$$

Here $\omega_{tot} = \omega_{sys}\otimes\omega_\beta$, where ω_{sys} is a generic state of the system while ω_β is a KMS-state of the reservoir, corresponding to an inverse temperature β. Notice that we have introduced two complex quantities

$$\Gamma_\alpha^{(a)} = \int_{-\infty}^0 d\tau\sum_{\vec{p}\in\Lambda_N}|f_m(\vec{p})|^2e^{-i\tau\nu_\alpha(\vec{p})},\qquad \Gamma_\alpha^{(b)} = \int_{-\infty}^0 d\tau\sum_{\vec{p}\in\Lambda_N}|f_n(\vec{p})|^2e^{i\tau\nu_\alpha(\vec{p})}, \tag{11}$$

$f_m(\vec{p}) = \sqrt{m(\vec{p})}f(\vec{p})$, $f_n(\vec{p}) = \sqrt{n(\vec{p})}f(\vec{p})$, where $m(\vec{p})$ and $n(\vec{p})$ are the following two point functions: $m(\vec{p}) = \omega_\beta(a_{\vec{p},j}a_{\vec{p},j}^\dagger)$ and $n(\vec{p}) = \omega_\beta(a_{\vec{p},j}^\dagger a_{\vec{p},j})$. Both $\Gamma_\alpha^{(a)}$ and $\Gamma_\alpha^{(b)}$ exist because of the standard regularity assumption on f under which the stochastic limit makes sense, [1, 3]. As we discuss in the Appendix, this analytic expression for $I(t)$ is the key result for us since it suggests the introduction of the following *stochastic limit hamiltonian*

$$H_N^{(sl)}(t) = \sum_{j=1}^N\sum_{\alpha=0,\pm}\{\rho_\alpha^j(c_{\alpha j}^{(a)}(t) + {c_{\alpha j}^{(b)}}^\dagger(t)) + h.c\}, \tag{12}$$

where the operators $c_{\alpha j}^{(\gamma)}(t)$ are assumed to satisfy the commutation rules

$$[c_{\alpha j}^{(\gamma)}(t), {c_{\beta k}^{(\mu)}}^\dagger(t')] = \delta_{jk}\,\delta_{\alpha\beta}\,\delta_{\gamma\mu}\delta(t-t')\Gamma_\alpha^{(\gamma)},\qquad \text{for } t > t', \tag{13}$$

see [3] and the Appendix below. The reason for this is that, using the hamiltonian (12) in the computation of

$$(-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, \Omega_{tot}(H_N^{(sl)}(t_1) H_N^{(sl)}(t_2)),$$

where $\Omega_{tot} = \omega_{sys} \otimes \Omega_\beta$, and Ω_β is a KMS-like state related to the operators $\{c_{\alpha j}^{(\gamma)}(t)\}$, we recover the same $I(t)$ as in (10), at least if the commutation rules in (13) are satisfied. We refer to [3] and [1] for further details concerning this procedure, and to [8] for a recent review on applications to many-body systems. After some algebraic computations, making use of the so-called *time consecutive principle* introduced in [1], following the procedure sketched in the Appendix, it is possible to associate to this hamiltonian a one parameter group of automorphisms of the observables of the system, representing its time evolution, whose generator L, when acting on the intensive operators S^0 and S^+S^-, looks like

$$L(S^0) := \mathcal{F}\text{-strong} \lim_{N\to\infty} L(S_N^0) = -\frac{8g^4 S^0 (S^+S^-)^2}{\omega^3} h(S^0, S^+S^-). \tag{14}$$

and

$$L(S^+S^-) := \mathcal{F}\text{-strong} \lim_{N\to\infty} L(S_N^+ S_N^-) = -\frac{16 g^4 (S^+S^-)^3}{\omega^3} h(S^0, S^+S^-), \tag{15}$$

where

$$\begin{aligned} h(S^0, S^+S^-) = {} & \Re\Gamma_+^{(a)} \frac{\omega - g}{(\omega + gS^0)^2} + \Re\Gamma_-^{(a)} \frac{\omega + g}{(\omega - gS^0)^2} \\ & + \Re\Gamma_+^{(b)} \frac{\omega + g}{(\omega + gS^0)^2} + \Re\Gamma_-^{(b)} \frac{\omega - g}{(\omega - gS^0)^2}. \end{aligned} \tag{16}$$

The phase structure of the model is given by the right-hand sides of equations (14) and (15), in particular, from the zeros of the functions

$$f_1(x,y) = -\frac{8g^4 x y^2}{\omega^3} h(x,y), \qquad f_2(x,y) = -\frac{16 g^4 y^3}{\omega^3} h(x,y), \tag{17}$$

where we have introduced, to simplify the notation, $x = S^0$ and $y = S^+S^-$. With this definitions we also have $\omega = g\sqrt{x^2 + 4y}$ and $\nu = 2\widetilde{\epsilon} + gx$. In particular, see [9], the existence of a super-conducting phase corresponds to the existence of a nontrivial zero of f_1 and f_2, that is, in our scheme, to a nontrivial zero of the function h: $h(x_o, y_o) = 0$. Following Buffet and Martin's original idea, we look for solutions corresponding to $\nu = 0$. We will discuss in the next section that this is not the only possibility. This means that, because of (8), the value of $x = S^0$ is fixed: $x = -2\widetilde{\epsilon}/g$. This also implies, see [3], that $\Re\Gamma_-^{(\gamma)} = 0$, $\gamma = a, b$, while

$$\Re\Gamma_+^{(a)} = \pi \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} \sum_{\vec{p}\in\mathcal{E}_N} |f(\vec{p})|^2, \quad \Re\Gamma_+^{(b)} = \pi \frac{1}{e^{\beta\omega} - 1} \sum_{\vec{p}\in\mathcal{E}_N} |f(\vec{p})|^2. \tag{18}$$

Therefore equation $h(x_o, y_o) = 0$ becomes

$$\pi \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} \sum_{\vec{p}\in\mathcal{E}_N} |f(\vec{p})|^2 \frac{\omega - g}{(\omega + gx)^2} + \pi \frac{1}{e^{\beta\omega} - 1} \sum_{\vec{p}\in\mathcal{E}_N} |f(\vec{p})|^2 \frac{\omega + g}{(\omega + gx)^2} = 0,$$

or, equivalently,

$$e^{\beta\omega} = \frac{g+\omega}{g-\omega}, \tag{19}$$

or, yet,

$$g \tanh\left(\frac{\beta\omega}{2}\right) = \omega,$$

which is exactly the equation found in [9]. The value of the critical temperature, under which superconductivity takes place, is therefore exactly the same as in [9], $T_c := \frac{g}{2k}$.

From the above treatment it is clear that the SL approach can be successfully used to analyze the phase structure of low temperature superconductivity as it follows from our open system interacting with a bosonic thermal bath.

The procedure discussed above is technically much simpler than the one used in the original paper, [9]. Among the other simplifications, for instance, a single equation $h(x,y) = 0$ must be solved instead of the system $f_1(x,y) = f_2(x,y) = 0$, which is the highly transcendental system which appears in [9].

3. More reservoirs. In this section we introduce some possible generalizations of the model discussed above which may let T_c to increase. This is important in concrete applications, of course, since it would suggest some possible mechanism giving rise to superconductivity at a *reasonably high* temperature.

The idea is very simple and is well put in evidence by the SL approach: suppose that the free evolution of the annihilation operator of the reservoir, $a_{\vec{p},i}(t) = a_{\vec{p},i}e^{-i\epsilon_{\vec{p}}t}$, is replaced, for some reason, by $a_{\vec{p},i}(t) = a_{\vec{p},i}e^{-i\gamma\epsilon_{\vec{p}}t}$, γ being some real constant less than one, $\gamma < 1$. As a consequence, the function $\nu_\alpha(\vec{p})$ in (9) will be replaced by $\nu_\alpha(\vec{p}) = \nu - \gamma\epsilon_{\vec{p}} + \alpha\omega$. All the other formulas are left unchanged, at least formally; $h(x,y)$ is the same as in (16), $\Re\Gamma_-^{(\rho)} = 0$, $\rho = a,b$, while $\Re\Gamma_+^{(a)} = \pi\frac{e^{\beta\omega/\gamma}}{e^{\beta\omega/\gamma}-1}\sum_{\vec{p}\in\mathcal{E}_N}|f(\vec{p})|^2$ and $\Re\Gamma_+^{(b)} = \pi\frac{1}{e^{\beta\omega/\gamma}-1}\sum_{\vec{p}\in\mathcal{E}_N}|f(\vec{p})|^2$, where, again, $\mathcal{E}_N = \{\vec{p}\in\Lambda_N : \epsilon_{\vec{p}} = \omega\}$. It is easy to check now that equation $h(x,y) = 0$ produces, taking $\nu = 0$ as before, the following equation:

$$e^{\beta\omega/\gamma} = \frac{g+\omega}{g-\omega},$$

which admits a nontrivial solution in $]0,g[$ if $g\beta/\gamma - 2 > 0$, that is under a *new* critical temperature $T_c^{(\gamma)} = \frac{g}{2k\gamma} = \frac{T_c}{\gamma}$, which is larger than T_c since $\gamma < 1$. This very easy procedure makes the value of the critical temperature to increase leaving unchanged all the physical parameters, in particular g. It is worth stressing that a similar mechanism was by no means evident in [9, 10].

The main point, therefore, is to find a possible way to change the free evolution of the boson operators as shown above. For that, one could first try to consider a reservoir obeying different statistical properties. However, it is very well known that both bosons and fermions produce the same free time evolution. For this reason, if we want to get a different result, we should try considering a reservoir made of quons, [11]. However this attempt has many technical difficulties and will not be considered here.

Another possibility to get a different time evolution for $a_{\vec{p},i}(t)$ consists in switching on an interaction between the boson reservoir in [3], which we will call $\mathcal{R}_1$ and another reservoir, $\mathcal{R}_2$, which only interacts with $\mathcal{R}_1$ and not with the system $\mathcal{S}$. This is the scheme of the rest of the paper: different choices of the second reservoir will be discussed below, with simple forms of interactions. These different choices all share a common output, that is the formal expression of $I(t)$, see (10). It should be clear that our results in this direction are really only a first step in our analysis, also because they produce an unexpected result: more nontrivial solutions of the equation $h(x,y)=0$ exist which make no reference to any critical temperature. The physical meaning of these solutions is still under investigation.

3.1. *A bosonic second reservoir.* Let

$$H_N = H_N^{(sys)} + H_N^{(res)} + \lambda H_N^{(I)} = H_N^0 + \lambda H_N^{(I)}, \tag{20}$$

where $H_N^{(sys)}$ is given in (1), $H_N^{(I)}$ in (4) and

$$H_N^{(res)} = H_N^{(R_1)} + H_N^{(R_2)} + \mu H_N^{(R_1,R_2)}. \tag{21}$$

We take $H_N^{(R_1)}$ as in (3), $H_N^{(R_1)} = \sum_{j=1}^N \sum_{\vec{p}\in\Lambda_N} \epsilon_{\vec{p}}\, a_{\vec{p},j}^\dagger a_{\vec{p},j}$, and

$$H_N^{(R_2)} = \sum_{j=1}^N \sum_{\vec{p}\in\Lambda_N} \epsilon_{\vec{p}}\, b_{\vec{p},j}^\dagger b_{\vec{p},j}, \qquad H_N^{(R_1,R_2)} = \sum_{j=1}^N \sum_{\vec{p}\in\Lambda_N} (a_{\vec{p},j}^\dagger b_{\vec{p},j} + a_{\vec{p},j} b_{\vec{p},j}^\dagger). \tag{22}$$

Here both the reservoirs satisfy a bosonic statistic and they are independent:

$$[a_{\vec{p},i}, a_{\vec{q},j}^\dagger] = [b_{\vec{p},i}, b_{\vec{q},j}^\dagger] = \delta_{ij}\delta_{\vec{p}\,\vec{q}}, \qquad [a_{\vec{p},i}^\sharp, b_{\vec{q},j}^\sharp] = 0. \tag{23}$$

With these definitions the free time evolution of $H_N^{(I)}$, $H_N^{(I)}(t) = e^{iH_N^0 t} H_N^{(I)} e^{-iH_N^0 t}$, depends on $\mathcal{R}_2$ only through its interaction with $\mathcal{R}_1$. We have, using the definition (7),

$$e^{iH_N^0 t}\sigma_j^+ e^{-iH_N^0 t} = e^{iH_N^{(sys)} t}\sigma_j^+ e^{-iH_N^{(sys)} t} = e^{i\nu t}\rho_0^j + e^{i(\nu+\omega)t}\rho_+^j + e^{i(\nu-\omega)t}\rho_-^j,$$

while

$$a_{\vec{p},i}(t) = e^{iH_N^0 t} a_{\vec{p},i} e^{-iH_N^0 t} = e^{iH_N^{(res)} t} a_{\vec{p},i} e^{-iH_N^{(res)} t} = e^{-i\epsilon_{\vec{p}} t}\left[a_{\vec{p},i}\cos(\mu t) - i b_{\vec{p},i}\sin(\mu t)\right].$$

Remark. The time evolution of the operator $b_{\vec{p},i}$ can be found easily, but it has no role in the computation of the stochastic limit of our model.

If we now introduce the following function:

$$\nu_{\alpha\beta}(\vec{p}) = \nu - \epsilon_{\vec{p}} + \alpha\omega + \beta\mu, \tag{24}$$

where $\alpha = 0, \pm$ and $\beta = \pm$, we get

$$H_N^{(I)}(t) = \frac{1}{2}\sum_{j=1}^N \sum_{\alpha=0,\pm} (\rho_\alpha^j [a_j(f e^{it\nu_{\alpha-}}) + a_j(f e^{it\nu_{\alpha+}}) + b_j(f e^{it\nu_{\alpha-}}) - b_j(f e^{it\nu_{\alpha+}})] + h.c.). \tag{25}$$

Let us now define the following operators:

$$A_{\vec{p},j} = \frac{a_{\vec{p},j} + b_{\vec{p},j}}{\sqrt{2}}, \qquad B_{\vec{p},j} = \frac{a_{\vec{p},j} - b_{\vec{p},j}}{\sqrt{2}}. \tag{26}$$

The only nontrivial commutation rules are now

$$[A_{\vec{p},i}, A^\dagger_{\vec{q},j}] = [B_{\vec{p},i}, B^\dagger_{\vec{q},j}] = \delta_{ij}\delta_{\vec{p}\,\vec{q}}. \tag{27}$$

With these definitions we get

$$H_N^{(I)}(t) = \frac{1}{\sqrt{2}} \sum_{j=1}^N \sum_{\alpha=0,\pm} (\rho^j_\alpha [A_j(fe^{it\nu_{\alpha-}}) + B_j(fe^{it\nu_{\alpha+}})] + h.c), \tag{28}$$

which produces, repeating the same computations as in [3], formally the same result as in (10):

$$I(t) = -t \sum_{j=1}^N \sum_{\alpha=0,\pm} \{\omega_{sys}(\rho^j_\alpha {\rho^j_\alpha}^\dagger)\Gamma^{(m)}_\alpha + \omega_{sys}({\rho^j_\alpha}^\dagger \rho^j_\alpha)\Gamma^{(n)}_\alpha\}, \tag{29}$$

where the only difference is in the expressions of the coefficients $\Gamma^{(m)}_\alpha$ and $\Gamma^{(n)}_\alpha$, which are now defined as

$$\Gamma^{(m)}_\alpha = \frac{\Gamma^{(m_A)}_{\alpha-} + \Gamma^{(m_B)}_{\alpha+}}{2}, \qquad \Gamma^{(n)}_\alpha = \frac{\Gamma^{(n_A)}_{\alpha-} + \Gamma^{(n_B)}_{\alpha+}}{2}. \tag{30}$$

Here we have introduced

$$\begin{cases} \Gamma^{(m_A)}_{\alpha-} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 m_A(\vec{p}) e^{-i\tau\nu_{\alpha-}(\vec{p})}, \\ \Gamma^{(m_B)}_{\alpha+} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 m_B(\vec{p}) e^{-i\tau\nu_{\alpha+}(\vec{p})}, \\ \Gamma^{(n_A)}_{\alpha-} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 n_A(\vec{p}) e^{+i\tau\nu_{\alpha-}(\vec{p})}, \\ \Gamma^{(m_B)}_{\alpha+} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 n_B(\vec{p}) e^{+i\tau\nu_{\alpha+}(\vec{p})}, \end{cases} \tag{31}$$

where $m_A(\vec{p}) = \omega_A(A_{\vec{p},j}A^\dagger_{\vec{p},j})$, $m_B(\vec{p}) = \omega_B(B_{\vec{p},j}B^\dagger_{\vec{p},j})$, $n_A(\vec{p}) = \omega_A(A^\dagger_{\vec{p},j}A_{\vec{p},j})$ and $n_B(\vec{p})$ $= \omega_B(B^\dagger_{\vec{p},j}B_{\vec{p},j})$. Here ω_A and ω_B are the KMS-states of respectively the A and the B operators. The *total* state is now given by the tensor product of three contributions, one for the system and two for the two reservoirs: $\omega_{tot} = \omega_{sys} \otimes \omega_A \otimes \omega_B$.

REMARK. For all our results to be meaningful, we have to require that all these integrals exist and are finite. This is a condition on $f(\vec{p})$, which extends the analogous one given in [3].

Due to the fact that $I(t)$ in (29) coincides formally with the one in (10), one easily recovers the same conclusions as in [3, 9, 10]: the system undergoes to a phase transition, from a normal to a superconducting phase, if the function $h(x,y)$ defined in analogy with (16) as

$$h(x,y) = \Re\Gamma^{(m)}_+ \frac{\omega-g}{(\omega+gx)^2} + \Re\Gamma^{(m)}_- \frac{\omega+g}{(\omega-gx)^2} + \Re\Gamma^{(n)}_+ \frac{\omega+g}{(\omega+gx)^2} + \Re\Gamma^{(n)}_- \frac{\omega-g}{(\omega-gx)^2}, \tag{32}$$

has a nontrivial zero (x_0, y_0).

Finding such a zero may be very hard, in general. First we observe that

$$\begin{cases} \Re\Gamma^{(m)}_+ = \frac{\pi}{2} \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 \left(m_A(\vec{p})\delta(\nu_{+-}(\vec{p})) + m_B(\vec{p})\delta(\nu_{++}(\vec{p}))\right), \\ \Re\Gamma^{(m)}_- = \frac{\pi}{2} \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 \left(m_A(\vec{p})\delta(\nu_{--}(\vec{p})) + m_B(\vec{p})\delta(\nu_{-+}(\vec{p}))\right), \\ \Re\Gamma^{(n)}_+ = \frac{\pi}{2} \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 \left(n_A(\vec{p})\delta(\nu_{+-}(\vec{p})) + n_B(\vec{p})\delta(\nu_{++}(\vec{p}))\right), \\ \Re\Gamma^{(n)}_- = \frac{\pi}{2} \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 \left(n_A(\vec{p})\delta(\nu_{--}(\vec{p})) + n_B(\vec{p})\delta(\nu_{-+}(\vec{p}))\right). \end{cases} \tag{33}$$

Using the usual suggestion contained in [9] we first consider briefly what happens if we fix $\nu = 0$. This assumption, which was very useful in [9, 10] and [3] to simplify the computations, has no immediate consequences here. In fact, if $\nu = 0$, it is possible to see that a sufficient condition for ω_0 to be a zero of the function h in (32) is that both these equations

$$e^{\beta_A(\omega_0-\mu)} = \frac{g+\omega_0}{g-\omega_0}, \qquad e^{\beta_B(\omega_0+\mu)} = \frac{g+\omega_0}{g-\omega_0}, \tag{34}$$

are satisfied, where β_A and β_B are the inverse temperatures of the two reservoirs A and B. Since A and B are different linear combinations of the original reservoirs $\mathcal{R}_1$ and $\mathcal{R}_2$, see formula (26), it would be reasonable to require $\beta_A = \beta_B$. But this is only compatible with one of these two choices: (a) if $\beta_A = \beta_B \neq 0$ then necessarily $\mu = 0$: therefore we go back, as it is expected, to [3] and we recover the same critical temperature. This is because $\mu = 0$ implies that $\mathcal{R}_1$ and $\mathcal{R}_2$ do not interact. (b) If $\beta_A = \beta_B = 0$, equations in (34) imply $\omega_0 = 0$, which is not what we want.

However, if we do not assume that $\beta_A = \beta_B$, it is possible to prove that, if our model admits a critical temperature, this must be necessarily lower than the one in [9], $T_c = \frac{g}{2k}$: it seems that the presence of this second reservoir can only decrease the value of T_c, which is exactly the opposite of our original aim. However, this is not the end of the story. In fact, these conclusions are a consequence of having chosen $\nu = 0$. In other words, we are looking for nontrivial solutions of the equation $h(x, y) = 0$ when $\nu = 0$. Other nontrivial solutions may exist, and our strategy is very flexible to discuss this new situation. This flexibility is lost in [9] and [10] for several reasons, and, among others, because the equation $h(x, y) = 0$ must be replaced by a system of equations.

Let us assume now that $\nu \neq 0$. We look for solutions such that only $\nu_{+-}(\vec{p})$ assume, for some $\vec{p}$, the value 0, while $\nu_{++}(\vec{p}), \nu_{--}(\vec{p})$ and $\nu_{-+}(\vec{p})$ are always different from zero. For such a solution to exists it is enough that the following inequalities are all satisfied:

$$\begin{cases} \nu + \omega + \mu < 0, \\ \nu - \omega - \mu < 0, \\ \nu - \omega + \mu < 0, \\ \nu + \omega - \mu \geq 0, \end{cases} \tag{35}$$

A trivial solution surely exists if we fix $\nu = \mu$ as far as $\omega \in]2\mu, -2\mu[$. This implies, because $0 \leq |\omega| \leq \sqrt{5}g$, that the coupling constant μ in $H^{(res)}$ must be negative and smaller than $-\frac{\sqrt{5}}{2}g$. With this choice we get $\Re\Gamma_-^{(m)} = \Re\Gamma_-^{(n)} = 0$, and $\Re\Gamma_+^{(m)} = \frac{\pi}{2}\sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 m_A(\vec{p})\delta(\nu_{+-}(\vec{p}))$ and $\Re\Gamma_+^{(n)} = \frac{\pi}{2}\sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 n_A(\vec{p})\delta(\nu_{+-}(\vec{p}))$. Now, as an easy consequence, we recover the same equation as in the previous section, $e^{\beta_A\omega} = \frac{g+\omega}{g-\omega}$ which implies that the critical temperature is not affected in this case.

More interesting is the situation when the system (35) holds true without having $\nu = \mu$. This is possible: for instance, the choice $\mu = -\omega$, $\nu = -\frac{\omega}{2}$ is a possible solution of (35) with $\nu \neq \mu$.

If (35) is satisfied we deduce that

$$e^{\beta_A(\omega+\nu-\mu)} = \frac{g+\omega}{g-\omega}. \tag{36}$$

In order to check whether this equation admits nontrivial solutions for some $\omega \in]0, g[$, we consider three different situations:

(i) if $\nu = \mu$ then we go back to the usual condition, [3], and we deduce the existence of a critical temperature which coincides with the *usual* one, $T_c = \frac{g}{2k}$.

(ii) if $\nu > \mu$ then the situation is different: since the function $F(\omega) := e^{\beta_A(\omega+\nu-\mu)} - \frac{g+\omega}{g-\omega}$ is such that $F(0) = e^{\beta_A(\nu-\mu)} - 1 > 0$ and $\lim_{\omega\to g^-} F(\omega) = -\infty$. Therefore, since $F(\omega)$ is continuous in $]0, g[$, then it surely exists a solution $F(\omega_0) = 0$ with $\omega_0 \in]0, g[$, independently of the values of β_A.

(iii) if $\nu < \mu$ then $F(0) = e^{\beta_A(\nu-\mu)} - 1 < 0$ and we cannot conclude as in (ii).

The conclusion of this analysis is therefore that, whenever (35) holds true, other solutions of equation $h(x, y) = 0$ different from those found in [9] may exist. Their physical meaning, however, is still to be understood. It may be worth stressing that we are not claiming that superconductivity exists independently of the temperature. We are just saying that more solutions of the equations considered in [9] exist, at least when the physical constants satisfy some peculiar conditions. This is a rather interesting feature of our model and surely deserves a deeper investigation.

3.2. *A fermionic second reservoir.* We consider now a different reservoir $\mathcal{R}_2$ and a different interaction between $\mathcal{R}_1$ and $\mathcal{R}_2$. In particular we assume that $H_N^{(R_2)} = \sum_{j=1}^N \sum_{p\in\Lambda_N} \eta_{\vec{p}}\, b_{\vec{p},j}^\dagger b_{\vec{p},j}$, where the operators $b_{\vec{p},j}$ satisfy the following CAR, $\{b_{\vec{p},j}, b_{\vec{q},i}^\dagger\} = \delta_{ij}\delta_{\vec{p},\vec{q}}$, $\{b_{\vec{p},j}, b_{\vec{q},i}\} = \{b_{\vec{p},j}^\dagger, b_{\vec{q},i}^\dagger\} = 0$, and commute with the $a_{\vec{p},j}$'s, and

$$H_N^{(R_1,R_2)} = \sum_{j=1}^N \sum_{\vec{p}\in\Lambda_N} a_{\vec{p},j}^\dagger a_{\vec{p},j} b_{\vec{p},j}^\dagger b_{\vec{p},j}. \tag{37}$$

The physical difference between this operator and the hamiltonian $H_N^{(R_1,R_2)}$ of the previous subsection, where if a boson a is created then a boson b is annihilated, is clear: here, in fact, $H_N^{(R_1,R_2)}$ only counts the number of bosons a and fermions b. A consequence of this different definition is that, while in the previous model the total number operator $\hat{N} = \hat{N}_a + \hat{N}_b = \sum_{j,\vec{p}\in\Lambda_N} a_{\vec{p},j}^\dagger a_{\vec{p},j} + \sum_{j,\vec{p}\in\Lambda_N} b_{\vec{p},j}^\dagger b_{\vec{p},j}$ commutes with $H_N^{(res)}$ even if $[H_N^{(res)}, \hat{N}_a] \neq 0$ and $[H_N^{(res)}, \hat{N}_b] \neq 0$, here we have $[H_N^{(res)}, \hat{N}_a] = [H_N^{(res)}, \hat{N}_b] = [H_N^{(res)}, \hat{N}] = 0$.

The free time evolution of the operator $a_{\vec{p},j}$, $a_{\vec{p},j}(t) = e^{iH_N^{(res)}t} a_{\vec{p},j} e^{-iH_N^{(res)}t}$, is again easily computed: $a_{\vec{p},j}(t) = a_{\vec{p},j} e^{-it(\epsilon_{\vec{p}} + \mu b_{\vec{p},j}^\dagger b_{\vec{p},j})}$, and the expression of $H_N^{(I)}(t)$ is quite similar to that in (6),

$$H_N^{(I)}(t) = \sum_{j=1}^N \sum_{\alpha=0,\pm} (\rho_\alpha^j\, a_j(f e^{it\nu_{\alpha j}}) + h.c). \tag{38}$$

Here $\nu_{\alpha j}$ is the following **operator**:

$$\nu_{\alpha j}(\vec{p}) = \nu + \alpha\omega - \epsilon_{\vec{p}} - \mu b_{\vec{p},j}^\dagger b_{\vec{p},j}, \tag{39}$$

and, analogously to what we did in (4), we have written

$$a_j(fe^{it\nu_{\alpha j}}) = \sum_{\vec{p}\in\Lambda_N} a_{\vec{p},j} e^{it(\nu+\alpha\omega-\epsilon_{\vec{p}})} e^{-it\mu b^\dagger_{\vec{p},j} b_{\vec{p},j}}.$$

If we now consider two KMS states, one for $\mathcal{R}_1$ and one for $\mathcal{R}_2$, corresponding to the inverse temperature β_1 and β_2 respectively, and we take $\omega_{tot} = \omega_{sys}\otimes\omega_{res} = \omega_{sys}\otimes\omega_{\beta_1}\otimes\omega_{\beta_2}$, we deduce for $I_\lambda(t) = (-i/\lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, \omega_{tot}(H_N^{(I)}(t_1/\lambda^2) H_N^{(I)}(t_2/\lambda^2))$ that

$$(40)\qquad I_\lambda(t) = -\sum_{j,\alpha,\beta}\sum_{\vec{p}\in\Lambda_N}\Bigg(\omega_{\mathcal{S}}(\rho^j_\alpha {\rho^j_\beta}^\dagger)|f_{m_a}(\vec{p})|^2$$

$$\times\frac{1}{\lambda^2}\int_0^t dt_1\int_0^{t_1} dt_2 e^{i\widetilde{\nu}_\alpha(\vec{p})\frac{t_1}{\lambda^2} - i\widetilde{\nu}_\beta(\vec{p})\frac{t_2}{\lambda^2}}\omega_{\beta_2}(e^{-i\mu b^\dagger_{\vec{p},j}b_{\vec{p},j}\frac{(t_1-t_2)}{\lambda^2}}) + \omega_{\mathcal{S}}({\rho^j_\alpha}^\dagger \rho^j_\beta)|f_{n_a}(\vec{p})|^2$$

$$\times\frac{1}{\lambda^2}\int_0^t dt_1\int_0^{t_1} dt_2 e^{-i\widetilde{\nu}_\alpha(\vec{p})\frac{t_1}{\lambda^2} + i\widetilde{\nu}_\beta(\vec{p})\frac{t_2}{\lambda^2}}\omega_{\beta_2}(e^{i\mu b^\dagger_{\vec{p},j}b_{\vec{p},j}\frac{(t_1-t_2)}{\lambda^2}})\Bigg),$$

where

$$f_{m_a}(\vec{p}) = f(\vec{p})\sqrt{m_a(\vec{p})},\qquad f_{n_a}(\vec{p}) = f(\vec{p})\sqrt{n_a(\vec{p})}$$

and

$$\widetilde{\nu}_\alpha(\vec{p}) = \nu + \alpha\omega - \epsilon_{\vec{p}} = \nu_\alpha(\vec{p}) + \mu b^\dagger_{\vec{p},j} b_{\vec{p},j}.$$

In order to compute $I(t) = \lim_{\lambda\to 0} I_\lambda(t)$ we first have to compute $\omega_{\beta_2}(e^{i\gamma b^\dagger_{\vec{p},j} b_{\vec{p},j}})$ for a generic value of γ. This is easily done recalling that $(b^\dagger_{\vec{p},j})^2 = (b_{\vec{p},j})^2 = 0$ and that ω_{β_2} is a KMS state. After some computations we get

$$(41)\qquad \omega_{\beta_2}(e^{\pm i\mu b^\dagger_{\vec{p},j}b_{\vec{p},j}\frac{(t_1-t_2)}{\lambda^2}}) = \frac{e^{\beta_2\eta_{\vec{p}}} + e^{\pm i\mu\frac{(t_1-t_2)}{\lambda^2}}}{e^{\beta_2\eta_{\vec{p}}}+1}.$$

It is now almost straightforward to deduce that

$$(42)\qquad I(t) = -t\sum_{j=1}^N\sum_{\alpha=0,\pm}\{\omega_{sys}(\rho^j_\alpha {\rho^j_\alpha}^\dagger)\Gamma^{(m)}_\alpha + \omega_{sys}({\rho^j_\alpha}^\dagger\rho^j_\alpha)\Gamma^{(n)}_\alpha\},$$

which is formally identical to equation (10), but for the definition of the coefficients which are now

$$(43)\qquad \begin{cases}\Gamma^{(m)}_\alpha = \int_{-\infty}^0 d\tau\sum_{\vec{p}\in\Lambda_N}|f(\vec{p})|^2 m_a(\vec{p})\frac{1}{e^{\beta_2\eta_{\vec{p}}}+1}(e^{-i\tau\widetilde{\nu}_\alpha(\vec{p})+\beta_2\eta_{\vec{p}}} + e^{-i\tau(\widetilde{\nu}_\alpha(\vec{p})-\mu)}),\\ \Gamma^{(n)}_\alpha = \int_{-\infty}^0 d\tau\sum_{\vec{p}\in\Lambda_N}|f(\vec{p})|^2 n_a(\vec{p})\frac{1}{e^{\beta_2\eta_{\vec{p}}}+1}(e^{+i\tau\widetilde{\nu}_\alpha(\vec{p})+\beta_2\eta_{\vec{p}}} + e^{+i\tau(\widetilde{\nu}_\alpha(\vec{p})-\mu)}).\end{cases}$$

Also for this model, therefore, the existence of a nontrivial zero of the function $h(x,y)$ defined as in (32) is the first step in order to analyze its superconducting features. To find this zero we first have to compute the following quantities:

$$(44)\qquad \begin{cases}\Re\Gamma^{(m)}_\alpha = \pi\sum_{\vec{p}\in\Lambda_N}\frac{1}{e^{\beta_2\eta_{\vec{p}}}+1}|f(\vec{p})|^2 m_a(\vec{p})(e^{\beta_2\eta_{\vec{p}}}\delta(\widetilde{\nu}_\alpha(\vec{p})) + \delta(\widetilde{\nu}_\alpha(\vec{p})-\mu)),\\ \Re\Gamma^{(n)}_\alpha = \pi\sum_{\vec{p}\in\Lambda_N}\frac{1}{e^{\beta_2\eta_{\vec{p}}}+1}|f(\vec{p})|^2 n_a(\vec{p})(e^{\beta_2\eta_{\vec{p}}}\delta(\widetilde{\nu}_\alpha(\vec{p})) + \delta(\widetilde{\nu}_\alpha(\vec{p})-\mu)).\end{cases}$$

We use now the same strategy as in the previous subsection, i.e., instead of fixing $\nu = 0$, we look for solutions of the following inequalities,

$$\left\{\begin{array}{l}\nu+\omega<0,\\ \nu-\omega<0,\\ \nu-\omega-\mu<0,\\ \nu+\omega-\mu\geq 0,\end{array}\right. \tag{45}$$

which is close but not identical to the system in (35), since if such a solution exists, then some interesting consequences arise, as we shall see. First, let us observe that if we fix $\nu=\mu$ then the system above is surely satisfied if we also take the coupling constant μ such that $\mu<-\sqrt{5}g$. However, as for the model described before, the equality $\nu=\mu$ produces again the same critical temperature as in [3], so that it is not very interesting for us. However, it is easy to see that other solutions of (45) exist corresponding to $\nu\neq\mu$. As an example we can choose $\mu=-2\omega$ and $\nu=-\frac{3\omega}{2}$.

After few computations we recover essentially the same equation as in (36):

$$e^{\beta_1(\omega+\nu-\mu)}=\frac{g+\omega}{g-\omega}. \tag{46}$$

It is clear, therefore, that the conclusions are exactly the same: (i) if $\nu=\mu$ then we get the same critical temperature as in [3]; (ii) if $\nu>\mu$ then we surely have a solution $F(\omega_0)=0$ with $\omega_0\in]0,g[$, for all the values of β_1. Again, this could suggest the existence of a superconducting phase for all values of the temperature, but for special values of the physical constants. However, as we have already stated several times, this is still to be better understood; (iii) if $\nu<\mu$ then a deeper analysis is required to conclude something about the critical temperature (and, again, we will not consider this situation here).

3.3. *A spin-like second reservoir.* We consider here another model whose structure is close to that of the previous models. The only differences wrt our previous definitions are again in $H_N^{(R_2)}$ and $H_N^{(R_1,R_2)}$. We put

$$H_N^{(R_2)}=\eta\sum_{j=1}^N\sum_{p\in\Lambda_N}\tau^0_{\vec{p},j},\quad H_N^{(R_1,R_2)}=\sum_{j=1}^N\sum_{\vec{p}\in\Lambda_N}a^\dagger_{\vec{p},j}a_{\vec{p},j}\tau^0_{\vec{p},j}, \tag{47}$$

where the operators $\tau^k_{\vec{p},j}$, $k=0,\pm$, satisfy the same algebra of the Pauli matrices $[\tau^+_{\vec{p},i},\tau^-_{\vec{q},j}]=\delta_{ij}\delta_{\vec{p}\vec{q}}\tau^0_{\vec{p},j}$, $[\tau^\pm_{\vec{p},i},\tau^0_{\vec{q},j}]=\mp 2\delta_{ij}\delta_{\vec{p}\vec{q}}\tau^\pm_{\vec{p},j}$ and commute with the $a_{\vec{p},j}$'s. The interpretation is not very different from that of the previous model: $H_N^{(R_1,R_2)}$ is a sort of *number operator*, as in the previous subsection, which counts the excitations of both $\mathcal{R}_1$ and $\mathcal{R}_2$, without creating or annihilating any of them. Once again we can find the exact free time evolution of the operators of $\mathcal{R}_1$:

$$a_{\vec{p},j}(t)=e^{iH_N^{(res)}t}a_{\vec{p},j}e^{-iH_N^{(res)}t}=a_{\vec{p},j}e^{-it(\epsilon_{\vec{p}}+\mu\tau^0_{\vec{p},i})},$$

so that

$$H_N^{(I)}(t)=\sum_{j=1}^N\sum_{\alpha=0,\pm}\left(\rho^j_\alpha\, a_j(fe^{it\nu_{\alpha j}})+h.c\right), \tag{48}$$

where we have introduced the operator

$$\nu_{\alpha j}(\vec{p})=\nu+\alpha\omega-\epsilon_{\vec{p}}-\mu\tau^0_{\vec{p},j}=\nu_\alpha(\vec{p})-\mu\tau^0_{\vec{p},j} \tag{49}$$

Most of the computations as in the previous section can be repeated here, and $a_j(fe^{it\nu_{\alpha j}})$ is defined analogously to the previous model. Again the state of the *complete system* is a product of three states, one general state for the system $\mathcal{S}$ and two KMS states for $\mathcal{R}_1$ and $\mathcal{R}_2$. The only major difference is now in the computation of $\omega_{\beta_2}(e^{i\alpha\tau^0_{\vec{p},j}})$ which can be obtained by introducing an o.n. basis of the spin operators $\tau^0_{\vec{p},i}$. We get

$$\omega_{\beta_2}(e^{i\alpha\tau^0_{\vec{p},j}}) = \frac{e^{i\alpha} + e^{-i\alpha}e^{2\beta_2\eta}}{1+e^{2\beta_2\eta}}, \tag{50}$$

which must replace the result in (41). Therefore, defining

$$\Gamma_\alpha^{(m)} = \frac{\Gamma_\alpha^{(m-)} + e^{2\beta_2\eta}\Gamma_\alpha^{(m+)}}{1+e^{2\beta_2\eta}}, \quad \Gamma_\alpha^{(n)} = \frac{\Gamma_\alpha^{(n-)} + e^{2\beta_2\eta}\Gamma_\alpha^{(n+)}}{1+e^{2\beta_2\eta}} \tag{51}$$

where

$$\begin{cases} \Gamma_\alpha^{(m-)} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 m_a(\vec{p}) e^{-i\tau\nu_\alpha^-(\vec{p})}, \\ \Gamma_\alpha^{(m+)} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 m_a(\vec{p}) e^{-i\tau\nu_\alpha^+(\vec{p})}, \\ \Gamma_\alpha^{(n-)} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 n_a(\vec{p}) e^{+i\tau\nu_\alpha^-(\vec{p})}, \\ \Gamma_\alpha^{(n+)} = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 n_a(\vec{p}) e^{+i\tau\nu_\alpha^+(\vec{p})}, \end{cases} \tag{52}$$

with $\nu_\alpha^\pm(\vec{p}) = \nu_\alpha(\vec{p}) \pm \mu$, $m_a(\vec{p}) = \omega_{\beta_1}(a_{\vec{p},j}a^\dagger_{\vec{p},j})$ and $n_a(\vec{p}) = \omega_{\beta_1}(a^\dagger_{\vec{p},j}a_{\vec{p},j})$, we get, once again, the same result as in (10):

$$I(t) = -t\sum_{j=1}^N \sum_{\alpha=0,\pm} \{\omega_{sys}(\rho_\alpha^j {\rho_\alpha^j}^\dagger)\Gamma_\alpha^{(m)} + \omega_{sys}({\rho_\alpha^j}^\dagger \rho_\alpha^j)\Gamma_\alpha^{(n)}\}, \tag{53}$$

Again, the critical temperature is related to the function $h(x,y)$ defined as in (32) with the above definition of the constants.

Let us now consider first the simplest situation, that is the limiting case $\beta_2 \to \infty$. In this case we see from (51) that

$$\begin{cases} \lim_{\beta_2\to\infty} \Gamma_\alpha^{(m)} = \Gamma_\alpha^{(m+)} \text{ and } \Re\Gamma_\alpha^{(m+)} = \pi \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 m_a(\vec{p})\delta(\nu_\alpha^+(\vec{p})), \\ \lim_{\beta_2\to\infty} \Gamma_\alpha^{(n)} = \Gamma_\alpha^{(n+)} \text{ and } \Re\Gamma_\alpha^{(n+)} = \pi \sum_{\vec{p}\in\Lambda_N} |f(\vec{p})|^2 n_a(\vec{p})\delta(\nu_\alpha^+(\vec{p})). \end{cases} \tag{54}$$

We will here consider only the following three different situations:

(1) first of all, assume $\mu < \nu = 0$. Then, due to (54), $\Re\Gamma_-^{(m+)} = \Re\Gamma_-^{(n+)} = 0$, and, introducing the set $\widetilde{\mathcal{E}}_N = \{\vec{p}\in\Lambda_N : \epsilon_{\vec{p}} = \omega+\mu\}$, we have $\Re\Gamma_+^{(m+)} = \pi\sum_{\vec{p}\in\widetilde{\mathcal{E}}_N} |f(\vec{p})|^2 \frac{e^{\beta_1(\omega+\mu)}}{e^{\beta_1(\omega+\mu)}-1}$ and $\Re\Gamma_+^{(n+)} = \pi\sum_{\vec{p}\in\widetilde{\mathcal{E}}_N} |f(\vec{p})|^2 \frac{1}{e^{\beta_1(\omega+\mu)}-1}$. Therefore equation $h(x,y)=0$ produces

$$e^{\beta_1(\omega+\mu)} = \frac{g+\omega}{g-\omega}. \tag{55}$$

Recalling that $\mu < 0$, the usual analysis can be repeated and we conclude that no solution exists for $\beta_1 \le \frac{2}{g}$, which means that the critical temperature can only decrease under these conditions: $T_c < \frac{g}{2k}$;

(2) let us now consider the case in which $\nu = -\mu$. Under this assumption we recover the usual equation, $e^{\beta_1\omega} = \frac{g+\omega}{g-\omega}$, so that the critical temperature coincides with the one found in [9] and [3];

(3) finally, let $\nu > -\mu$. Then, if also $\nu + \mu - \omega < 0$ is satisfied, the equation to be considered is $e^{\beta_1(\omega+\mu+\nu)} = \frac{g+\omega}{g-\omega}$, which has always a *superconducting* solution since $\lim_{\omega\to g^-} F(\omega) = \lim_{\omega\to g^-} e^{\beta_1(\omega+\mu+\nu)} - \frac{g+\omega}{g-\omega} = -\infty$ and $F(0) = e^{\beta_1(\nu+\mu)} - 1 > 0$.

Let us now remove the hypothesis $\beta_2 \to \infty$. Without going into details it is possible to see that if the following system is satisfied:

$$\begin{cases} \nu + \omega - \mu \geq 0, \\ \nu + \omega + \mu < 0, \\ \nu - \omega - \mu < 0, \\ \nu - \omega + \mu < 0, \end{cases} \tag{56}$$

(which holds true for instance if $\nu = \frac{3\mu}{2}$ and $\mu = -\omega$), then $h(x, y) = 0$ produces the equation $e^{\beta_1(\omega-\mu+\nu)} = \frac{g+\omega}{g-\omega}$ and the existence of a solution depends on the difference $\nu - \mu$: a solution always exists (for all β_1) for $\nu - \mu > 0$, exists only for $\beta_1 > \frac{2}{g}$ for $\nu - \mu = 0$ while for $\nu - \mu < 0$ the question is still open.

As we see, the situation is very close to that of the other models considered in the previous sections: more nontrivial solutions of the equation $h(x, y) = 0$ may exist which were not obtained in [9].

4. Conclusions and outcome. We have shown how the SL approach can be successfully used to simplify the treatment of the open BCS model of low superconductivity. We have also considered extended versions of the Martin-Buffet model, and from the analysis above it turns out that the choice $\nu = 0$ in [9, 10] and [3] is a very particular one and that many different solutions of the equation $h(x, y) = 0$ may be lost fixing this value. This suggests to work out a deeper analysis in order to understand the physical meaning of these different solutions which, by the way, also appear in the original, single-reservoir model, [4].

Moreover, since the free time evolution of the operators of $\mathcal{R}_1$ that we have obtained in the models discussed here is not the one we originally asked for, $a_{\vec{p},i}(t) = a_{\vec{p},i}e^{-i\gamma\epsilon_{\vec{p}}t}$, we are also interested in finding different soluble models, like some other version of a double reservoir open BCS model, which can produce this time behavior.

Appendix: A few results on the stochastic limit. In this Appendix we will briefly summarize some of the basic facts and properties concerning the SLA which are used all throughout the paper. We refer to [1] and references therein for more details.

Given an open system $\mathcal{S} + \mathcal{R}$ we write its hamiltonian H as the sum of two contributions, the free part H_0 and the interaction λH_I. Here λ is a coupling constant, H_0 contains the free evolution of both the system $\mathcal{S}$ and the reservoir $\mathcal{R}$, while H_I contains the interaction between $\mathcal{S}$ and $\mathcal{R}$. Working in the interaction picture, we define $H_I(t) = e^{iH_0t}H_Ie^{-iH_0t}$ and the so called wave operator $U_\lambda(t)$ which is the solution of the following differential equation

$$\partial_t U_\lambda(t) = -i\lambda H_I(t)U_\lambda(t), \tag{A.57}$$

with the initial condition $U_\lambda(0) = 1\!\!1$. Using the van-Hove rescaling $t \to \frac{t}{\lambda^2}$, see [10, 1] for instance, we can rewrite the same equation in a form which is more convenient for our perturbative approach, that is

$$\partial_t U_\lambda\left(\frac{t}{\lambda^2}\right) = -\frac{i}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) U_\lambda\left(\frac{t}{\lambda^2}\right), \tag{A.58}$$

with the same initial condition as before. Its integral counterpart is

$$U_\lambda\left(\frac{t}{\lambda^2}\right) = \mathbb{1} - \frac{i}{\lambda}\int_0^t H_I\left(\frac{t'}{\lambda^2}\right) U_\lambda\left(\frac{t'}{\lambda^2}\right) dt', \tag{A.59}$$

which is the starting point for a perturbative expansion, which works in the following way.

Suppose, to begin with, that we are interested in the zero temperature situation. Then let φ_0 be the ground vector of the reservoir and ξ a generic vector of the system. Now we put $\varphi_0^{(\xi)} = \varphi_0 \otimes \xi$. We want to compute the limit, for λ going to 0, of the first nontrivial order of the mean value of the perturbative expansion of $U_\lambda(t/\lambda^2)$ above in $\varphi_0^{(\xi)}$, that is the limit of

$$I_\lambda(t) = \left(-\frac{i}{\lambda}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \left\langle H_I\left(\frac{t_1}{\lambda^2}\right) H_I\left(\frac{t_2}{\lambda^2}\right)\right\rangle_{\varphi_0^{(\xi)}}, \tag{A.60}$$

for $\lambda \to 0$. Under some regularity conditions on the functions which are used to smear out the (typically) bosonic fields of the reservoir, this limit is shown to exist for many relevant physical models, see [1], and [2, 5, 8] for a few recent applications to quantum many body theory. It is at this stage that all the complex quantities like the $\Gamma_\alpha^{(\gamma)}$'s we have introduced in the main body of this paper appear. We define $I(t) = \lim_{\lambda\to 0} I_\lambda(t)$. In the same sense of the convergence of the (rescaled) wave operator $U_\lambda(\frac{t}{\lambda^2})$ (the convergence in the sense of correlators), it is possible to check that also the (rescaled) reservoir operators converge and define new operators which do not satisfy canonical commutation relations but a modified version of these. This is, for instance, the genesis of the commutation rules in (13). Moreover, these limiting operators depend explicitly on time and they live in a Hilbert space which is different from the original one. In particular, they annihilate a vacuum vector, η_0, which is no longer the original one, φ_0.

It is not difficult to deduce the form of a time dependent self-adjoint operator $H_I^{(sl)}(t)$, which depends on the system operators and on the limiting operators of the reservoir, such that the first nontrivial order of the mean value of the expansion of $U_t = \mathbb{1} - i\int_0^t H_I^{(sl)}(t')U_{t'}dt'$ on the state $\eta_0^{(\xi)} = \eta_0 \otimes \xi$ coincides with $I(t)$. The operator U_t defined by this integral equation is called again the *wave operator*.

The form of the generator of the reduced dynamics follows now from an operation of normal ordering. More in details, we start defining the flux of an observable $\widetilde{X} = X \otimes \mathbb{1}_r$, where $\mathbb{1}_r$ is the identity of the reservoir and X is an observable of the system, as $j_t(\widetilde{X}) = U_t^\dagger \widetilde{X} U_t$. Then, using the equation of motion for U_t and $U_t^\dagger$, we find that $\partial_t j_t(\widetilde{X}) = iU_t^\dagger[H_I^{(sl)}(t), \widetilde{X}]U_t$. In order to compute the mean value of this equation on the state $\eta_0^{(\xi)}$, so to get rid of the reservoir operators, it is convenient to compute first the commutation relations between U_t and the limiting operators of the reservoir. At this stage the so called time consecutive principle is used in a very heavy way to simplify the computation. This principle, which has been checked for many classes of physical models,

[1], states that, if $\beta(t)$ is any of these limiting operators of the reservoir, then

$$[\beta(t), U_{t'}] = 0, \text{ for all } t > t'. \tag{A.61}$$

Using this general result and recalling that η_0 is annihilated by the limiting annihilation operators of the reservoir, it is now a simple exercise to compute $\langle \partial_t j_t(X) \rangle_{\eta_0^{(\xi)}}$ and, by means of the equation $\langle \partial_t j_t(X) \rangle_{\eta_0^{(\xi)}} = \langle j_t(L(X)) \rangle_{\eta_0^{(\xi)}}$, to identify the form of the generator L.

Let us now briefly consider the case in which $T > 0$. In this case the state of the reservoir is no longer given by φ_0. It is now convenient to use the so-called *canonical representation of thermal states*, [1]. Any annihilator operator $a_{\vec{p},j}$ can be written as the following linear combination

$$a_{\vec{p},j} = \sqrt{m(\vec{p})}\, c^{(a)}_{\vec{p},j} + \sqrt{n(\vec{p})}\, c^{(b),\dagger}_{\vec{p},j}, \tag{A.62}$$

where $m(\vec{p})$ and $n(\vec{p})$ are the following two-points functions,

$$m(\vec{p}) = \omega_\beta(a_{\vec{p},j} a^\dagger_{\vec{p},j}) = \frac{1}{1 - e^{-\beta\epsilon_{\vec{p}}}}, \qquad n(\vec{p}) = \omega_\beta(a^\dagger_{\vec{p},j} a_{\vec{p},j}) = \frac{e^{-\beta\epsilon_{\vec{p}}}}{1 - e^{-\beta\epsilon_{\vec{p}}}}, \tag{A.63}$$

for a bosonic reservoir, if ω_β is a KMS state corresponding to an inverse temperature β. The operators $c^{(\alpha)}_{\vec{p},j}$ are assumed to satisfy the following commutation rules

$$[c^{(\alpha)}_{\vec{p},j}, {c^{(\gamma)}_{\vec{q},k}}^\dagger] = \delta_{jk}\delta_{\vec{p}\,\vec{q}}\delta_{\alpha\gamma}, \tag{A.64}$$

while all the other commutators are trivial. Let moreover Φ_0 be the vacuum of the operators $c^{(\alpha)}_{\vec{p},j}$:

$$c^{(\alpha)}_{\vec{p},j}\Phi_0 = 0, \qquad \forall \vec{p}, j, \alpha.$$

Then it is immediate to check that the results in (A.63) for the KMS state can be found, using these new variables, representing ω_β as the following vector state $\omega_\beta(\cdot) = \langle \Phi_0, \cdot\Phi_0 \rangle$. With this GNS-like representation it is trivial to check that both the CCR and the two-point functions are easily recovered. Once this representation is introduced, all the same steps as for the situation with $T = 0$ can be repeated, and the expression for the generator can be deduced using exactly the same strategy.

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