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A NOTE ON CERTAIN PARTIAL SUM OPERATORS

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Abstract. We show that for the t-deformed semicircle measure, where $\frac{1}{2} < t \leq 1$, the expansions of L_p functions with respect to the associated orthonormal polynomials converge in norm when $\frac{3}{2} < p < 3$ and do not converge when $1 \leq p < \frac{3}{2}$ or $3 < p$. From this we conclude that natural expansions in the non-commutative L_p spaces of free group factors and of free commutation relations do not converge for $1 \leq p < \frac{3}{2}$ or $3 < p$.

1. Introduction. In a recent publication [7] Junge, Nielsen, Ruan and Xu develop a theory of a certain class of operator L_p spaces. They are interested in operator spaces which can be paved out by complemented copies of finite dimensional non-commutative L_p spaces in such a way that the completely bounded distance (of the copies, to the finite dimensional L_p spaces) and the completely bounded norms of the projections are uniformly bounded. Spaces of this class they call $\mathcal{COL}_p$ spaces.

Among other results they prove that for a von Neumann algebra N with separable predual and QWEP, the non-commutative L_p space $L_p(N)$, where $1 < p < \infty$, is a $\mathcal{COL}_p$ space iff it has the completely bounded approximation property. They prove that in this case the space $L_p(N)$ has a Schauder basis with uniformly controlled completely bounded norm of the basis projections.

The regular von Neumann algebra $VN(\mathbb{F}_r)$ of the free group on $r \in \{1, 2, \ldots, \infty\}$ generators admits a canonical trace $tr(T) = \langle T\delta_e, \delta_e \rangle$ (here, and in the sequel, e denotes the identity of the group and for $x \in \mathbb{F}_r$, δ_x the point mass one at x). It, together with this

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functional, is a standard example of a non-commutative probability space. It fulfils the above conditions and for $1 < p < \infty$ the spaces $L_p(VN(\mathbb{F}_r)) = L_p(tr)$ possess a cb-basis. The completely bounded approximation property of $L_p(VN(\mathbb{F}_r))$ is a consequence of the weak amenability of $\mathbb{F}_r$ and to establish this, use is made of the natural length function with respect to the generating set. Let $\mathbb{F}_r$ be generated by $S = \{x_1, \ldots, x_r\}$, then the length function $\mathbf{l}$ is defined as: $\mathbf{l}(w) = \inf\{n : w = x_{i_1}^{\epsilon_1} \cdots {}_{i_n}^{\epsilon_n}, x_{i_j} \in S, \epsilon_j \in \{1, -1\}\}$.

Since acting on $l_2(\mathbb{F}_r)$ the von Neumann algebra $VN(\mathbb{F}_r)$ is in its standard representation $l_2(\mathbb{F}_r) = L_2(VN(\mathbb{F}_r))$ and we may represent $T \in L_2(VN(\mathbb{F}_r))$ as $T \sim \sum_{x\in\mathbb{F}_r} \langle T\delta_e, \delta_x\rangle\delta_x$ as an L_2 norm convergent sum. When $r < \infty$ we clearly can arrange to sum according to the length, i.e.

$$T \sim \sum_{n=0}^{\infty} \sum_{\{x:\, \mathbf{l}(x)=n\}} \langle T\delta_e, \delta_x\rangle\delta_x.$$

Moreover, when $r = 1$ then $\mathbb{F}_r = \mathbb{Z}$ and it is a rather basic fact of Fourier analysis that an element $T \in L_p(VN(\mathbb{F}_1)) = L_p(\mathbb{T}),\ 1 < p < \infty$ can be represented by an L_p-norm convergent Fourier sequence

$$T \sim \sum_{n=0}^{\infty} (a_n e^{in\theta} + a_{-n} e^{-in\theta})$$

(in fact much more is known about these representations). Phrasing this in another way, the point masses in the order according to the length of the group elements are a bounded Schauder basis of $L_p(VN(\mathbb{F}_1)),\ 1 < p < \infty$.

A question pointed out to us by Xu[1] is if for finite r the point masses, arranged according to the length of the group elements, still constitute a (completely) bounded basis of $L_p(VN(\mathbb{F}_r)),\ 1 < p < \infty$.

In this note we show that for $r > 1$, and $p > 3$ or $p < \frac{3}{2}$ the partial sum operators

$$S_N(T) = \sum_{\{x\in\mathbb{F}_r\,:\,\mathbf{l}(x)\leq N\}} \langle T\delta_e, \delta_x\rangle\delta_x$$

are not uniformly bounded on $L_p(VN(\mathbb{F}_r))$, which answers the question in the negative for this range of p.

THEOREM 1. *Assume $r \geq 2$ and $1 \leq p < \frac{3}{2}$ or $p > 3$. Then there exists $T \in L_p(VN(\mathbb{F}_r))$ such that*

$$\sup_{N\in\mathbb{N}} \|S_N(T)\|_p = \infty,$$

accordingly $S_N(T)$ does not converge in norm as $N \to \infty$.

The proof of this will be given after considering radial functions on $\mathbb{F}_r$ and properties of expansions with respect to certain orthogonal polynomials in section 3.

In section 4 we show that this phenomenon also appears for the L_p spaces associated to von Neumann algebras of free commutation relations.

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[1]We thank Professor Quanhua Xu for pointing out the problem to us.

2. Radial functions. A function $f : \mathbb{F}_r \to \mathbb{C}$ is called radial if its values $f(w)$ depend only on the length $\mathbf{l}(w)$ of the group elements w. Accordingly we call an operator $T \in VN(\mathbb{F}_r)$ radial if $T\delta_e$ is a radial l_2 function. $VN(\mathbb{F}_r)_{rad}$ denotes the radial operators in $VN(\mathbb{F}_r)$, and moreover for a space of functions E on $\mathbb{F}_r$ we denote E_{rad} the subspace of its radial functions.

For $n \in \mathbb{N}$ let χ_n be the characteristic function of the set $E_n = \{w \in \mathbb{F}_r : \mathbf{l}(w) = n\}$ of elements of length n.

For this section we shall assume $r < \infty$. Then, as is well known, the summable radial functions $l_1(\mathbb{F}_r)_{rad}$ are, under convolution, a commutative $*$-algebra generated by χ_1. This is seen from recursion relations:

$$\begin{aligned} \chi_1 * \chi_1 &= \chi_2 + 2r\chi_0, \\ \chi_1 * \chi_n &= \chi_{n+1} + (2r-1)\chi_n, \quad n = 2, 3, \ldots . \end{aligned}$$

The operator norm closure Cv_{rad} of $\lambda(l_1(\mathbb{F}_r)_{rad})$ is a commutative C^*-algebra isomorphic to $C(I)$, where the interval $I = [-2\sqrt{2r-1}, 2\sqrt{2r-1}]$ is the spectrum of $\lambda(\chi_1)$.

Let $P_n, n \in \mathbb{N}$ denote the sequence of polynomials defined by the recurrence

$$\begin{aligned} P_0(x) &= 1, \quad P_1(x) = x, \\ xP_1(x) &= P_2(x) + (2r)P_0(x), \\ xP_n(x) &= P_{n+1}(x) + (2r-1)P_{n-1}(x), \quad n = 2, 3, \ldots . \end{aligned}$$

Then the $*$-homomorphism $\hat{}: \chi_n \to P_n$ extends to the Gelfand transform of Cv_{rad}. The measure $d\mu(x) = \frac{r}{\pi}\frac{\sqrt{4(2r-1)-x^2}}{4r^2-x^2}dx$ represents the trace in the sense that for $f \in l_1(\mathbb{F}_r)_{rad}$

$$tr\lambda(f) = \int_I \widehat{f}(x)\, d\mu(x). \tag{1}$$

Using $\|f\|_2^2 = f * f^*(e) = tr\lambda(f * f^*) = \int_I (f * f^*)\hat{}(x)\, d\mu(x) = \int_I |f(x)|^2\, d\mu(x)$ we see that the Gelfand transform extends to an isometric isomorphism, still denoted $\hat{}$ from $l_2(\mathbb{F}_r)_{rad}$ onto $L_2(I, \mu)$.

For our purposes it is suitable to renormalise, first with respect to the l_2 norm, and further to the interval $[-1, 1]$ to obtain a sequence of orthonormal polynomials with respect to the transferred measure. We choose $t = 1 - \frac{1}{2r}$ as a parameter and have for $s \in [-1, 1]$ that $Q_{t,n}(s) = \|\chi_n\|_2^{-1} P_n(2\sqrt{2r-1}\, s)$ and

$$d\nu_t(s) = \frac{1}{\pi}\frac{r(2r-1)\sqrt{1-s^2}}{r^2-(2r-1)s^2}\, ds = \frac{1}{2\pi}\frac{4t\sqrt{1-s^2}}{1-4(1-t)ts^2}\, ds.$$

These polynomials then satisfy

$$Q_{t,0}(s) = 1, \quad Q_{t,1}(s) = 2\sqrt{t}\, s, \quad Q_{t,2}(s) = 4\sqrt{t}s^2 - \frac{1}{\sqrt{t}}, \tag{2}$$

$$s\, Q_{t,n}(s) = \frac{1}{2}\, Q_{t,n+1}(s) + \frac{1}{2}\, Q_{t,n-1}(s), \quad n = 2, 3, \ldots . \tag{3}$$

The measures ν_t are examples of t-transformed measures in the sense of Bożejko and Wysoczański. Here we are interested only in the parameter range $[\frac{1}{2}, 1]$ and we refer for further discussion to [5].

LEMMA 1. *If $r < \infty$ then the Banach spaces $L_p(VN(\mathbb{F}_r))_{rad}$ and $L_p([-1,1], \nu_{1-\frac{1}{2r}})$ are isometrically isomorphic.*

Proof. The operator $\mathcal{E} : f \mapsto \sum_n \frac{1}{\# E_n} \sum_{x \in E_n} f(x)\chi_n$ initially defined on finitely supported functions $f : \mathbb{F}_r \to \mathbb{C}$ extends to a conditional expectation from $VN(\mathbb{F}_r)$ onto $VN(\mathbb{F}_r)_{rad}$ and to an orthogonal projection from $l_2(\mathbb{F}_r)$ onto $l_2(\mathbb{F}_r)_{rad}$. Hence for the complex interpolation method we have $[VN(\mathbb{F}_r)_{rad}, l_2(\mathbb{F}_r)_{rad}]_\theta = ([VN(\mathbb{F}_r), l_2(\mathbb{F}_r)]_\theta)_{rad}$ for any $\theta \in [0,1]$. When $2 < p$ then by [8], for $\theta = \frac{2}{p}$ the right hand side equals $L_p(VN(\mathbb{F}_r))_{rad}$. On the other hand (1), with our normalisation, shows that $VN(\mathbb{F}_r)_{rad} = L_\infty([-1,1], \nu_{1-\frac{1}{2r}})$ and $l_2(\mathbb{F}_r)_{rad} = L_2([-1,1], \nu_{1-\frac{1}{2r}})$. This proves the assertion when $2 \le p \le \infty$. For $1 \le p \le 2$ it now follows by duality. ∎

3. Orthogonal polynomials. In this section we shall discuss L_p-boundedness of partial sums of expansions with respect to the orthonormal polynomials associated to the measures ν_t.

For $t = \frac{1}{2}$ the measure ν_t has the density $\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$, with respect to Lebesgue measure on the interval $[-1,1]$, and for $t = 1$ the density is $\frac{2}{\pi}\sqrt{1-x^2}$. In the first case the Tchebyscheff polynomials of first kind

$$T_n(x) = \cos n\theta, \text{ where } \cos\theta = x.$$

are orthogonal with respect to the measure $\nu_{\frac{1}{2}}$. In the second case we find the Tchebyscheff polynomials of second kind:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \text{ where } \cos\theta = x.$$

In both cases they satisfy the recursion

$$2xY_n(x) = Y_{n+1}(x) + Y_{n-1}(x), \qquad n = 1,2,3\ldots \tag{4}$$

and are then determined by the respective values

$$T_0(x) = 1,\ T_1(x) = x; \qquad U_0(x) = 1,\ U_1(x) = 2x. \tag{5}$$

When $t \in (\frac{1}{2}, 1)$ the orthonormal polynomials $Q_{t,n}$, see (3), satisfy this recursion relation except for the first step $n = 1$. If we redefine $\tilde{Q}_{t,0}(s) = 1/\sqrt{t}$ and $\tilde{Q}_{t,n} = Q_{t,n}$, $n = 1,2,\ldots$ then these polynomials satisfy the recursion (4) for all $n = 1,2,\ldots$ and they are still orthogonal with respect to ν_t. With the exception of $\tilde{Q}_{t,0}$ they are still normalised. From the values (3) and (5) we conclude

$$\tilde{Q}_{t,n} = \frac{2-2t}{\sqrt{t}}T_n + \frac{2t-1}{\sqrt{t}}U_n,\ n = 0,1,2\ldots.$$

For $t \in [\frac{1}{2}, 1]$ and ν_t-integrable $f : [-1,1] \to \mathbb{C}$ define

$$a_{t,n}(f) = \int_{-1}^{1} f(s)\tilde{Q}_{t,n}(s)\, d\nu_t(s)$$

and, for $N \in \mathbb{N}$ let

$$S_{t,N}(f) = \sum_{n=0}^{N} a_{t,n}(f)\tilde{Q}_{t,n}.$$

We are interested in the range of p, $1 \le p \le \infty$, for which the partial sum operators are uniformly L_p-bounded, that is there exists a constant $C_{t,p}$ such that for all $f \in L_p([-1,1],\nu_t)$:

$$\sup_{N\in\mathbb{N}}\left(\int_{-1}^{1} |S_{t,N}(f)(s)|^p \, d\nu_t(s)\right)^{\frac{1}{p}} \le C_{t,p}\left(\int_{-1}^{1} |f(s)|^p \, d\nu_t(s)\right)^{\frac{1}{p}}. \qquad (6)$$

If $t = \frac{1}{2}$ this range is the open interval $(1,\infty)$. This may be seen from the classical theorem of M. Riesz on Fourier series, because, by the substitution $x = \cos\theta$ we are led to the Fourier expansions of even functions on the circle.

For the case that $t = 1$ Pollard [9] proved that the range of validity of (6) includes $(\frac{3}{2},3)$ and that the partial sum operators are not uniformly bounded for $p < \frac{3}{2}$ or $p > 3$. The negative assertion on the convergence of the partial sums at the points $p = \frac{3}{2}$ and $p = 3$ is due to Askey and Hirschman [1, Theorem 4c].

Evidently, for any polynomial q, $q(x) = \sum_{n=0}^{\deg(q)} a_{t,n}(q)\tilde{Q}_{t,n}(x)$, $x \in [-1,1]$. Hence for elements of a dense subspace in $L_p([-1,1],\nu_t)$, $1 \le p < \infty$, the partial sums converge in L_p. If $1 \le p < \infty$, by the uniform boundedness principle the uniform boundedness of the partial sums is equivalent to the L_p convergence:

Theorem 2. *Assume $t \in (\frac{1}{2},1]$. Then for p in the range $\frac{3}{2} < p < 3$*

$$\lim_{N\to\infty}\left(\int_{-1}^{1} |f(s) - S_{t,N}(f)(s)|^p \, d\nu_t(s)\right)^{\frac{1}{p}} = 0, \qquad \forall f \in L_p([-1,1],\nu_t),$$

whereas for $p < \frac{3}{2}$ or $p > 3$ the L_p convergence fails.

Proof. The proof of this theorem relies on the above cited work of Pollard and that of Askey and Hirschman.

For the positive part we cite Pollard's theorem 5.1 and have to verify its hypotheses (H1)–(H(7) for the weight function $w(x) = \frac{1}{2\pi}\frac{4t\sqrt{1-x^2}}{1-4(1-t)tx^2} = \frac{1}{2\pi}t(x)(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}$, where $t(x) = \frac{4t}{1-4(1-t)tx^2}$ is positive and two times continuously differentiable on $[-1,1]$. Hence w belongs to the class $\mathbf{B}$ of weights considered already by Bernstein [2]. (H1)–(H6) are verified in section 6 of [9].

To verify (H7) we consider the kernels

$$k_\pm(x,y) = \left|\frac{\left(\frac{1-y^2}{1-x^2}\right)^{\pm\frac{1}{4}}\left(\frac{w(y)}{w(x)}\right)^{\frac{1}{2}-\frac{1}{p}} - 1}{x-y}\right|.$$

Lemma 2. *Assume $\frac{3}{2} < p < 3$. For $f \in L_p([-1,1],dx)$ let*

$$K_\pm f(x) = \int_{-1}^{1} k_\pm(x,y)f(y)\,dy.$$

Then the operators $K_\pm$ are bounded on $L_p([-1,1],dx)$.

Proof of the Lemma. As in the proof of lemma 7.1 of [9] it suffices to show the uniform boundedness, with respect to y, of the integrals

$$\int_{-1}^{1}\left|\frac{\left(\frac{1-y^2}{1-x^2}\right)^a\left(\frac{1-\alpha x^2}{1-\alpha y^2}\right)^\gamma - \left(\frac{1-x^2}{1-y^2}\right)^b}{x-y}\right| dx,$$

where $\alpha = 4(1-t)t$, $0 < a < 1$, $0 < b < 1$, and $\gamma = \pm(\frac{1}{2} - \frac{1}{p})$. This can be done, but it is a little easier to argue as follows: Write

$$k_{\pm}(x,y) = \left| \frac{(\frac{1-y^2}{1-x^2})^{c_{\pm}} \Phi(x,y) - 1}{x-y} \right|$$

where $c_{\pm} = \pm\frac{1}{4} + \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$ and $\Phi(x,y) = (\frac{1-\alpha_t y^2}{1-\alpha_t x^2})^{\frac{1}{2}-\frac{1}{p}} = \frac{f(y)}{f(x)}$, with $\alpha_t = 4t(1-t)$ and $f(x) = (1-\alpha_t x^2)^{\frac{1}{2}-\frac{1}{p}}$. Notice, that for $t \in (\frac{1}{2}, 1)$ we have $0 < \alpha_t < 1$ and $0 < \inf\{f(x) : x \in [-1,1]\} < 1$, furthermore f is infinitely differentiable with bounded derivatives. Now

$$k_{\pm}(x,y) \le \left| \frac{(\frac{1-y^2}{1-x^2})^{c_{\pm}} \Phi(x,y) - \Phi(x,y)}{x-y} \right| + \left| \frac{\Phi(x,y) - 1}{x-y} \right| =: k^1_{\pm}(x,y) + k^2_{\pm}(x,y).$$

First

$$k^1_{\pm}(x,y) = f(x)^{-1} \left| \frac{(\frac{1-y^2}{1-x^2})^{c_{\pm}} - 1}{x-y} \right| f(y)$$

and, by [9, Lemma 7.1], the integral operator

$$h \mapsto \int_{-1}^{1} k^1_{\pm}(.,y) h(y)\, dy$$

is bounded on $L_p([-1,1], dx)$, whenever $\frac{3}{2} < p < 3$.

Second

$$k^2_{\pm}(x,y) = f(x)^{-1} \left| \frac{f(y) - f(x)}{x-y} \right| < \text{const}$$

since f is bounded away from zero and has a bounded derivative. Hence the integral operator with this kernel is bounded on $L_p([-1,1], dx)$ for any $1 \le p \le \infty$. ∎

The negative assertion of the theorem is based on the following lemmata. We will continue the proof of the theorem first and postpone the proof of the lemmata.

LEMMA 3. *There exists $c_t > 0$ such that for all $N \in \mathbb{N}$:*

$$\int |\tilde{Q}_{t,N}(x)|\, d\nu_t(x) > c_t.$$

LEMMA 4. *Assume $1 \ge t > \frac{1}{2}$. If $1 \le p < \frac{3}{2}$ then $f(x) = (1-x)^{-1}(1-4(1-t)tx^2) \in L_p(\nu_t)$ and $\lim_{N\to\infty} a_{t,N}(f) > 0$.*

For $1 \le p < \frac{3}{2}$ and this function f we have

$$\begin{aligned} \|S_{t,N}(f) - S_{t,N-1}(f)\|_{L_p(\nu_t)} &\ge \left(\int |a_{t,N}(f) \tilde{Q}_{t,N}(x)|^p\, d\nu_t(x) \right)^{\frac{1}{p}} \\ &\ge a_{t,N}(f) \int |\tilde{Q}_{t,N}(x)|\, d\nu_t(x) \;>\; c_t{}' \;>\; 0. \end{aligned}$$

That the L_p convergence fails in the range $p > 3$ now follows by duality. ∎

Proof of Lemma 3. The trigonometric relation: $2\cos n\theta \sin\theta = \sin(n+1)\theta - \sin(n-1)\theta$ implies that

$$\tilde{Q}_{t,n}(\cos\theta) \;=\; \frac{t}{\sqrt{t}} \frac{\sin(n+1)\theta}{\sin\theta} - \frac{1-t}{\sqrt{t}} \frac{\sin(n-1)\theta}{\sin\theta}, \quad n = 1, 2, \ldots.$$

For $t > \frac{1}{2}$ on the $n+1$ intervals

$$I_k = \left(\frac{(2k+1)\pi}{2(n+1)} - \frac{\pi}{n+1}\left(\frac{1}{2} - \frac{1}{4t}\right), \frac{(2k+1)\pi}{2(n+1)} + \frac{\pi}{n+1}\left(\frac{1}{2} - \frac{1}{4t}\right)\right), \qquad k = 0, \dots, n,$$

in $[0, \pi]$ we have $|\sin(n+1)\theta| > \frac{1}{2t}$. Set $E = \bigcup_k I_k$. Then,

$$\begin{aligned}
&\int_{-1}^{1} |\tilde{Q}_{t,n}(x)| \, d\nu_t(x) \\
&\geq \int_0^{\pi} \left| \frac{t}{\sqrt{t}} \frac{\sin(n+1)\theta}{\sin\theta} - \frac{1-t}{\sqrt{t}} \frac{\sin(n-1)\theta}{\sin\theta} \right| \frac{1}{2\pi} \frac{4t\sin^2\theta}{1-4(1-t)t\cos^2(\theta)} \, d\theta \\
&\geq \int_E \left| \frac{t}{\sqrt{t}} \frac{\sin(n+1)\theta}{\sin\theta} - \frac{1-t}{\sqrt{t}} \frac{\sin(n-1)\theta}{\sin\theta} \right| \frac{1}{2\pi} \frac{4t\sin^2\theta}{1-4(1-t)t\cos^2(\theta)} \, d\theta \\
&\geq \frac{1}{2\pi} \int_E \left(\frac{t}{\sqrt{t}} \frac{1}{2t} - \frac{1-t}{\sqrt{t}} \right) \frac{4t\sin\theta}{1-4(1-t)t\cos^2(\theta)} \, d\theta \\
&\geq \frac{1}{\pi}(2t-1)\sqrt{t} \int_E \sin\theta \, d\theta \; > \; 0.
\end{aligned}$$

The integral $\int_E \sin\theta \, d\theta$ may be estimated from below by a positive constant, independent of n, since roughly $\frac{n+1}{3}$ of the intervals I_k are in $[\frac{\pi}{3}, \frac{2\pi}{3}]$. ∎

Proof of Lemma 4. It is clear that $f \in L_p([-1,1], \nu_t)$, when $1 \leq p < \frac{3}{2}$. Let

$$b_{t,n} = \int_{-1}^{1} f(x) U_n(x) \, d\nu_t(x).$$

Then it follows from (9.3.11) and (4.3.3) in [10] that $b_{t,n}$ converges to a limit $b_t > 0$, as $n \to \infty$. Hence,

$$\begin{aligned}
a_{t,n} \; &= \int_{-1}^{1} \sqrt{t} f(x) U_n(x) - \frac{1-t}{\sqrt{t}} f(x) U_{n-2}(x) \, d\nu_t(x) \\
&= \sqrt{t} b_{t,n} - \frac{1-t}{\sqrt{t}} b_{t,n-2} \; \to \; \frac{2t-1}{\sqrt{t}} b_t \; > \; 0. \; ∎
\end{aligned}$$

Now the proof of theorem 1 is evident: By lemma 1 we identified a subspace of $L_p(VN(\mathbb{F}_r))$ as the L_p space $L_p([-1,1], \nu_{1-\frac{1}{2r}})$ and showed in the corresponding section that an expansion of radial L_p operators amounts to an expansion of L_p functions in the latter space with respect to the orthonormal polynomials $Q_{t,n}$. For those the negative assertion of theorem 2 disproves the norm convergence for $1 \leq p < \frac{3}{2}$ or $p > 3$.

4. Von Neumann algebras of free commutation relations. Let $\mathcal{H}$ be an N-dimensional Hilbert space. The free (or full) Fock space is $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ for some vacuum vector Ω. On it the commutation relations $a_i a_j^+ = \delta_{i,j}$ may be represented, by taking (left) annihilation and creation operators with respect to an orthonormal basis $e_1, \dots, e_N$ of $\mathcal{H}$.

The von Neumann algebra generated by $G_1 = a_1 + a_1^+, \dots, G_N = a_N + a_N^+$ we shall denote $\mathfrak{A}$. The vacuum expectation

$$\varepsilon(T) = \langle T\Omega, \Omega \rangle, \qquad T \in B(\mathcal{F}(\mathcal{H})),$$

is a faithful normal trace on $\mathfrak{A}$. Moreover, $\mathfrak{A}$ acting on $\mathcal{F}(\mathcal{H})$ is in its standard representation and identifying $L_2(\mathfrak{A},\varepsilon)$ with $\mathcal{F}(\mathcal{H})$ any $T \in L_2(\mathfrak{A},\varepsilon)$ is of the form

$$T = \sum_{\underline{i}} \alpha_{\underline{i}}\omega(\underline{i}), \qquad \text{where } \underline{i} = (i_1, \dots, i_n),\ 1 \le i_j \le N,$$

and $\omega(\underline{i}) \in \mathfrak{A}$ acts by $\omega(\underline{i})\Omega = e_{i_1} \otimes \dots \otimes e_{i_n}$. One can show (see [6, 4, 3]) that

$$\omega(\underline{i}) = a_{i_1} \dots a_{i_n} + a_{i_1}^{+} a_{i_2} \dots a_{i_n} + \dots + a_{i_1}^{+} \dots a_{i_n}^{+}.$$

It follows (see [4, proposition 2.9]) that for $\underline{i} = (\overbrace{i_1, \dots, i_1}^{k_1}, \overbrace{i_2, \dots, i_2}^{k_2}, \dots, \overbrace{i_n, \dots, i_n}^{k_n})$

$$\omega(\underline{i}) = U_{k_1}(\omega(i_1))U_{k_2}(\omega(i_2)) \dots U_{k_n}(\omega(i_n)),$$

where U_k are the Tchebyscheff polynomials of second kind.

In this case it is natural to define the length of a tensor $\underline{e} = e_{i_1} \otimes \dots \otimes e_{i_n}$, and of the multi-index $\underline{i} = (i_1, \dots, i_n)$, $1 \le i_j \le N$, as $\mathbf{l}(\underline{e}) = n$ respectively as $\mathbf{l}(\underline{i}) = n$.

We can also ask when for $T = \sum_{\underline{i}} \alpha_{\underline{i}}\omega_{\underline{i}} \in L_p(\mathfrak{A},\varepsilon)$ we have convergence of the partial sums in $L_p(\mathfrak{A},\varepsilon)$:

$$S_N(T) = \sum_{\mathbf{l}(\underline{i}) \le N} \alpha_{\underline{i}}\omega_{\underline{i}} \to T.$$

THEOREM 3. *If for all $T \in L_p(\mathfrak{A},\varepsilon)$*

$$\|S_N(T) - T\|_{L_p(\mathfrak{A},\varepsilon)} \to 0,$$

then

$$\frac{3}{2} < p < 3.$$

Proof. We consider the radial elements of $\mathfrak{A}$:

$$\chi_0 = \omega(\emptyset) = \mathrm{id}, \quad \chi_1 = \sum_{\mathbf{l}(\underline{i})=1} \omega(\underline{i}), \quad \chi_n = \sum_{\mathbf{l}(\underline{i})=n} \omega(\underline{i}),\ n = 2, 3, \dots.$$

It is easy to see that

$$\chi_1\chi_n = \chi_{n+1} + N\chi_{n-1},\ n = 1, 2, \dots.$$

Then it follows that

$$\chi_n = \left(\frac{1}{\sqrt{N}}\right)^n U_n\left(\frac{\chi_1}{\sqrt{N}}\right).$$

The distribution of χ_1 with respect to ε is known to be the measure

$$d\mu_N = \frac{2}{\pi N}\sqrt{N - x^2}\, dx.$$

Hence convergence of $S_N(T)$, for radial T, implies the convergence of expansions with respect to the orthogonal polynomials in $L_p([-\sqrt{N}, \sqrt{N}], \mu_N)$. But by a theorem of Pollard [9] this is false for p not in the range $3/2 \le p \le 3$. If $p = \frac{3}{2}$ or $p = 3$, then Askey and Hirschman [1, Theorem 4c] show that the partial sum operators are not bounded uniformly in n. ∎

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