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GENERALIZED q-DEFORMED GAUSSIAN RANDOM VARIABLES

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Abstract. We produce generalized q-Gaussian random variables which have two parameters of deformation. One of them is, of course, q as for the usual q-deformation. We also investigate the corresponding Wick formulas, which will be described by some joint statistics on pair partitions.

1. Introduction. Bożejko and Speicher studied the q-analogues of Brownian motions in [BS1], [BS2] and investigated q-Gaussian processes with Kümmerer in [BKS], which are governed by usual independence for $q = 1$ and free independence for $q = 0$. Their constructions were based on the q-Fock space over a Hibert space $\mathcal{H}$. Let $a^+(f)$ be the q-creation operator and $a^-(f)$ be the q-annihilation operator associated with $f \in \mathcal{H}$, respectively. They satisfy the q-commutation relation $a^-(f)a^+(f) - qa^+(f)a^-(f) = \|f\|^2 \cdot \mathbf{1}$, which interpolates between the bosonic, canonical commutation relation (CCR), at $q = 1$, and the fermionic, canonical anti-commutation relation (CAR), at $q = -1$. Then the q-deformed centered Gaussian random variables are given by the position operators $a^+(f) + a^-(f)$ $(f \in \mathcal{H})$.

A certain general method for the construction of deformed Fock spaces was developed in [BS3], which is based on self-adjoint contractions on the tensor product space $\mathcal{H} \otimes \mathcal{H}$,

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and the braid relations play an important role. Their construction includes the q-Fock space as a special case which suggested to make it more general.

We shall here present the generalized q-deformed Fock space according to their construction together with an auxiliary positive sequence and give the associated generalized q-deformed Gaussian random variables. The special choices of the positive sequence will yield interesting two-parameter, (q, s) and (q, t)-deformed, Gaussian random variables which include the known examples, the t-free, the t-classical (see [BW2]), and the s-free Gaussian random variables.

We also investigate the corresponding Wick formulas. One of the deformation parameters is, of course, $q \in (-1, 1)$ as for the usual q-deformation. On the q-deformed Wick formula, the set partition statistic of pair partitions cr, the number of crossings, is used for q-counting (see, for instance, [An], [EP]). The other parameters of our deformations will require statistics other than cr in order to describe the Wick formulas in combinatorial terms. Namely, the number of inner points, ip, and the number of outer connected components, oc, will be used for the (q, s) and the (q, t)-deformations, respectively. The non one-mode interacting Fock space will be also presented as special case.

Although the partition statistic cr is strongly multiplicative on pair partitions, the joint statistics (cr, ip) and (cr, oc) are not strongly multiplicative functions any more. We give an example of the model of the strongly multiplicative function by the joint statistics of cr, the number of crossings, and cc, the number of connected components. This model will be given by the free compression in [NS]. At the end of paper, we will propose some problems for future work.

2. Deformed Fock spaces. In this section, we shall recall the general method for construction of deformed Fock spaces based on a self-adjoint contraction in [BS3].

Let $\mathcal{H}$ be a real Hilbert space. Consider a self-adjoint contraction T in $B(\mathcal{H} \otimes \mathcal{H})$ such that

$$(\mathbf{1} \otimes T)(T \otimes \mathbf{1})(\mathbf{1} \otimes T) = (T \otimes \mathbf{1})(\mathbf{1} \otimes T)(T \otimes \mathbf{1}),$$

where $T \otimes \mathbf{1}$ and $\mathbf{1} \otimes T$ are the natural amplifications of T to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. We define

$$T_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{i-1 \text{ times}} \otimes T \text{ on } \mathcal{H}^{\otimes (i+1)}$$

and by amplification also on all $\mathcal{H}^{\otimes n}$ with $n > i + 1$. Then the operators $\{T_i\}$ are self-adjoint contractions and satisfy the braids relations:

$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \quad \text{with} \quad |i - j| \geq 2. \end{cases}$$

The authors of [BS3] defined, for a vector $f \in \mathcal{H}$, a creation operator $d^*(f)$ and an annihilation operator $d(f)$ on a dense subset $\mathcal{F}$ of the full Fock space $\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ ($\|\Omega\| = 1$), $\mathcal{F}$ being the set of finite linear combinations of the elementary vectors. On the full Fock space, we have the canonical free creation operator $\ell^*(f)$ given by

$$\ell^*(f)\,\Omega = f,$$
$$\ell^*(f)\,x_1 \otimes \cdots \otimes x_n = f \otimes x_1 \otimes \cdots \otimes x_n$$

and the free annihilation operator $\ell(f)$ by

$$\ell(f)\,\Omega = 0,$$
$$\ell(f)\,x_1 \otimes \cdots \otimes x_n = \langle x_1|f\rangle\, x_2 \otimes \cdots \otimes x_n.$$

They put

$$d^*(f) = \ell^*(f)$$

and

$$d(f) = \ell(f)(\mathbf{1} + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}) \text{ on } \mathcal{H}^{\otimes n}.$$

Of course, $d^*(f)$ and $d(f)$ are not adjoint to each other with respect to the original scalar product $(\cdot\,|\,\cdot)_0$ on the full Fock space. Hence, they have introduced a new scalar product $(\cdot\,|\,\cdot)_T$ which makes $d^*(f)$ and $d(f)$ adjoint to each other.

The new scalar product is defined by

$$(\xi\,|\,\eta)_T = \delta_{m,n}(\xi\,|\,P^{(n)}\eta)_0 \text{ for } \xi \in \mathcal{H}^{\otimes m}, \eta \in \mathcal{H}^{\otimes n},$$

where the operator

$$P^{(n)} = \sum_{\sigma\in S_n} \varphi(\sigma)$$

on $\mathcal{H}^{\otimes n}$ is the operator corresponding to the function $\varphi : S_n \to B(\mathcal{H}^{\otimes n})$ given by $\varphi(e) = \mathbf{1}$ and $\varphi(\pi_i) = T_i \quad (i = 1, 2, \ldots, n-1)$. Here $\{\pi_1, \pi_2, \ldots, \pi_{n-1}\}$ is, of course, the set of generators for the permutation group of n elements S_n, where $\pi_i : (i, i+1) \mapsto (i+1, i)$.

Remark 1. It can be checked without much difficulty that if we put

$$R^{(n)} = \mathbf{1} + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-2}T_{n-1}$$

then $\{P^{(n)}\}$ satisfy the recurrence relation

$$P^{(n+1)} = (\mathbf{1} \otimes P^{(n)})R^{(n+1)}$$

with $P^{(1)} = \mathbf{1}$.

In the above situation, the following was shown (see [BS3]):

Theorem 2. (i) *If $\|T\| < 1$ then the operators $P^{(n)}$ are strictly positive for all n and we can take the completion of $\mathcal{F}$ with respect to the new scalar product $(\cdot\,|\,\cdot)_T$ as $\mathcal{F}_T$, the T-deformed Fock space.*

(ii) *For a vector $f \in \mathcal{H}$, $d^*(f)$ and $d(f)$ are adjoint to each other on $\mathcal{F}_T$, that is, for all $k \in \mathbb{N}$ and $\xi, \eta \in \bigoplus_{k=0}^n \mathcal{H}^{\otimes k}$ we have*

$$(d^*(f)\xi\,|\,\eta\,)_T = (\xi\,|\,d(f)\eta\,)_T.$$

3. Generalized q-deformed Fock space. The q-deformed Fock space introduced in [BKS] can be obtained in the above manner by defining the contraction operator T as

$$T : x \otimes y \mapsto q(y \otimes x) \text{ on } \mathcal{H} \otimes \mathcal{H}, \text{ for } q \in (-1, 1).$$

In this case, the operator $R^{(n)}$ for the recurrence relation in Remark 1 can be reduced to

$$Q^{(n)} = \mathbf{1} + q\Pi_1 + q^2\Pi_1\Pi_2 + \cdots + q^{n-1}\Pi_1 \cdots \Pi_{n-2}\Pi_{n-1},$$

where Π_i is the natural flip operator for the ith and the $(i+1)$st factors on a tensor product space.

Now we generalize these operators $\{Q^{(n)}\}$ and introduce generalized q-deformed Fock spaces. Given a sequence of *positive* numbers $\{\tau_n\}_{n\geq 1}$, we put

$$\widehat{R}^{(n)} = \tau_n Q^{(n)} \quad \text{for} \quad n \geq 1,$$

and define the operators $\{\widehat{P}^{(n)}\}$ by the same recurrence relation for $\{P^{(n)}\}$ as in Remark 1. Namely, we define the operators $\{\widehat{P}^{(n)}\}$ by the recurrence relation

$$\widehat{P}^{(n+1)} = (\mathbf{1} \otimes \widehat{P}^{(n)})\widehat{R}^{(n+1)},$$

with $\widehat{P}^{(1)} = \mathbf{1}$.

If we define the operator $\widetilde{P}^{(n)}$ by the recurrence relation

$$\widetilde{P}^{(n+1)} = (\mathbf{1} \otimes \widetilde{P}^{(n)})Q^{(n+1)},$$

then the operator $\widehat{P}^{(n)}$ is given in the form

$$\widehat{P}^{(n)} = \Big(\prod_{i=1}^{n} \tau_i\Big)\widetilde{P}^{(n)}.$$

We know that $\widetilde{P}^{(n)}$ is positive by Theorem 2 and τ_i, $i = 1, 2, \ldots, n$ are positive. Hence the operator $\widehat{P}^{(n)}$ remains positive.

Now we shall introduce the new scalar product $(\,\cdot\,|\,\cdot\,)_{(q,\{\tau_n\})}$ in the same manner as above, that is,

$$(\xi \,|\, \eta)_{(q,\{\tau_n\})} = \delta_{m,n}(\xi \,|\, \widehat{P}^{(n)}\eta)_0 \text{ for } \xi \in \mathcal{H}^{\otimes m}, \eta \in \mathcal{H}^{\otimes n}.$$

It can be seen that this new scalar product behaves on the elementary vectors as follows:

$$(x_1 \otimes \cdots \otimes x_n \,|\, y_1 \otimes \cdots \otimes y_m)_{(q,\{\tau_n\})} = \delta_{n,m}\Big(\prod_{i=1}^{n} \tau_i\Big) \sum_{\pi \in S_n} q^{inv(\pi)} \langle x_1 | y_{\pi(1)}\rangle \cdots \langle x_n | y_{\pi(n)}\rangle,$$

where $inv(\pi)$ is *the number of inversions of permutation* $\pi \in S_n$ defined by

$$inv(\pi) = \#\{(i,j) : 1 \leq i < j \leq n,\, \pi(i) > \pi(j)\}.$$

Proposition 3. *Given $q \in (-1, 1)$ and a positive sequence $\{\tau_n\}$, we define, for a vector $f \in \mathcal{H}$, the operator $a^-_{(q,\{\tau_n\})}(f)$ (simply denoted by $a^-(f)$) by*

$$\begin{aligned}
&a^-(f)\,\Omega = 0,\\
&a^-(f)\,x_1 = \tau_1\langle x_1|f\rangle\,\Omega,\\
&a^-(f)\,x_1 \otimes \cdots \otimes x_n = \tau_n \sum_{k=1}^{n} q^{k-1}\langle x_k|f\rangle\, x_1 \otimes \cdots \otimes \overset{\vee}{x}_k \otimes \cdots \otimes x_n \ (n \geq 2),
\end{aligned}$$

where the symbol $\overset{\vee}{x}_k$ means that x_k has to be deleted in the tensor product.

Then the operator $a^-(f)$ is adjoint to the canonical creation operator $a^+(f)$ on the full Fock space with respect to the new scalar product $(\,\cdot\,|\,\cdot\,)_{(q,\{\tau_n\})}$.

Since this proposition can be proved a similar way to [BW2] we omit the details.

By analogy with the Boson and the q-cases, for a vector $f \in \mathcal{H}$, we refer to the position operator

$$\omega(f) = a^+(f) + a^-(f)$$

as the generalized q-deformed centered Gaussian random variable of variance $\|f\|^2$.

4. The (q,s)-deformation. We shall consider a special choice of the sequence $\{\tau_n\}$, namely $\tau_n = s^{2(n-1)}$ $(n \geq 1)$. We shall call such a deformation *the (q,s)-deformation.*

It can be easily seen that the scalar product for the (q,s)-deformed Fock space will be given by

$$(x_1 \otimes \cdots \otimes x_n \,|\, y_1 \otimes \cdots \otimes y_m)_{(q,s)} = \delta_{n,m}\, s^{n(n-1)} \sum_{\pi \in S_n} q^{inv(\pi)} \langle x_1 | y_{\pi(1)} \rangle \cdots \langle x_n | y_{\pi(n)} \rangle,$$

and the (q,s)-deformed annihilation operator by

$$\begin{aligned} &a^-(f)\,\Omega = 0, \\ &a^-(f)\,x_1 = \langle x_1 | f \rangle\,\Omega, \\ &a^-(f)\,x_1 \otimes \cdots \otimes x_n = s^{2(n-1)} \sum_{k=1}^{n} q^{k-1} \langle x_k | f \rangle\, x_1 \otimes \cdots \otimes \overset{\vee}{x_k} \otimes \cdots \otimes x_n \ (n \geq 2). \end{aligned}$$

For a unit vector $f \in \mathcal{H}$, we shall evaluate the moments of the (q,s)-Gaussian random variable $\omega(f) = a^+(f) + a^-(f)$ with respect to the vacuum expectation $(\cdot\,|\,\cdot)_{(q,s)}$. We expand the nth power of the Gaussian random variable $\omega(f)^n = (a^+(f) + a^-(f))^n$ as

$$(a^+(f) + a^-(f))^n = \sum_{\varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1} a^{\varepsilon_n}(f) a^{\varepsilon_{n-1}}(f) \cdots a^{\varepsilon_1}(f),$$

where we regard $a^{+1}(f)$ and $a^{-1}(f)$ as $a^+(f)$ and $a^-(f)$, respectively.

It can be obtained by the routine argument (see, for instance, [EP]) that the summand

$$a^{\varepsilon_n}(f) a^{\varepsilon_{n-1}}(f) \cdots a^{\varepsilon_1}(f)$$

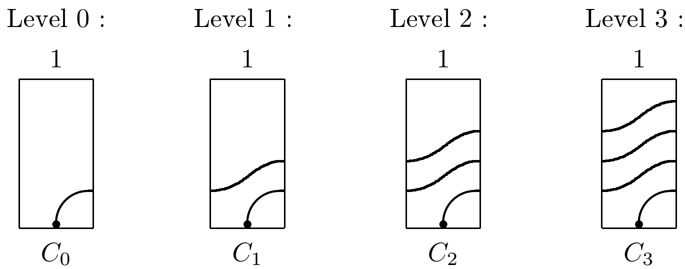
has non-zero vacuum expectation if and only if $\{\varepsilon_i\}_{i=1}^n$ is a Catalan sequence. In particular, n is even, say $n = 2m$. For the Catalan sequence $\{\varepsilon_i\}_{i=1}^{2m}$, we put

$$\begin{aligned} \ell_1 &= 0, \\ \ell_2 &= \varepsilon_1 = 1, \\ \ell_i &= \varepsilon_1 + \cdots \varepsilon_{i-1} \geq 0 \quad (2 \leq i \leq 2m-1), \\ \ell_{2m} &= \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{2m-1} = 1, \end{aligned}$$

and we shall call $\{\ell_i\}_{i=1}^{2m}$ *the associated level sequence.*

In order to evaluate the vacuum expectation, we shall use the cards arrangement, which is a similar technique as in [ER] for juggling patterns but we have to prepare different kinds of cards and we should introduce weights for the cards.

Creation cards. We make the cards C_i $(i = 0, 1, 2, \ldots)$ for the creation operator. The card C_i has i inflow lines from the left and $(i+1)$ outflow lines to the right, where one new line starts from the middle point on the ground level. For each $j \geq 1$, the inflow line of the jth level goes to the $(j+1)$st level without any crossing. We call the card C_i *the* *creation card of level i.* Each creation card has weight 1. We will illustrate the cards of first few levels:

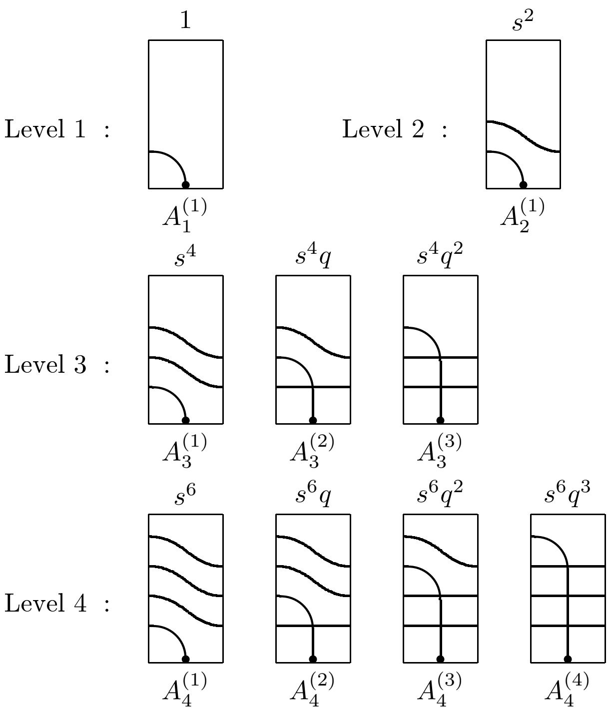


The weight of each card is indicated over the upper edge of the card.

Annihilation cards. Next we shall make the cards for the annihilation operator. For $i \geq 1$, we consider the cards $A_i^{(j)}$ $(j = 1, 2, \dots, i)$ which have i inflow lines from the left and $i-1$ outflow lines to the right. On the card $A_i^{(j)}$, only the inflow line of the jth level goes down to the middle point on the ground level and will be annihilated. The lines flowing into levels lower than the jth go on horizontally parallel and keep their levels. Hence $j-1$ crossings will occur. Moreover if the line flows into the $k(> j)$th level, it will flow out to the $(k-1)$st level without any crossing. We call the cards $A_i^{(j)}$ *the annihilation cards of level i.* The weight of the card will be q to the number of crossings on the card times s to twice the number of passes through lines, that is,

$$wt(A_i^{(j)}) = s^{2(i-1)} q^{j-1}.$$

We illustrate the annihilation cards of the first few levels:



REMARK 4. The creation cards represent the relations $a^{+}(f)f^{\otimes i} = f\otimes f^{\otimes i}$ $(i \geq 0)$, where the number of lines corresponds to the number of tensor factors. The annihilation cards reflect the relations

$$a^{-}(f)\, f^{\otimes i} = \sum_{j=1}^{i} s^{i-1}q^{j-1}\underbrace{(f\otimes\cdots\otimes \overset{\vee}{f}\otimes\cdots\otimes f)}_{\text{The } j\text{th factor is deleted}},$$

where the annihilated line indicates the position of the factor in the tensor product which should be deleted.

Given a Catalan sequence $\{\varepsilon_i\}_{i=1}^{2m}$, we shall arrange the cards depending both on $\{\varepsilon_i\}$ and on the associated level sequence $\{\ell_i\}_{i=1}^{2m}$ in the following manner: If $\varepsilon_i = +1$ then we put the creation card of level ℓ_i, C_{ℓ_i} at the ith position. If $\varepsilon_i = -1$ then we put one of the annihilation cards of level ℓ_i, $A_{\ell_i}^{(j)}$. There are ℓ_i possibilities of the choice of annihilation cards. We call such arrangements of cards *the admissible arrangements.*

By our construction, for a given Catalan sequence $\{\varepsilon_i\}_{i=1}^{2m}$, the sum of the products of the weight for the cards in all the admissible arrangements is equal to the vacuum expectation of the monomial

$$a^{\varepsilon_n}(f)a^{\varepsilon_{n-1}}(f)\cdots a^{\varepsilon_1}(f).$$

Furthermore, there is one-to-one correspondence between all the admissible arrangements for all the Catalan sequences of length $2m$ and $\mathcal{P}_2(2m)$, the set of pair partitions of $2m$ elements.

We shall now introduce the set partition statistics on pair partitions, which will enable us to describe the moments in combinatorial terms.

Let π be a pair partition of the set $\{1, 2, \ldots, 2m\}$ of $2m$ elements. For a block $(i,j) \in \pi$, we define $inpt(i,j)$ to be the number of k with $i < k < j$ (inner points) and we call $ip(\pi) = \sum_{(i,j)\in\pi} inpt(i,j)$ *the sum of the inner points* of the pair partition π (see [Yo]). Namely, if $\pi = \{(i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m)\}$, where $i_k < j_k$, then we have

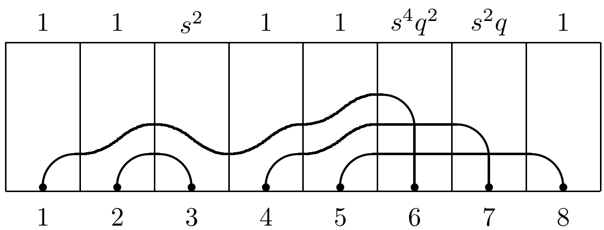
$$ip(\pi) = \sum_{k=1}^{m}(j_k - i_k - 1).$$

Note that the same statistic ip can be found in [EP] as the sum of gaps.

We also adopt the well-known statistic cr, *the number of crossings* (see, for instance, [Bi], [EP]), which is given by the number of pairs of blocks which will cross, that is,

$$cr(\pi) = {}^{\#}\{((a,b),(c,d)) : (a,b),(c,d) \in \pi \text{ with } a < c < b < d\}.$$

EXAMPLE. For the Catalan sequence $\{\varepsilon_i\}_{i=1}^{8} = \{+1, +1, -1, +1, +1, -1, -1, -1\}$, we can have, for instance, the following admissible arrangement:



The product of the weights of the cards is $s^8 q^3$ and the corresponding pair partition is

$$\pi = \{\{1,6\},\{2,3\},\{4,7\},\{5,8\}\}.$$

For the statistics we have $ip(\pi) = 8$ and $cr(\pi) = 3$.

Now we shall illustrate how the statistic ip will appear in the product of the weights of the cards in an admissible arrangement. We note first that each pass through line on the annihilation card is the segment of some connected line which will make a pair, and it is obvious that this site of annihilation is an inner point of the pair. Secondly, in an admissible arrangement, the annihilation cards of level $k+1$ and the creation cards of level k are completely parenthesized, and both the annihilation card of level $k+1$ and the creation card of level k have k passes through lines. Hence, the sum of the number of passes through lines on the annihilation cards in an admissible arrangement is just half the sum of the inner points of the corresponding pair partition. Furthermore, it is clear that the crossings of blocks of pair partitions will be counted only on the annihilation cards. Consequently, we can have the following evaluation:

THEOREM 5. *For a vector $f \in \mathcal{H}$ with $\|f\| = 1$, the moments of the (q,s)-Gaussian random variable $\omega(f) = a^+(f) + a^-(f)$ is given by*

$$(\omega(f)^n\,\Omega\,|\,\Omega)_{(q,s)} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \displaystyle\sum_{\pi \in \mathcal{P}_2(2m)} s^{ip(\pi)} q^{cr(\pi)}, & \text{if } n = 2m. \end{cases}$$

With help of the combinatorial arguments in [Fl], it is not so difficult to see that the Stieltjes transform of the standardized (q,s)-Gaussian measure $\nu_{(q,s)}$ can be expanded into the following continued fraction:

$$\int \frac{d\,\nu_{(q,s)}(t)}{z-t} = \cfrac{1}{z - \cfrac{[1]_q}{z - \cfrac{s^2[2]_q}{z - \cfrac{s^4[3]_q}{z - \cfrac{s^6[4]_q}{\ddots}}}}},$$

where $[n]_q$ stands for the q-integer that is $[n]_q = \frac{1-q^n}{1-q}$.

We can apply the cards arrangements to a more general situation and obtain the following (q,s)-Wick formula:

THEOREM 6. *Let $\omega(f_j) = a^+(f_j) + a^-(f_j)$ be the (q,s)-Gaussian random variables. Then*

$$(\omega(f_{2m})\cdots\omega(f_2)\omega(f_1)\,\Omega\,|\,\Omega)_{(q,s)} = \sum_{\pi \in \mathcal{P}_2(2m)} \Bigl(\prod_{(i,j)\in\pi} \langle f_i | f_j \rangle\Bigr) s^{ip(\pi)} q^{cr(\pi)}.$$

5. Other special cases. We shall consider another special choice of the sequence $\{\tau_n\}$ by putting, for $t > 0$, $\tau_1 = 1$ and $\tau_n = t$ if $n \geq 2$. We shall call such a deformation *the (q,t)-deformation.* In this case, the scalar product and the annihilation operator will be reduced as follows: The scalar product for the (q,t)-deformed Fock space will be given by

$$(x_1 \otimes \cdots \otimes x_n \,|\, y_1 \otimes \cdots \otimes y_m\,)_{(q,t)} = \delta_{n,m} t^{n-1} \sum_{\pi \in S_n} q^{inv(\pi)} \langle x_1 | y_{\pi(1)} \rangle \cdots \langle x_n | y_{\pi(n)} \rangle,$$

and the (q,t)-deformed annihilation operator by

$$\begin{aligned} &a^-(f)\,\Omega = 0,\\ &a^-(f)\,x_1 = \langle x_1 | f \rangle\,\Omega,\\ &a^-(f)\,x_1 \otimes \cdots \otimes x_n = t \sum_{k=1}^{n} q^{k-1} \langle x_k | f \rangle\, x_1 \otimes \cdots \otimes \overset{\vee}{x_k} \otimes \cdots \otimes x_n \ (n \geq 2). \end{aligned}$$

Remark 7. In the case of $q = 0$, the $(0,t)$-deformed Fock space is nothing else but the t-free Fock space investigated in [BW2] (see also [BW1]). If we take the limit $q \to 1$ then we can also obtain the t-classical Fock space. Another example of a Fock space representation of (q,t)-Gaussian random variables was given by Wojakowski in [Wo].

In order to obtain the (q,t)-Wick formula we shall arrange the cards again. The figures of cards are the same as before and the weights of the creation cards will not be changed. But we have to give different weights to the annihilation cards to indicate the (q,t)-annihilation, that is,

$$wt(A_i^{(j)}) = \begin{cases} 1 & \text{if } i = 1, \\ tq^{j-1} & \text{if } i \geq 2. \end{cases}$$

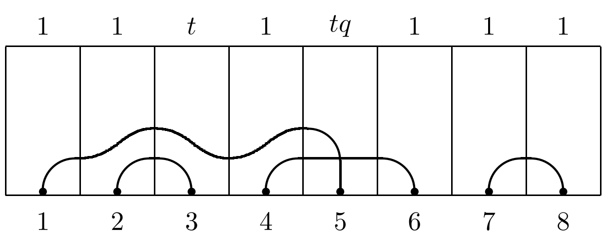
We shall adopt the set partition statistic oc, *the number of outer connected components*, introduced in [BW2]. We will regard, of course, that pairs which cross each other are contained in the same connected components and the outerness can be defined as in non-crossing case (see [BLS]).

It is an obvious combinatorial fact that outer connected components should be closed by the annihilation card of level 1. Furthermore, only the annihilation card of level 1 has weight 1 and the annihilation cards other than of level 1 have the factor t in their weights. Of course, there are m annihilation cards in the admissible cards arrangement of length $2m$.

Example. If the pair partition is

$$\pi = \{\{1,5\},\{2,3\},\{4,6\},\{7,8\}\}$$

then the corresponding admissible card arrangement is given as follows:



There are two outer connected components $\{1,5\} \cup \{4,6\}$ and $\{7,8\}$, which are closed by the annihilation of level 1 at the sites 6 and 8, respectively. On the other hand, the connected component $\{2,3\}$ is inner, which is closed at the site 3 by the annihilation card of level 2. The product of the weights of the cards is t^2q, which, of course, equals $t^{4-oc(\pi)}q^{cr(\pi)}$.

Combining the above arguments, we have the following (q,t)-Wick formula:

Theorem 8. *Let $\omega(f_j)=a^+(f_j)+a^-(f_j)$ be the (q,t)-Gaussian random variables. Then*

$$(\omega(f_{2m})\cdots\omega(f_2)\omega(f_1)\,\Omega\,|\,\Omega)_{(q,t)}=\sum_{\pi\in\mathcal{P}_2(2m)}\Big(\prod_{(i,j)\in\pi}\langle f_i|f_j\rangle\Big)\,t^{m-oc(\pi)}q^{cr(\pi)}.$$

In the generalized q-deformation introduced in section 3, if we consider the case of $q=0$ then it yields the interacting Fock space which is, however, not one mode but more general. The scalar product for this case will be reduced to

$$(x_1\otimes\cdots\otimes x_n\,|\,y_1\otimes\cdots\otimes y_m)_{(0,\{\tau_n\})}=\delta_{n,m}\Big(\prod_{i=1}^{n}\tau_i\Big)\langle x_1|y_1\rangle\langle x_2|y_2\rangle\cdots\langle x_n|y_n\rangle,$$

and the annihilation operator will be

$$\begin{aligned}&a^-(f)\,\Omega=0,\\&a^-(f)\,x_1=\tau_1\langle x_1|f\rangle\,\Omega,\\&a^-(f)\,x_1\otimes\cdots\otimes x_n=\tau_n\langle x_1|f\rangle\,x_2\otimes x_3\otimes\cdots\otimes x_n\ (n\geq 2).\end{aligned}$$

In this case we can also obtain the Wick formula by the same argument for non-crossing case in [AB].

Theorem 9. *Let $\omega(f_j)=a^+(f_j)+a^-(f_j)$ be the Gaussian random variables. Then*

$$(\omega(f_{2m})\cdots\omega(f_2)\omega(f_1)\,\Omega\,|\,\Omega)_{(0,\{\tau_n\})}=\sum_{\pi\in\mathcal{NCP}_2(2m)}\Big(\prod_{(i,j)\in\pi}\langle f_i|f_j\rangle\Big)\,\mathbf{t}(\pi),$$

where $\mathcal{NCP}_2(2m)$ denotes the set of non-crossing pair partitions of $2m$ elements. Here the function $\mathbf{t}(\pi)$ is defined as follows:

$$\mathbf{t}(\pi)=\prod_{B_i\in\pi}\tau_{d(B_i)},$$

where $d(B_i)$ denotes the depth of the block B_j in the non-crossing pair partition π given by

$$d(B_i)={}^{\#}\{k:B_i\subset B_k\}.$$

6. The strongly multiplicative function on pair partitions. We denote the set of pair partitions on $\{1,2,\ldots,2n\}$ by $\mathcal{P}_2(1,2,\ldots,2n)$ and put

$$\mathcal{P}_2(\infty)=\bigcup_{n=1}^{\infty}\mathcal{P}_2(1,2,\ldots,2n).$$

Definition. (i) A function $\mathbf{t}$ on $\mathcal{P}_2(\infty)$ is called *weakly multiplicative*, if we have for all $k,m\in\mathbb{N}$ with $k<m$ and all $\pi_1\in\mathcal{P}_2(1,\ldots,k)$ and $\pi_2\in\mathcal{P}_2(k+1,\ldots,m)$

$$\mathbf{t}(\pi_1\cup\pi_2)=\mathbf{t}(\pi_1)\cdot\mathbf{t}(\pi_2).$$

(ii) A function $\mathbf{t}$ on $\mathcal{P}_2(\infty)$ is called *strongly multiplicative*, if we have for all $k, \ell, m \in \mathbb{N}$ with $k < \ell < m$ and all $\pi_1 \in \mathcal{P}_2(1, \dots, k, \ell+1, \dots, m)$ and $\pi_2 \in \mathcal{P}_2(k+1, \dots, \ell)$

$$\mathbf{t}(\pi_1 \cup \pi_2) = \mathbf{t}(\pi_1) \cdot \mathbf{t}(\pi_2).$$

REMARK 10. The function $\mathbf{t}_q(\pi) = q^{cr(\pi)}$ on $\mathcal{P}_2(\infty)$ is strongly multiplicative. The functions $\mathbf{t}_s(\pi) = s^{ip(\pi)}$ and $\mathbf{t}_t(\pi) = t^{|\pi| - oc(\pi)}$, where $|\pi|$ stands for the number of blocks (pairs) in π, are not strongly multiplicative although they are weakly multiplicative. Hence, the joint statistics $s^{ip(\pi)} q^{cr(\pi)}$ and $t^{|\pi|-oc(\pi)} q^{cr(\pi)}$ would not be strongly multiplicative any more.

Here we consider the partition statistic cc, *the number of connected components.* It is obvious that the statistic cc will yield the strongly multiplicative function by $\mathbf{t}_\alpha(\pi) = \alpha^{-cc(\pi)}$.

Since the product of strongly multiplicative functions is again strongly multiplicative, for instance, the function

$$\mathbf{t}_{(q,\alpha)}(\pi) = \mathbf{t}_\alpha(\pi)\, \mathbf{t}_q(\pi) = \alpha^{-cc(\pi)} q^{cr(\pi)}$$

is strongly multiplicative.

Now we shall give an example of a construction of deformed Gaussian random variables for which the tracial and strongly multiplicative positive definite function such as above will appear in a moment formula. The model for this example is given by the free compression investigated in [NS]. We shall start with a tracial and strongly multiplicative example, like the q-case.

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, where φ is a trace. Let $p \in \mathcal{A}$ is a projection such that $\varphi(p) = \alpha\, (\neq 0)$. We consider deformed Gaussian random variable $\omega(f)$ in $\mathcal{A}$, where f is in a real Hilbert space $\mathcal{H}$ and p is free from $\omega(f)$. We make free compression by p, that is, we have the system of non-commutative probability space $(p\,\mathcal{A}\,p, \widetilde{\varphi})$ where $\widetilde{\varphi} = \frac{1}{\alpha}\, \varphi|_{p\,\mathcal{A}\,p}$, and the random variable $\widetilde{\omega}(f) = \frac{1}{\alpha}\, p\,\omega(f)\,p$ in $(p\,\mathcal{A}\,p, \widetilde{\varphi})$. This is our desired deformed Gaussian random variable. Indeed we have the following evaluation:

THEOREM 11. *For a vector $f \in \mathcal{H}$ with $\|f\| = 1$, we consider deformed Gaussian random variable $\omega(f)$. Let $\mathbf{t}$ be the corresponding positive definite function on $\mathcal{P}_2(\infty)$ for $\omega(f)$ and we assume that it is tracial and strongly multiplicative. Then the moments of the induced deformed Gaussian random variable $\widetilde{\omega}(f)$ are given by*

$$\widetilde{\varphi}\big(\widetilde{\omega}(f)^n\big) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{\pi \in \mathcal{P}_2(2m)} \alpha^{-cc(\pi)} \mathbf{t}(\pi), & \text{if } n = 2m. \end{cases}$$

Proof. By the assumption, the $2m$th moment of the deformed Gaussian random variable $\omega(f)$ is written in the form

$$m_{2m} = \sum_{\pi \in \mathcal{P}_2(2m)} \mathbf{t}(\pi).$$

Concerning the moments of odd orders, they should vanish and, hence, the free cumulants of odd orders of $\omega(f)$ also vanish.

If a non-crossing partition has no block of odd size, that is, all blocks are even, then we call such a non-crossing partition *even*. We denote the set of even non-crossing partitions of $2m$ elements by $\mathcal{NC}_e(2m)$.

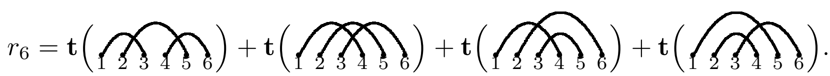
Given a pair partition $\pi \in \mathcal{P}_2(2m)$, the connected components of π will induce the even non-crossing partitions $\nu \in \mathcal{NC}_e(2m)$ canonically. We write this correspondence by $\Phi(\pi) = \nu$. For example, if the pair partition is $\pi = \{\{1,5\},\{2,3\},\{4,6\}\}$ then π has two connected components $\{1,5\} \cup \{4,6\}$ and $\{2,3\}$, thus the corresponding even non-crossing partition becomes $\Phi(\pi) = \{\{1,4,5,6\},\{2,3\}\}$. Pair partitions π_1 and π_2 in $\mathcal{P}_2(2m)$ are said to be *equivalent in connected components* if $\Phi(\pi_1) = \Phi(\pi_2)$. By this equivalence, we can rewrite the above formula on the $2m$th moment as

$$m_{2m} = \sum_{\nu \in \mathcal{NC}_e(2m)} \sum_{\substack{\pi \in \mathcal{P}_2(2m) \\ \Phi(\pi) = \nu}} \mathbf{t}(\pi).$$

Here we put

$$r_{2k} = \sum_{\substack{\rho \in \mathcal{P}_2(2k), \\ cc(\rho) = 1}} \mathbf{t}(\rho),$$

that is, r_{2k} is the sum of $\mathbf{t}$-values for the pair partitions in $\mathcal{P}_2(2k)$ which are constituted from only one connected component. We illustrate the case of r_6 below:



Then the strong multiplicativity of $\mathbf{t}$ guarantees that, for a given non-crossing partition $\nu \in \mathcal{NC}_e(2m)$,

$$\sum_{\Phi(\pi) = \nu} \mathbf{t}(\pi) = \prod_{V \in \nu} r_{|V|},$$

where $|V|$ denotes the size of the block V in the even non-crossing partition $\nu \in \mathcal{NC}_e(2m)$. Hence we have the equality

$$m_{2m} = \sum_{\nu \in \mathcal{NC}_e(2m)} \prod_{V \in \nu} r_{|V|},$$

which means that r_{2k} is nothing else than the $2k$th free cumulant of deformed Gaussian random variable $\omega(f)$.

By the formula (1.15) in [NS], it follows that the $2k$th free cumulant $\widetilde{r}_{2k}$ of the induced deformed Gaussian random variable $\widetilde{\omega}(f)$ is given by $\widetilde{r}_{2k} = \frac{1}{\alpha} r_{2k}$. Using the free moment-cumulant formula again, we obtain

$$\widetilde{m}_{2m} = \sum_{\nu \in \mathcal{NC}_e(2m)} \prod_{V \in \nu} \widetilde{r}_{|V|} = \sum_{\nu \in \mathcal{NC}_e(2m)} \alpha^{-|\nu|} \prod_{V \in \nu} r_{|V|},$$

where $|\nu|$ stands for the number of blocks in the even non-crossing partition $\nu \in \mathcal{NC}_e(2m)$. Under the map Φ, it is clear that the number of connected components of a pair partition π, $cc(\pi)$, equals the number of blocks of the corresponding even non-crossing partition $\nu(=\Phi(\pi))$. Consequently we have for the $2m$th moment of the induced deformed Gaussian random variable $\widetilde{\omega}(f)$

$$\widetilde{m}_{2m} = \sum_{\nu\in\mathcal{NC}_e(2m)} \alpha^{-|\nu|} \sum_{\Phi(\pi)=\nu} \mathbf{t}(\pi) = \sum_{\pi\in\mathcal{P}_2(2m)} \alpha^{-cc(\pi)}\mathbf{t}(\pi).$$

Hence the corresponding positive definite function $\widetilde{\mathbf{t}}$ on $\mathcal{P}_2(\infty)$ for the induced Gaussian random variable $\widetilde{\omega}(f)$ is given by

$$\widetilde{\mathbf{t}}(\pi) = \alpha^{-cc(\pi)}\mathbf{t}(\pi)$$

and it is again tracial and strongly multiplicative. ∎

Starting with $\mathbf{t}(\pi) = q^{cr(\pi)}$ we have the (q,α)-deformed Gaussian random variables and the following remarks naturally arise:

REMARK 12. (i) The function $\mathbf{t}_{(q,\alpha)}(\pi) = \alpha^{-cc(\pi)}q^{cr(\pi)}$ is strongly multiplicative and the vacuum state is trace so we can think about the second quantization and ultracontractivity or hypercontractivity of corresponding Ornstein-Uhlenbeck semigroups, like in the q-case (see [Bo1], [Bo2]).

(ii) There is an open problem about factoriality of the von Neumann algebra generated by the (q,α)-Gaussian random variables (see [Hi], [No], [Ri], and [Si]).

(iii) How about connections with the classical Markov processes like in [BKS]?

(iv) How about the orthogonal (q,α)-Hermite polynomials?

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