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COMPACTNESS PROPERTIES FOR MULTIPLICATION OPERATORS ON VON NEUMANN ALGEBRAS AND THEIR PREDUALS

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Abstract. We consider compactness, weak compactness and complete continuity for multiplication operators on von Neumann algebras and their preduals.

1. Introduction. Our primary interest in this paper is to demonstrate how difficult it is, even in the noncommutative world, for multiplication operators on analogues of an L_∞ and L_1 spaces to be compact, weakly compact or completely continuous. Recall that under the term *(weakly) compact operator* one understands a bounded linear operator T from one Banach space into another which maps bounded sets onto relatively (weakly) compact sets. A Banach space operator will be said to be *completely continuous* if it maps weakly convergent sequences onto norm convergent sequences. A Banach space X is said to have

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Dunford-Pettis property if every weakly compact operator with domain X is completely continuous. The function spaces $C(K)$ and $L_1(\mu)$ are among the best-known examples of such spaces. Now in the noncommutative world the Dunford-Pettis property is known to fail for almost all types of von Neumann algebras and their preduals [CI, B]. Hence in our analysis we cannot rely on the convenient connection between weakly compact and completely continuous operators afforded by this property in the classical setting. However despite this drawback, at least as far as multiplication operators are concerned, one still finds a remarkable similarity between the results pertaining to weakly compact multiplication operators and those pertaining to completely continuous ones.

The results on compactness and weak compactness of multiplication operators have been proved by Akemann and Wright in [AW] for general C^*-algebras. In their arguments they use the reduced atomic representation of a C^*-algebra. Those arguments are very elegant, but they do not show clearly enough what is happening in the von Neumann algebra case. In particular, one cannot easily see why there are no nonzero (weakly) compact multiplication operators on von Neumann algebras without minimal projections, like algebras of type II and III. Their techniques also make implicit use of the fact that the operators under consideration behave well with regard to duality. This seriously limits the application of their techniques to the investigation of complete continuity, which is not preserved under duality. We decided to give new and very straightforward arguments in support of the facts. As such these arguments not only provide new information with regard to completely continuous multiplication operators, but also give additional insight into the case of (weakly) compact ones. They follow from a couple of lemmas of independent interest, which we shall use in our subsequent paper generalizing the results to noncommutative L^p-spaces [GJL]. Note that several other generalizations have already been obtained by a series of authors, see for example [M, BC]. One should also consult the very interesting work of Pfitzner on weak compactness in C^*-algebras [P].

We use standard notation and terminology for von Neumann algebra theory, as found, for example, in [T, KR1, KR2]. In particular, we denote by $B(H)$ the algebra of all bounded operators on a Hilbert space H, and by $K(H)$ the ideal of compact operators on that space. We call an element of $\mathcal{M}$ *compact* if its image in some (though obviously not every) faithful normal representation of $\mathcal{M}$ on a Hilbert space H belongs to $K(H)$. We denote by $\mathcal{Z}(\mathcal{M})$ the *center* of the algebra $\mathcal{M}$ and by $c(f)$ the *central support (*or *carrier)* of f in $\mathcal{M}$. A projection e of $\mathcal{M}$ is *of finite rank* (or *finite dimensional*) in $\mathcal{M}$ if the algebra $e\mathcal{M}e$ is finite dimensional.

We use the same symbol M_f for *left multiplication operators* acting on the algebra: $M_f : \mathcal{M} \to \mathcal{M}, a \mapsto fa$ and on its predual $M_f : \mathcal{M}_* \to \mathcal{M}_*, \psi \mapsto f\psi$, where f belongs to $\mathcal{M}$. Similarly, we denote the *right multiplication operators* by ${}_fM$. Note that if $f = u|f|$ is the polar decomposition of f, then $M_f = M_u M_{|f|}$ and $M_{|f|} = M_{u^*} M_f$. The ideal property of compact, weakly compact and completely continuous operators now implies that M_f has one of the three properties if and only if $M_{|f|}$ has the same property. (Hence, we can always assume that the symbol we use for the multiplication is positive.) Since $f^* = |f|u^*$, the same can be said of M_f and M_{f^*}. Note that ${}_fM$ can be obtained by consecutive application of the involution of the algebra and M_{f^*}. Since the $*$-operation is norm, weak and weak* continuous on $\mathcal{M}$, M_f is compact, weakly compact or completely continuous if and only if ${}_fM$ has the same property. Analogous statements are true of multiplication operators on the predual $\mathcal{M}_*$.

In a few places, we were able to demonstrate two different kinds of strategies—the more straightforward ones, based on the properties of von Neumann algebras, and the more general ones, using tensor products and 'diagram chasing'.

2. Weak compactness. Note that $M_f : \mathcal{M}_* \to \mathcal{M}_*$ is weakly compact if and only if so is its adjoint $(M_f)^* = {}_fM : \mathcal{M} \to \mathcal{M}$. We shall use the fact without further notice.

We start with a simple but very useful lemma, which may well belong to the mathematical folklore. Since we were not able to trace it down in the literature, we provide it here with a proof.

LEMMA 2.1. *Let $\mathcal{M}$ be a von Neumann algebra with no minimal projections. Then no maximal abelian von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$ has minimal projections.*

Proof. Let $\mathcal{M}$ be a von Neumann algebra with no minimal projections and let $\mathcal{M}_0$ be a commutative von Neumann subalgebra of $\mathcal{M}$. Suppose that e_0 is a minimal projection in $\mathcal{M}_0$. By hypothesis, there must exist a projection $f_0 \in \mathcal{M} \setminus \mathcal{M}_0$ with $0 < f_0 < e_0$. Now given any other projection e in $\mathcal{M}_0$, we have by commutativity that $e_0e \in \mathcal{M}_0$ is a subprojection of e_0. So by minimality

$$\text{either} \quad e_0e = 0 \quad (\text{i.e. } e_0 \perp e) \qquad \text{or} \quad e_0e = e_0 \quad (\text{i.e. } e_0 \le e).$$

Thus since $f_0 < e_0$ we also have that

$$\text{either} \quad f_0 \perp e \qquad \text{or} \quad f_0 < e$$

for any projection e in $\mathcal{M}_0$. But this means that f_0 commutes with all the projections in $\mathcal{M}_0$. Since the span of these projections is dense in $\mathcal{M}_0$, f_0 commutes with $\mathcal{M}_0$. Therefore $\mathcal{M}_0$ cannot be maximal abelian, since $\{f_0, \mathcal{M}_0\}$ generates a commutative subalgebra which is strictly larger than $\mathcal{M}_0$. ∎

The next lemma uses standard arguments of the structure theory of von Neumann algebras.

LEMMA 2.2. *Let $\mathcal{M}$ be a von Neumann algebra. (1) Assume $\mathcal{M}$ is not a finite direct sum of finite discrete factors (i.e. it is not finite dimensional). Then there is in $\mathcal{M}$ an orthogonal sequence of non-zero projections. (2) Assume $\mathcal{M}$ has no minimal projections. If $e \in \mathcal{M}$ is a non-zero projection, then there exists in $\mathcal{M}$ an orthogonal sequence of non-zero subprojections of e. (3) Let $e \in \mathcal{M}$ be a projection which is not of finite rank in $\mathcal{M}$. Then there exists in $\mathcal{M}$ an orthogonal sequence of non-zero subprojections of e. In all cases, we can choose each of the projections to be σ-finite.*

Proof. (1) The conditions on $\mathcal{M}$ imply that it can be represented as a direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4$, where $\mathcal{M}_1$ is continuous (i.e. type *II* or *III*), $\mathcal{M}_2$ is type I_∞ (in particular, properly infinite), $\mathcal{M}_3$ is finite discrete with non-atomic center (i.e. a direct sum of type I_n, $n < \infty$ algebras with non-atomic centers) and $\mathcal{M}_4$ is a direct sum of finite discrete factors; with all but one of the four summands possibly zero, but if

$\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = \{0\}$, then $\mathcal{M}_4$ is an infinite direct sum of finite discrete factors. We consider the different cases:

(a) $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = \{0\}$ and $\mathcal{M}_4$ is an infinite direct sum of finite discrete factors. Let $z_n \in \mathcal{Z}(\mathcal{M}_4)$ be such that each of $\mathcal{M}_4 z_n$ is a different type I_n factor. Then (z_n) constitutes the desired sequence of non-zero projections.

(b) $\mathcal{M}_1 \neq \{0\}$ and/or $\mathcal{M}_2 \neq \{0\}$. There exists a projection $e \in \mathcal{M}_1$ (resp. $e \in \mathcal{M}_2$) such that $e \sim 1 - e$. We can put $e_1 = e$ and continue the procedure for the algebra $(1-e)\mathcal{M}_1(1-e)$ (resp. $(1-e)\mathcal{M}_2(1-e)$), which is obviously continuous (resp. type I_∞), to obtain $e_2 \leq 1 - e_1$. Then $e_3 \leq 1 - e_1 - e_2$ and so on.

(c) $\mathcal{M}_3 \neq \{0\}$. If $\mathcal{M}_3$ is not σ-finite, then $1 \in \mathcal{M}_3$ can be written as an orthogonal sum of (central) non-zero σ-finite projections. In this case, we can choose any countable subfamily from the collection. Thus, we may assume that M_3 is σ-finite. Since $\mathcal{M}_3$ is a direct sum of type I_n algebras, it is enough to consider a non-zero direct summand of type I_n, that is an algebra of the form $\mathbb{F} \otimes \mathcal{Z}$, where $\mathbb{F}$ is type I_n factor and $\mathcal{Z}$ is a σ-finite non-atomic commutative algebra. Let τ be a faithful normal tracial state on $\mathcal{Z}$. Since $\mathcal{Z}$ is non-atomic, we can find a projection $e_1 \in \mathcal{Z}$ with $0 < \tau(e_1) < 1$, then $e_2 \leq 1 - e_1$, $e_2 \in \mathcal{Z}$ with $0 < \tau(e_2) < \tau(1 - e_1)$ and so on. By passing to $(1 \otimes e_n)$, this gives an orthogonal sequence of non-zero projections in $\mathbb{F} \otimes \mathcal{Z}$.

(2) This is a direct consequence of (1) applied to $e\mathcal{M}e$.

(3) Again, $e\mathcal{M}e$ cannot be a finite direct sum of finite discrete factors, since then it would be finite-dimensional in some faithful representation of $\mathcal{M}$. Thus, part (1) of the lemma applies to $e\mathcal{M}e$.

The validity of the last remark follows from the fact that any non-σ-finite projection is an orthogonal sum of σ-finite ones, of which we can use any countable subset. Indeed, this is an immediate consequence of Zorn's lemma. A non-σ-finite projection e must have a non-zero σ-finite subprojection e_0: take any non-zero $\varphi \in (e\mathcal{M}e)_{*,+}$ and put $e_0 = \mathrm{supp}(\varphi)$. ∎

The following lemma contains the essence of what we are going to prove.

Lemma 2.3. *Let $\mathcal{M}$ be a von Neumann algebra and let $f \in \mathcal{M}$. If there exists an infinite sequence of projections $e_n \in \mathcal{M}$ and a number $\lambda > 0$ satisfying the following conditions:*

$$\begin{aligned} |f|e_n = e_n|f| &\quad \textit{for all } n, \\ |f|e_n \geq \lambda e_n &\quad \textit{for all } n, \\ e_n \to 0 &\quad \textit{strongly}, \\ e_n &\quad \textit{are } \sigma\textit{-finite and non-zero}, \end{aligned}$$

then the operator $M_f : \mathcal{M}_* \to \mathcal{M}_*$ is not weakly compact.

Proof. We may assume that $f \in \mathcal{M}_+$. Observe also that $e = \bigvee_n e_n$ is σ-finite, hence there exists $\varphi \in (\mathcal{M}_*)_+$ such that $\mathrm{supp}(\varphi) = e$. We have $\varphi(e_n) > 0$ for all n, so that we can define a sequence of states (φ_n) by

$$\varphi_n(x) = \frac{1}{\varphi(e_n)}\varphi(e_n x e_n) \quad \text{for all } x \in \mathcal{M} \text{ and all } n$$

(i.e. $\varphi_n = \frac{1}{\varphi(e_n)} e_n \varphi e_n$). The φ_n's are states since $\|\varphi_n\| = \varphi_n(1) = 1$. Now,

$$\begin{aligned}(M_f \varphi_n)(e_n) = (f\varphi_n)(e_n) = \varphi_n(e_n f) &= \frac{1}{\varphi(e_n)} \varphi(f e_n) \\ &\geq \frac{1}{\varphi(e_n)} \lambda \varphi(e_n) = \lambda > 0.\end{aligned}$$

Hence, the set $\{M_f \varphi_n\}$ is not relatively weakly compact (see [T], Lemma III.5.5), which ends the proof. ∎

The next two theorems show that weak compactness of the multiplication operators can happen only for very "discrete" algebras and only if the element f which we use is compact.

PROPOSITION 2.4. *Let $\mathcal{M}$ be a von Neumann algebra without minimal projections, $f \in \mathcal{M}$ and $f \neq 0$. Then $M_f : \mathcal{M}_* \to \mathcal{M}_*$ is not weakly compact.*

Proof. Assume as in the proof of Lemma 2.3 that f is positive (and non-zero). Let $\mathcal{A}$ denote a maximal abelian von Neumann subalgebra of M containing the element f. Let $\lambda > 0$ be such that $e = \chi_{[\lambda,\infty[}(f) \neq 0$. By Lemma 2.1, $\mathcal{A}$ has no minimal projections. Now we can apply Lemma 2.2(2) to e and obtain an orthogonal sequence (e_n) of subprojections of e satisfying all four conditions of Lemma 2.3. Consequently, the operator $M_f : \mathcal{M}_* \to \mathcal{M}_*$ is not weakly compact. ∎

THEOREM 2.5. *Let $\mathcal{M}$ be a von Neumann algebra and $f \in \mathcal{M}$. The operator M_f is weakly compact on either $\mathcal{M}$ or $\mathcal{M}_*$ if and only if f is compact in $\mathcal{M}$. If this is the case, the algebra $\mathcal{M}c(f)$ is σ-finite and atomic. Moreover, if (z_n) is a sequence of central projections such that $c(f) = z_1 + z_2 + \ldots$ and $\mathcal{M}z_k$'s are factors, then $(\|fz_n\|) \in c_0$.*

Proof. "$\Rightarrow$" Assume f is positive and non-zero and that $c(f) = 1$. If, for some $\lambda > 0$, the spectral projection $e_\lambda = \chi_{[\lambda,\infty[}(f)$ is not of finite rank in $\mathcal{M}$, then Lemma 2.2(3) gives a sequence (e_n) satisfying the assumptions of Lemma 2.3 and M_f is not weakly compact. Hence, for each $\lambda > 0, e_\lambda$ is of finite rank. Consequently, there exists a (finite or infinite) decreasing sequence (λ_n) of strictly positive real numbers such that the spectrum of f consists of λ_n's and, possibly, zero. We only need to show that in some representation of $\mathcal{M}$ all the spectral projections e_n corresponding to eigenvalues λ_n have finite-dimensional ranges. Now, if for some central projection $z \in \mathcal{M}$ the algebra $\mathcal{M}(1-z)$ has no minimal projections, then $\operatorname{supp}(f) \leq z$, since otherwise, by Proposition 2.4, $f(1-z)$ would not be weakly compact. Hence, we can assume that $\mathcal{M}$ is a direct sum of discrete factors. Evidently, $\mathcal{M}$ cannot have more than a countable number of non-zero summands, and if it is an infinite direct sum of such factors, say $\mathcal{M} = \mathcal{M}z_1 + \mathcal{M}z_2 + \ldots$ with $z_1, z_2, \ldots \in \mathcal{Z}(M)$ and $\mathcal{M}z_1, \mathcal{M}z_2, \ldots$ discrete, then the norms $\|fz_n\|$ tend to zero when n tends to infinity. Otherwise, we could easily build a sequence satisfying all the assumptions of Lemma 2.3 as in the proof of Lemma 2.2(1)(a). Our result is obvious for a finite discrete factor, so assume for a moment that $\mathcal{M}c(f)$ (or simply $\mathcal{M}$) is a factor of type I_∞ (where ∞ stands for some cardinal number). Such an algebra can be represented as $B(H)$ for a suitable Hilbert space H. In such a case, all the spectral projections e_n must have finite-dimensional ranges, otherwise we would easily get a sequence of projections satisfying the assumptions of Lemma 2.3. This means that in a suitable representation each of $fz_1, fz_2, \dots$ is compact. Moreover, as the norms of fz_n go to zero, each non-zero spectral value of f is a spectral value of a finite number of fz_n's only. Hence, in this representation the spectrum of f is either finite or its non-zero elements can be arranged into a sequence (λ_n) tending to zero and such that the spectral subspaces corresponding to λ_n's are finite dimensional. Obviously, this is enough to guarantee that f is also compact.

"$\Leftarrow$" Assume that f is positive and compact.

Suppose first that $\mathcal{M} = B(H)$ for some infinite dimensional Hilbert space H. Let $\xi \in H$, $\|\xi\| = 1$ and let f be a one-dimensional projection onto the subspace generated by ξ. Choose now any sequence (e_n) of orthogonal projections in $\mathcal{M}$ converging strongly to zero. Then, for any φ in the unit ball of $\mathcal{M}_*$, there is a trace-class operator h such that $\varphi(\cdot) = \mathrm{tr}(h\cdot)$. Let (ξ_i) be an orthonormal basis in H with one of the vectors equal to ξ. Then

$$(f\varphi)(e_n) = \mathrm{tr}(he_nf) = \sum (he_nf\xi_i, \xi_i) = (he_n\xi, \xi) = (e_n\xi, h^*\xi)$$

and

$$|(f\varphi)(e_n)| \le \|e_n\xi\| \|h\| \le \|e_n\xi\| \|h\|_1 \le \|e_n\xi\|.$$

Thus, the convergence $(f\varphi)(e_n) \to 0$ is uniform with respect to φ from the unit ball of the predual. This means that finite-rank f's and consequently also compact ones are such that M_f's are weakly compact.

Using the well-known fact that a Banach space operator $u : X \to Y$ is weakly compact if and only its second adjoint u^{**} maps X^{**} into Y, we can also argue as follows. If we use trace duality to identify $B(H)$ and $K(H)^{**}$, then $M_f : B(H) \to B(H)$ is just the second adjoint of M_f as an operator on $K(H)$. So weak compactness is equivalent to M_f acting from $B(H)$ to $K(H)$, which is readily seen to be equivalent to compactness of f. This kind of argument seems to trace back to K. Vala [V]; see also C. A. Akemann and S. Wright [AS].

Observe now that the algebra $\mathcal{M}c(f)$ must be atomic and σ-finite. In fact, if $\mathcal{M}z$ has no minimal projections for some central projection z, then the spectral projections of fz corresponding to its strictly positive eigenvalues cannot be of finite rank. Also, if $\|fz\| > 0$ for uncountable number of central projections z such that $\mathcal{M}z$ is a factor, then, for some $\epsilon > 0$, the number of such projections with $\|fz\| > \epsilon$ would be infinite, which is impossible for a compact f. Hence, $\mathcal{M}c(f)$ is σ-finite. Let (z_n) be a sequence of central projections such that all $\mathcal{M}z_n$'s are factors and $c(f) = z_1 + z_2 + \dots$. Then all the factors are discrete and $\|fz_n\| \to 0$. Recall that each infinite discrete factor is $*$-isomorphic to $B(H)$ for some infinite dimensional Hilbert space H. Hence, by what we have just proved, M_f is weakly compact on each discrete factor, the result for finite discrete factors being obvious. Now, if we choose k_0 large enough, we can make the norm of $f(z_{k_0+1} + z_{k_0+2} + \dots)$ arbitrarily small. Hence, with e_n and φ as before,

$$|(f\varphi)(e_n)| \le |(fz_1\varphi)(e_n)| + \dots + |(fz_{k_0}\varphi)(e_n)| + \|f(z_{k_0+1} + z_{k_0+2} + \dots)\|,$$

which can be made arbitrarily small uniformly w.r.t. φ from the unit ball, by what we have already proved.

Therefore, if f is compact, then M_f is weakly compact. ∎

3. Complete continuity. In this section we show that a multiplication operator M_f on $\mathcal{M}$ or $\mathcal{M}_*$ can be completely continuous only if the algebra $\mathcal{M}c(f)$ is finite and atomic. Moreover, if M_f is completely continuous on the algebra, then f must be compact.

THEOREM 3.1. *Let $\mathcal{M}$ be a von Neumann algebra and $f \in \mathcal{M}$. Then M_f acting on $\mathcal{M}$ is completely continuous if and only if $\mathcal{M}c(f)$ is finite and f is compact.*

Proof. Assume that M_f is completely continuous and that $c(f) = 1$. Consider first the case when $\mathcal{M}$ has no minimal projections. Let $\mathcal{M}_0$ be a maximal abelian von Neumann subalgebra of $\mathcal{M}$ containing f. By Lemma 2.1, $\mathcal{M}_0$ has no minimal projection, either. Note that M_f restricts to a completely continuous map from $\mathcal{M}_0$ to $\mathcal{M}_0$. Since $\mathcal{M}_0$ is commutative (and hence to all intents of purposes a $C(K)$ space), the restriction of M_f to $\mathcal{M}_0$ is weakly compact [DU, p 160, Corollary 17]. Hence we may apply Proposition 2.4 to conclude that $f = 0$ in this case.

The next case to consider is that of an arbitrary I_∞ factor. We can assume that $\mathcal{M} = B(H)$ for some Hilbert space H. As above, the restriction of M_f to some maximal abelian von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$ is weakly compact. By 2.5, f is compact in $\mathcal{M}_0$. Consequently, if f is not zero, there are a non-zero projection e in $\mathcal{M}$ and a number λ such that $fe = \lambda e$. We can assume that e is one-dimensional as an operator on H. Let (e_n) be a sequence of pairwise orthogonal one-dimensional projections on H such that $e_1 = e$. Let v_n be partial isometries such that $v_n^* v_n = e_n$ and $v_n v_n^* = e$. Choose any $\varphi \in \mathcal{M}^*$. The Cauchy-Schwarz inequality gives

$$|\varphi(v_n)| \leq \varphi(e_n)^{1/2} \varphi(1)^{1/2} \to 0.$$

Hence v_n is weakly null. On the other hand, the norms

$$\|M_f v_n\| = \|M_f e v_n\| = |\lambda| \|v_n\| = |\lambda|$$

do not converge to zero, which yields a contradiction. Hence, f must be zero.

In this case, we could again proceed by means of 'soft analysis'. If M_f acts on $\mathcal{M}$ in a completely continuous manner, then it does so on $K(H)$. When we identify, in a canonical fashion, $K(H)$ and the completed injective tensor product $H \check{\otimes} H$, then M_f becomes $1_H \otimes f$. Suppose now that $f \neq 0$. Pick $x \in H$ with $f(x) \neq 0$ and identify the linear span of x as well as of $f(x)$ with $\mathbb{C}$. Recall that H is just a copy of $H \check{\otimes} \mathbb{C}$. Combine all this to see 1_H can be considered as being induced by $1_H \otimes f$ through these identifications. We conclude that 1_H is compact, whence $\dim H < \infty$.

Suppose now that $\mathcal{M}$ is an infinite direct sum of finite type I factors. Let (z_n) be a sequence of pairwise orthogonal non-zero central projections such that each $\mathcal{M}z_n$ is a factor. Obviously, the sequence (z_n) is weakly (i.e. $\sigma(\mathcal{M}, \mathcal{M}^*)$) null. Hence $\|fz_n\| = \|M_f z_n\| \to 0$, so that, for each ϵ, the set of these nonzero central projections z for which $\mathcal{M}z$ is a factor and $\|fz\| \geq \epsilon$, is finite. Hence, the set of all nonzero central projections z for which $fz \neq 0$ and $\mathcal{M}z$ is a factor is at most countable and f is compact (cf. the proof of Theorem 2.5).

We are left with the case of a finite direct sum of finite type I factors. It is clear that in this case f must be compact in $\mathcal{M}$.

For the converse, assume that f is compact. The result is obvious if $\mathcal{M}$ is a finite direct sum of finite type I factors, so let $\mathcal{M}$ be an infinite direct sum of such factors. Let also the z_n's be selected as before. Given $\epsilon > 0$, we can find k_0 such that $\|fz_k\| < \epsilon$ for any $k > k_0$. Take an arbitrary weakly null sequence (a_n) in $\mathcal{M}$. We can assume that its elements are taken from the unit ball. Obviously, $\|a_n z_k\| \to 0$ as $n \to \infty$, since the weak and norm topologies coincide on any finite type I factor. Hence $\|a_n z_k\| < \epsilon/\|f\|$ for $k \le k_0$ and n sufficiently large. Consequently, for all such n, $\|fa_n z_k\| < \epsilon$ for all k, which means that $\|fa_n\| \le \epsilon$, so that M_f is completely continuous. ∎

THEOREM 3.2. *Let $\mathcal{M}$ be a von Neumann algebra and $f \in \mathcal{M}$. The operator M_f acting on $\mathcal{M}_*$ is completely continuous if and only if $\mathcal{M}c(f)$ is finite atomic.*

Proof. Assume that, for some $f \in \mathcal{M}_+$, M_f is completely continuous and $c(f) = 1$. Consider first the case when $\mathcal{M}$ is properly infinite. Let (e_n) be a sequence of pairwise orthogonal projections in $\mathcal{M}$, all equivalent to 1, and let (v_n) be such that $v_n^* v_n = e_n$ and $v_n v_n^* = 1$. Choose any state φ on $\mathcal{M}$. As in the proof of Theorem 3.1, v_n is $\sigma(\mathcal{M}, \mathcal{M}^*)$-null, hence also $\sigma(\mathcal{M}, \mathcal{M}_*)$-null. This implies that (φv_n) is weakly null. Thus $M_f(v_n\varphi) \to 0$ in norm. At the same time, $\|M_f(\varphi v_n)\| \ge \varphi(v_n v_n^* f) = \varphi(f)$. Hence $f = 0$.

Assume now that $\mathcal{M}$ has no minimal projections. Fix a state φ on $\mathcal{M}$. Since, by Lemma 2.1, maximal abelian subalgebras containing f cannot have minimal projections, there is in $\mathcal{M}$ a Rademacher sequence (r_n) consisting of symmetries commuting with f and such that $\varphi(r_k r_l) = 0$ for $k \ne l$. To see this, recall that each such subalgebra is $*$-isomorphic, as a von Neumann algebra, to the ℓ_∞-direct sum of L^∞-spaces over some nonatomic finite measure spaces. Note that the mapping $x \mapsto x\varphi$ from $\mathcal{M}$ into $\mathcal{M}_*$ factorizes through the Hilbert space H_φ of the GNS representation of $\mathcal{M}$ w.r.t. φ: $x \mapsto x\xi \mapsto x\varphi$, where $\xi \in H_\varphi$ is such that $\varphi = \omega_\xi$. Since $(r_n\xi)$ forms an orthonormal sequence in H_φ, it is weakly null there. Consequently, $(r_n\varphi)$ is weakly null in $\mathcal{M}_*$. Thus, $M_f(r_n\varphi) \to 0$ in norm. At the same time $\|M_f(r_n\varphi)\| \ge \varphi(r_n f r_n) = \varphi(f)$. Hence $f = 0$.

Assume finally that $\mathcal{M}$ is finite atomic. Then $\mathcal{M}_*$ has Schur's property. In fact, by Proposition III.5.10 in [T], it is enough to show that if (φ_n) is weakly null, then the sets $\{|\varphi_n|\}$ and $\{|\varphi_n^*|\}$ are both relatively weakly compact. To this end, note that since $\mathcal{M}$ is finite, the $*$-operation is σ-strongly continuous on bounded parts of $\mathcal{M}$ (see for example [S], Theorem 2.5.6), so that by Theorem III.5.7 in [T], the Arens-Mackey topology $\tau(\mathcal{M}, \mathcal{M}_*)$ coincides with the σ-strong topology on bounded parts of $\mathcal{M}$. This yields relative weak compactness of the set of absolute values of any relatively compact subset of $\mathcal{M}_*$, as explained in detail in Exercise V.2.5(d) from [T]. Hence, every bounded operator on $\mathcal{M}_*$ is completely continuous, which ends the proof. ∎

4. Compactness. The criteria for compactness of a multiplication operator are easy to read off from our results on completely continuous operators. In particular, we obtain that a multiplication operator on the algebra is compact if and only if it is completely continuous.

THEOREM 4.1. *Let $\mathcal{M}$ be a von Neumann algebra and $f \in \mathcal{M}$. Then M_f acting on $\mathcal{M}$ or $\mathcal{M}_*$ is compact if and only if $\mathcal{M}c(f)$ is finite and f is compact.*

Proof. It is enough to prove the result for $\mathcal{M}$, since the other one follows from Schauder's theorem—the operator on $\mathcal{M}$ is compact if and only if its preadjoint operator on $\mathcal{M}_*$ is compact. If M_f is compact, then it is also completely continuous. Thus, by Theorem 3.1, $\mathcal{M}c(f)$ is finite and f is compact. For the converse, assume that the two conditions are satisfied. Let (a_n) be a bounded sequence in $\mathcal{M}$ and let (z_k) be the sequence of central projections from the proof of Theorem 3.1. Obviously, the sequence (a_n) has a subsequence which converges in norm on $\mathcal{M}z_1$, then a subsequence of this subsequence which converges in norm on $\mathcal{M}z_2$ and so on. Thus, we can construct a diagonal subsequence of (a_n) which is norm convergent on each finite factor $\mathcal{M}z_k$, $k = 1, 2, \ldots$. The same type of reasoning as in the proof of Theorem 3.1 shows that the image of this subsequence under M_f is in fact norm convergent on $\mathcal{M}$. ∎

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