

FEYNMAN DIAGRAMS AND THE QUANTUM STOCHASTIC CALCULUS

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Abstract. We present quantum stochastic calculus in terms of diagrams taking weights in the algebra of observables of some quantum system. In particular, we note the absence of non-time-consecutive Goldstone diagrams. We review recent results in Markovian limits in these terms.

1. Introduction. Quantum stochastic calculus (QSC) involves an analysis of the fundamental quantum processes of creation/annihilation/conservation [1] and intuitively this is somehow related to emission/absorption/scattering of physical quanta as described by quantum field theory (QFT). Quantum stochastic theory has the advantages, as well as the limitations, of having a mathematically rigorous setting. It also has the theory of classical probability to fall back on for much of its inspiration. So much so that the relationship with QFT, which was originally a major motivating factor, is now frequently overlooked. Effectively, the fundamental quantum processes should be idealizations of quantum fields for some suitable “Markovian” regime. They were introduced to describe open systems dynamics: here the quantum noise couples to some quantum system and so, in some sense, we are dealing not just with traditional quantum fields, but with quantum fields taking values in the algebra of observables of some quantum system.

When presenting his famous list of problems, Hilbert is supposed to have quoted an unnamed colleague as saying that “a mathematical theory should not be considered complete until one can walk out into the street and explain it to the first person you meet”. Let us suppose we did this and, as luck would have it, the first person we meet is a physicist. Would we succeed in explaining quantum stochastic calculus? My contention is that we should, though we might have to make do with some formal mathematics (presumably Hilbert wouldn’t have objected?). There are many fundamental ideas, familiar to physi-

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cists, hidden (sometimes too well-hidden!) in the mathematical formalism of quantum stochastic calculus. In this article, I've tried to present some basic results from quantum stochastic calculus in the language of QFT and, in particular, in terms of Feynman-type diagrams: the hope being that mathematicians and physicists will learn something from the cross-over. Diagrammatic techniques are central to QFT [2], [3], yet also of independent interest from it; they still remain an essential tool for investigating mathematical aspects of quantum theory and continue to yield some of the most illuminating insights (see, for instance, the papers [4], [5], [6]). I also want to present an account of a recent paper [7] which deals with the QSC approximation and which was originally formulated in diagrammatic language.

1.1. Expansions of evolutions. A free-particle (Bosonic) quantum field Φ_t living on a Hilbert (Fock) space \mathfrak{H} can be decomposed into positive and negative frequency terms as $\Phi_t = \Phi_t^{(+)} + \Phi_t^{(-)}$: here we suppress all dependence other than time and understand the time label to refer to Heisenberg picture of the free dynamics. We take $\Phi_t^{(+)}$ to be an annihilation field and $\Phi_t^{(-)}$ to be a creation field. If Ω is the Fock vacuum vector, then we have the identity

$$(1) \quad \Phi_t^{(+)} \Omega = 0,$$

along with the canonical commutation relations

$$(2) \quad [\Phi_t^{(+)}, \Phi_s^{(-)}] = G(t, s).$$

Here $G(t, s) \equiv \langle \Omega | \Phi_t \Phi_s \Omega \rangle$. Related to this is the propagator K defined as

$$K(t, s) = \langle \Omega | \vec{T} \Phi_t \Phi_s \Omega \rangle = G(t, s)\theta(t - s) + G(s, t)\theta(s - t),$$

where $\theta(\cdot)$ is the Heaviside step function. As usual \vec{T} is Dyson's chronological operation placing Heisenberg picture operators in increasing time-order from right to left.

Let $\{\Upsilon_t : t \geq 0\}$ be a family of self-adjoint operators on \mathfrak{H} with Υ_t being some function of $\Phi_t^{(\pm)}$. We are then interested in the evolution operator

$$(3) \quad U_t = \vec{T} \left\{ \exp -i \int_0^t \Upsilon_s ds \right\},$$

by which we mean the solution to the Schrödinger equation $i\partial_t U_t = \Upsilon_t U_t$, $U_0 = 1$. Suppose that we have a polynomial interaction

$$\Upsilon_t = \sum_{\nu} \frac{1}{\nu!} \lambda_{\nu} (\Phi_t)^{\nu}.$$

The standard device of quantum field theory is to expand U_t in terms of diagrams, see, e.g. [2] or [8]. A *Wick diagram* D is constructed as follows: choose $n = n(D)$ labelled points (vertices), each vertex will have some labelled legs attached (we let $m_{\nu} = m_{\nu}(D)$ denote the number of vertices having ν legs so that $n = \sum_{\nu} m_{\nu}$), we join several pairs of legs to form (undirected) edges, the result is a graph having several external lines and we now ignore the labelling. A Wick diagram is then the class of all topological equivalent

graphs. We let $c = c(D)$ denote the number of ways we could have originally connected the various legs to get the same graph. Now denote by \mathcal{D}_W the set of all Wick diagrams and define, for each $D \in \mathcal{D}_W$, the operator

$$(4) \quad \hat{D}(t) = (-i)^n \frac{c}{n!} \prod_{\nu} \left(\frac{\lambda_{\nu}}{\nu!} \right)^{m_{\nu}} \vec{N} \int_{[0,t]^n} \prod_D K \prod_D (\Phi^{(+)} + \Phi^{(-)})$$

where \vec{N} is normal ordering (placing all creation fields $\Phi^{(-)}$ to the left of all annihilation fields $\Phi^{(+)}$) and under the integral we have a factor $K(t_i, t_j)$ for each edge (i, j) occurring and a factor Φ_{t_k} for each external line at vertex (k) . It is then a basic result of QFT that U_t admits the expansion

$$U_t = \sum_{D \in \mathcal{D}_W} \hat{D}(t).$$

Next let \mathcal{P}_W denote the subset of *connected* Wick diagrams then we may list the elements as P_1, P_2, \dots and each $D \in \mathcal{D}_W$ can be decomposed as $D \equiv P_1^{n_1} \times P_2^{n_2} \times \dots$. Now one readily checks that $\hat{D}(t) = \vec{N} \frac{\hat{P}_1(t)^{n_1}}{n_1!} \frac{\hat{P}_2(t)^{n_2}}{n_2!} \dots$ and so

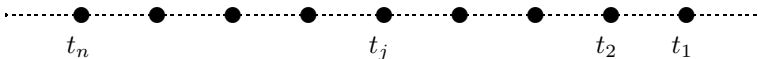
$$(5) \quad U_t = \vec{N} \sum_{n_1, n_2, \dots} \prod_{j=0}^{\infty} \frac{\hat{P}_j(t)^{n_j}}{n_j!} = \vec{N} \prod_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{\hat{P}_j(t)^n}{n!} = \vec{N} \exp \sum_{P \in \mathcal{P}_W} \hat{P}(t).$$

What we have managed to do is to express the evolution operator U_t as a normal ordered exponential of a sum over connected Wick diagrams. The connected Wick diagrams play the role of the ‘primes’ amongst the set of all Wick diagrams—indeed the trick of replacing a sum of products by a product of sums is just the one that goes on when we develop a prime number expansion of the Dirichlet series of a multiplicative function, the Riemann zeta function being perhaps the best known example; it is also the trick used to compute the grand canonical partition function for the free Bose gas. The result should be understood as an operator theoretic version of the usual cumulant moment expansion.

Now let \mathfrak{h}_S be a fixed Hilbert space. We move the action up to the Hilbert space $\mathfrak{h} \otimes \mathfrak{H}$ and set

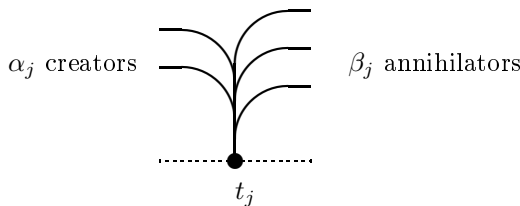
$$(6) \quad \Upsilon_t = \sum_{\alpha, \beta} E_{\alpha\beta} \otimes (\Phi_t^{(-)})^{\alpha} (\Phi_t^{(+)})^{\beta}$$

where we take $E_{\alpha\beta}^{\dagger} = E_{\beta\alpha}$. We now introduce a class of diagrams known as *Goldstone diagrams*—they differ from the previous ones in that the vertices are placed in time order [2]. Consider times $t_n > \dots > t_2 > t_1$ in the interval $[0, t]$ and draw these as vertices as shown below:



Suppose that at vertex j we have β_j legs coming in from the right, representing annihilators, and α_j legs going out to the left, representing creators. For example, the

vertex for $E_{23} \otimes [\Phi_{t_j}^{(-)}]^2 [\Phi_{t_j}^{(+)}]^3$ where we have $\alpha_j = 2$ creators and $\beta_j = 3$ annihilators is sketched as



We construct a Goldstone diagram D as follows. We take an arbitrary number $n = n(D)$ vertices and draw in an ordered line as above. We then draw creation / annihilation lines at each vertex corresponding to one of the terms appearing in Υ . We then connect selected creation legs to (necessarily later time) annihilation legs: the remaining uncontracted legs are then *directed* external lines. We consider the family \mathcal{D}_G of all (topologically distinct) diagrams obtained in this way. To each Goldstone diagram D we associate the operator

$$(7) \quad \hat{D}(t) = (-i)^n E_{\alpha_n \beta_n} \cdots E_{\alpha_1 \beta_1} \otimes \int_{\Delta_n(t)} \prod_D \Phi^{(-)} \prod_D G \prod_D \Phi^{(+)}$$

where $\Delta_n(t)$ is the simplicial region $\{(t_n, \dots, t_1) : t > t_n > \dots > t_2 > t_1 > 0\}$ and we have a factor $G(t_i - t_j)$ for each edge (i, j) , note $t_i > t_j$, a factor $\Phi_{t_k}^{(+)}$ for each incoming external line to a vertex k and a factor $\Phi_{t_k}^{(-)}$ for each outgoing external line. We then find the expansion

$$(8) \quad U_t = \sum_{D \in \mathcal{D}_G} \hat{D}(t).$$

In QFT one is used to switching between an expansion in terms of Goldstone diagrams (8) and one in terms of Wick diagrams (5). In the present case however we have an obstruction: the $E_{\alpha\beta}$'s do not necessarily commute! This complication means that the Goldstone diagrams are more fundamental in the present case. The problem, of course, is that the Dyson operator \vec{T} is reordering the Heisenberg fields only, while the $E_{\alpha\beta}$'s remain in their original order.

1.2. Zero dimensional QFT. Let a and a^\dagger be annihilation operators for a single mode harmonic oscillator, We have the commutation relations $[a, a^\dagger] = 1$ and $a\Omega = 0$. Let us consider the observable $q = za^\dagger + z^*a$ where z is a complex number. The Baker-Campbell-Hausdorff theorem says that $\exp\{itq\} = \exp\{itza^\dagger\} \exp\{-\frac{1}{2}|z|^2 t^2\} \exp\{itz^*a\}$ which here has the same content as the expansion (5). We can use a diagrammatic presentation based on two types of vertex: the creation type $\cdots \overbrace{\bullet}^{\curvearrowright} \cdots$, which has weight z , and the annihilation type $\cdots \underbrace{\bullet}_{\curvearrowleft} \cdots$ which has weight z^* . If we take a vacuum expectation of $\exp\{itq\}$ then we need only consider connected diagrams having no external lines—and there is only the one! The cumulant expansion is then

$$\langle \Omega | \exp\{itq\} \Omega \rangle = \exp\left\{ \frac{(it)^2}{2!} \cdots \overbrace{\bullet \cdots \bullet}^{\curvearrowright} \cdots \right\} \equiv \exp\left\{ -\frac{1}{2} t^2 |z|^2 \right\},$$

and we see that q is Gaussian in the vacuum state. As is well known, the odd moments vanish while the even moments are $\langle \Omega | q^{2n} \Omega \rangle = |z|^{2n} \frac{(2n)!}{2^n n!}$ where the combinatorial factor counts the number of ways to partition the $2n$ (time-ordered) vertices into n (contraction) pairs. For instance, the fourth moment involves three disconnected diagrams (our convention is that we only consider the contractions as edges; the thin horizontal base line does not affect connectivity!)

$$\langle q^4 \rangle = \text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\text{---} + \text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\text{---} + \text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\text{---} = 3|z|^4.$$

We may also consider the variable $N = (a+z)^\dagger(a+z) = a^\dagger a + z a^\dagger + z^* a + |z|^2$. We now introduce two extra vertices: a scattering vertex $\text{---}\overset{\frown}{\bullet}\text{---}\text{---}$, with weight unity and a constant vertex $\text{---}\bullet\text{---}\text{---}$ with weight $|z|^2$. To determine the vacuum expectation of $\exp\{itN\}$, we once again sum over all connected diagrams with no external lines. This gives

$$\langle \Omega | \exp\{itN\} \Omega \rangle = \exp \left\{ \frac{(it)}{1!} \text{---}\bullet\text{---}\text{---} + \frac{(it)^2}{2!} \text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\text{---} + \frac{(it)^3}{3!} \text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\text{---} + \dots \right\}$$

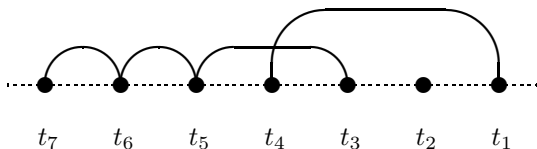
The n^{th} term in the exponential will look like $\text{---}\overset{\frown}{\bullet}\text{---}\overset{\frown}{\bullet}\text{---}\dots\text{---}\overset{\frown}{\bullet}\text{---}\text{---}$ and each such term has weight $|z|^2$ —since the scattering vertices all have weight unity. All cumulants are equal and we therefore have

$$\langle \Omega | \exp\{itN\} \Omega \rangle = \exp \left\{ \sum_{n \geq 1} \frac{(it)^n}{n!} |z|^2 \right\} = \exp \{ |z|^2 (e^{it} - 1) \}$$

and we recognize N as having a Poisson distribution of intensity $|z|^2$. The moments of the variable N are given as a polynomial of degree n in $|z|^2$, vis.

$$\langle \Omega | N^n \Omega \rangle = \sum_{m=1}^n S(n, m) |z|^{2m}$$

and, as is well-known in combinatorial analysis [9], the coefficients $S(n, m)$ are the Stirling numbers of the second kind: they count the number of ways to partition n items into m non-empty subsets. To see why they arise here, consider the following diagram contributing to $\langle N^7 \rangle$:



This diagram partitions the 7 vertices into 3 subsets, namely $\{t_7, t_6, t_5, t_3\}$, $\{t_4, t_1\}$ and $\{t_2\}$, with each subset forming a connected sub-diagram. This contributes $(|z|^2)^3$ to $\langle N^7 \rangle$. Consulting a textbook on combinatorics to get the Stirling numbers, we find

$$\langle N^7 \rangle = \sum_{m=1}^7 S(7, m) (|z|^2)^m = |z|^2 + 63|z|^4 + 301|z|^6 + 350|z|^8 + 140|z|^{10} + 21|z|^{12} + |z|^{14}.$$

Alternatively, we could draw all $B_7 = 877$ diagrams out! The numbers $B_n = \sum_{m=1}^n S(n, m)$ counting the total number of ways to partition the n vertices into non-empty subsets (of connected Goldstone diagrams) are known as the Bell numbers.

2. Quantum stochastic calculus. Remarkably, the equivalence between expansions (8) and (5) is restored in the non-commutative case in one very important situation. This is when we consider the cases α and β taking only the values 0, 1 in (6) and when the two-point function G is replaced by a delta-function. Effectively the field is some form of quantum white noise in time. As α, β is restricted to either 0 or 1, we shall have only four types of vertex: a constant vertex $E_{00} \otimes 1$, an emission vertex $E_{10} \otimes \Phi^{(-)}$, an absorption vertex $E_{01} \otimes \Phi^{(+)}$ and a scattering vertex $E_{11} \otimes \Phi^{(-)}\Phi^{(+)}$.

The reason for the algebraic equivalence, despite the fact that the $E_{\alpha\beta}$ need not commute, is that many of the Goldstone diagrams vanish identically. This is due to the singular nature of the two-point function with respect to the simplicial integration in (7). We note that absence of certain diagrams describing moments of quantum noises has occurred elsewhere, in particular, there is an elegant description of the various forms of independent quantum processes in these terms [11].

Let us introduce some formal symbolic notations [10]. We make the replacements

$$\Phi_t^{(+)} \hookrightarrow \mathbf{a}_t, \Phi_t^{(-)} \hookrightarrow \mathbf{a}_t^\dagger, G(t, s) \hookrightarrow \mathbf{g}(t - s),$$

where $\mathbf{g}(t - s) = \kappa \mathfrak{d}_+(t - s) + \kappa^* \mathfrak{d}_-(t - s)$. Here κ is a complex damping constant with $\gamma = 2\text{Re}\{\kappa\} > 0$. The objects $\mathfrak{d}_\pm(t - s)$ are one-sided delta functions defined (for functions f possessing left and right hand limits) by

$$(9) \quad \int_{-\infty}^{\infty} f(s) \mathfrak{d}_\pm(s - t) ds = \int_{-\infty}^{\infty} f(t + u) \mathfrak{d}_\pm(u) du = f(t^\pm).$$

Let us briefly indicate how to convert $U_t = \vec{T} \exp\{-i \int_0^t \Upsilon_s ds\}$ to normal order [10] where $\Upsilon_t = E_{\alpha\beta} \otimes (\mathbf{a}_t^\dagger)^\alpha (\mathbf{a}_t)^\beta$ (we use a convention from now on that repeated Greek indices are summed over values 0 and 1). When evaluating Goldstone diagrams, we find that if the contractions are not time-consecutive, that is, if we encounter $\mathbf{g}(t_i - t_j)$ with $i > j + 1$, then we force the multiple equalities $t_i = t_{i-1} = \dots = t_{j+1} = t_j$ due to the time ordering, and so the contribution vanishes. Only Goldstone diagrams with time-consecutive contractions are non-zero.

Starting from the integro-differential equation $U_t = 1 - i \int_0^t \Upsilon_s U_s ds$, we have

$$[\mathbf{a}_t, U_t] = -i \int_0^t [\mathbf{a}_t, \Upsilon_s] U_s ds = -i \int_0^t \mathbf{g}(t - s) E_{1\beta}(\mathbf{a}_t)^\beta U_s = -i \kappa E_{1\beta}(\mathbf{a}_t)^\beta U_t$$

or $\mathbf{a}_t U_t = (1 + i \kappa E_{11})^{-1} [U_t \mathbf{a}_t - i \kappa E_{10} U_t]$ and so

$$(10) \quad \partial_t U_t = -i E_{\alpha\beta} \otimes (\mathbf{a}_t^\dagger)^\alpha (\mathbf{a}_t)^\beta U_t \equiv (\mathbf{a}_t^\dagger)^\alpha L_{\alpha\beta} U_t (\mathbf{a}_t)^\beta$$

where

$$(11) \quad L_{\alpha\beta} = -i E_{\alpha\beta} - \kappa E_{\alpha 1} \frac{1}{1 + i \kappa E_{11}} E_{1\beta}.$$

We may interpret the conversion of the Schrödinger equation to normal ordered form as

a change from a Stratonovich to an Itô description. This agrees with the interpretation given originally by von Waldenfels for emission-absorption interactions [12].

Having normal-ordered the Schrödinger equation, we now iterate to get

$$\begin{aligned}
 (12) \quad U_t &= 1 + \int_0^t (\mathbf{a}_s^\dagger)^\alpha L_{\alpha\beta} U_s (\mathbf{a}_s)^\beta ds \\
 &= \sum_{n \geq 0} \int_{\Delta_n(t)} (\mathbf{a}_{t_n}^\dagger)^{\alpha_n} \cdots (\mathbf{a}_{t_1}^\dagger)^{\alpha_1} (L_{\alpha_n \beta_n} \cdots L_{\alpha_1 \beta_1}) (\mathbf{a}_{t_1})^{\beta_1} \cdots (\mathbf{a}_{t_n})^{\beta_n}.
 \end{aligned}$$

This is reasonably familiar to quantum field theorists and such expressions can be found for instance in Berezin’s book [13]. If f is a suitable test function, we may consider its coherent (i.e. exponential) vector $|\varepsilon(f)\rangle$ and take $\mathbf{a}_t |\varepsilon(f)\rangle = f(t) |\varepsilon(f)\rangle$ and $\langle \varepsilon(f) | \mathbf{a}_t^\dagger = \langle \varepsilon(f) | f(t)^*$. As (12) is normal ordered, we have no difficulty in assigning a meaning to $\langle \varepsilon(f) | U_t \varepsilon(g)\rangle$. At this stage we could just as well take (12) as the definition of the process, this is the starting point of the Maassen kernel calculus [14]. As such the time-consecutive contraction property is built into QSC, though in a way that is not readily apparent.

For the benefit of quantum probabilists, who may well be a little lost at this stage, we convert (10) into more familiar language [1]. Let $\Lambda_t^{\alpha\beta} = \int_0^t (\mathbf{a}_s^\dagger)^\alpha (\mathbf{a}_s)^\beta ds$ and we interpret these as the four fundamental quantum processes: Λ_t^{00} is time, Λ_t^{10} is creation, Λ_t^{01} is annihilation and Λ_t^{11} is conservation. Loosely speaking, we say $\{X_t, t \geq 0\}$ is adapted if $[\mathbf{a}_s^\dagger, X_t] = 0$ whenever $s > t$. Setting $X_t^{(j)} = \int_0^t (\mathbf{a}_s^\dagger)^\alpha x_{\alpha\beta}^{(j)}(s) (\mathbf{a}_s)^\beta ds$ where the $x_{\alpha\beta}^{(j)}(\cdot)$ are adapted, we see that putting to normal order yields

$$X_t^{(1)} X_t^{(2)} = \int_0^t (\mathbf{a}_s^\dagger)^\alpha [X_s^{(1)} x_{\alpha\beta}^{(2)}(s) + x_{\alpha\beta}^{(1)}(s) X_s^{(2)} + x_{\alpha 1}^{(1)}(s) x_{1\beta}^{(2)}(s)] (\mathbf{a}_s)^\beta ds.$$

The basic idea goes back to Hudson and Streater [15]. In QSC, we usually write

$$dX_t = x_{\alpha\beta}(t) d\Lambda_t^{\alpha\beta}$$

and the above result is presented as the quantum Itô formula

$$d(X^{(1)} X^{(2)}) = X^{(1)} d(X^{(2)}) + d(X^{(1)}) X^{(2)} + d(X^{(1)}) d(X^{(2)})$$

along with the quantum Itô table $d\Lambda_t^{\alpha 1} d\Lambda_t^{1\beta} = d\Lambda_t^{\alpha\beta}$. The equation (10) is then interpreted as the Itô quantum stochastic differential equation $dU_t = L_{\alpha\beta} U_t d\Lambda_t^{\alpha\beta}$ with $U_0 = 1$. The coefficients satisfy the identities $L_{\alpha\beta} + L_{\beta\alpha}^\dagger + \gamma L_{1\alpha}^\dagger L_{1\beta} = 0$ which are necessary and sufficient for U_t to be an adapted, unitary quantum stochastic process. The formula for the product of several quantum integrals comes down to a normal ordering problem which can ultimately be presented as a sum over diagrams, or equivalently, a sum over partitions of the time indices: for the classical case, see [16].

3. Markov limits. Finally, we wish to comment on how regular quantum fields can approximate the singular fields considered above. Let $\lambda \neq 0$ be a parameter and consider fields $\Phi_t^{(\pm)}(\lambda)$ with a regular two-point function $G_\lambda(\cdot)$ which becomes a delta-function in the limit $\lambda \rightarrow 0$. In particular, we may take $G(\cdot)$ to be an integrable function with $\gamma = \int_{-\infty}^\infty G$ and $\kappa = \int_0^\infty G$ and assume that $G(-t) = G(t)^*$. Then set $G_\lambda(t, s) = \lambda^{-2} G((t-s)/\lambda^2)$. We would then argue that in the limit G_λ converges to

the singular function \mathbf{g} consider above. We consider the regular unitary evolution operators

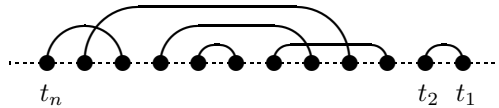
$$U_t(\lambda) = \vec{T} \exp \left\{ -i \int_0^t E_{\alpha\beta} \otimes (\Phi_s^{(-)}(\lambda))^\alpha (\Phi_s^{(+)}(\lambda)) \right\}$$

and we claim that for bounded $E_{\alpha\beta}$, with $\|\kappa E_{11}\| < 1$, $U_t(\lambda)$ converges to the singular process U_t considered in the last section.

The first remark that we make is that the $\lambda \rightarrow 0$ limit leads to the vanishing of each non-time-consecutive Goldstone diagram contributing to $U_t(\lambda)$. Moreover, when the surviving terms are computed and re-summed, we formally get the correct Itô expansion (12). Note that $L_{\alpha\beta} = -iE_{\alpha\beta} - i \sum_{r=1}^\infty E_{\alpha 1} (-i\kappa E_{11})^{r-1} E_{1\beta}$ giving the contribution to a time-consecutive block with α outgoing, β incoming lines and a sum over r successive scatterings in between. We see that the condition $\|\kappa E_{11}\| < 1$ is necessary to sum the geometric series.

The re-summation is rather tedious, though it helps that we know what answer to expect! We also have the issue of convergence, however, we settle this below. We remark that it is sufficient to consider only the vacuum convergence as the more general situation can be inferred from this when we look at convergence in arbitrary but appropriately scaled coherent vector states.

3.1. Pulé inequalities (Gaussian). Let us start with the case where we have emission and absorption only in the interaction. The vacuum Goldstone diagrams, as we have seen in section 2, consist of n_2 , say, pair contractions only. A typical diagram, one of $\frac{(2n_2)!}{2^{n_2} n_2!}$ having $2n_2$ vertices, is sketched below for $n_2 = 6$:



There exists a permutation σ of the $n = 2n_2$ time indices which re-orders to the diagram $D_0(n)$ shown below



The permutation is moreover unique if it has the induced ordering of the emission times. Not all permutations on the n time indices will arise this way, but the ones that do will be termed admissible. We now consider an estimate of the n -th term in the Dyson series:

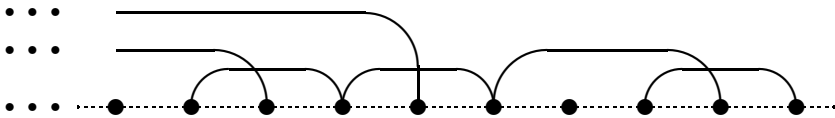
$$\begin{aligned} \sum_{D \in \mathcal{D}_G} \int_{\Delta_n(t)} \prod_D |G_\lambda| &= \sum_{\text{Admissible permutations}} \int_{\Delta_n(t)} \prod_{D_0(n)} |G_\lambda \circ \sigma| \\ &= \int_{R(t)} \prod_{k=1}^{n_2} |G_\lambda(t_{2k} - t_{2k-1})| \end{aligned}$$

where $R(t)$ is the union of simplices $\{(t_n, \dots, t_1) : t > t_{\sigma^{-1}(n)} > \dots > t_{\sigma^{-1}(1)} > 0\}$ over all admissible permutations σ . $R(t)$ will be a subset of $[0, t]^{2n_2}$ and if we introduce variables t_{2k} and $s_{2k} = t_{2k} - t_{2k-1}$ for $k = 1, \dots, n_2$ it is easily seen that the above is

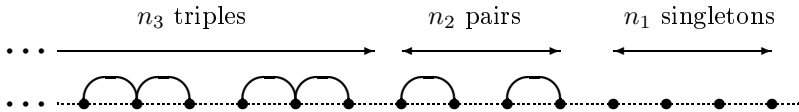
majorized by $|\kappa|^{n_2} \times \frac{\max(t, 1)^{n_2}}{n_2!}$. This the Pulé inequality [17] and clearly gives the uniform absolute estimate required to sum the series.

3.2. Pulé inequalities (Poissonian). We now consider scattering, and constant, terms in the interaction [7]. As we have seen there will be B_n (the n -th Bell number) Goldstone diagrams contributing to the n -th term in the vacuum Dyson series expansion. The Bell numbers grow rapidly and have a complicated asymptotic behaviour. The proliferation of diagrams is due mainly to the multiple scattering that now may take place.

Let us consider a typical Goldstone diagram. We shall assume that within the diagram there are n_1 singleton vertices $[\dots \dots \bullet \dots \dots]$, n_2 contraction pairs $[\dots \dots \overset{\frown}{\bullet} \dots \dots \smile \dots \dots]$, n_3 contraction triples $[\dots \dots \overset{\frown}{\bullet} \dots \overset{\frown}{\bullet} \dots \smile \dots \smile \dots \dots]$, etc. That is the Goldstone diagram has a total of $n = \sum_j j n_j$ vertices which are partitioned into $m = \sum_j n_j$ connected subdiagrams. For instance, we might have an initial segment of a diagram looking like the following:



There will exist a permutation σ of the n vertices which will reorder the vertices so that we have the singletons first, then the pair contractions, then the triples, etc., so that we obtain a picture of the following type



The permutation is again unique if we retain the induced ordering of the first emission times for each connected block. We now wish to find a uniform estimate for the n -th term in the Dyson series, we have

$$(13) \quad \sum_{\text{Goldstone diagrams}} \int_{\Delta_n(t)} \prod |G_\lambda| \times \text{“weights”}$$

where the weights are the operator norms of various products of the type $E_{\alpha_n \beta_n} \dots E_{\alpha_1 \beta_1}$. In general, the weight is bounded by

$$\|E_{11}\|^{n_1+2n_2+3n_3+\dots} \times C^{n_1+n_2+n_3+\dots}$$

where $C = \max_{\alpha\beta} \|E_{\alpha\beta}\|$. This is because each connected diagram of j vertices will typically have one emission and one absorption, but $j - 2$ scattering vertices. The Pulé argument of rearranging the sum over diagrams into a single integral over a region $R(t)$ of $[0, t]^n$ again applies and by similar reason we arrive at the upper bound for (13) this time of the type

$$\sum'_{n_1, n_2, n_3, \dots} \| \kappa E_{11} \|^{n_1+2n_2+3n_3+\dots} \times C^{n_1+n_2+n_3+\dots} \times \frac{\max(t, 1)^{n_1+n_2+n_3+\dots}}{n_1! n_2! n_3! \dots}$$

Here the sum is restricted so that $\sum_j j n_j = n$. A uniform estimate for the entire series is then given by removing this restriction:

$$\Xi(A, B) = \sum_{n_1, n_2, n_3, \dots} \frac{\exp\{\sum_j (Aj + B)n_j\}}{n_1! n_2! n_3! \dots}$$

where $e^A = \|\kappa E_{11}\|$ and $e^B = C \max(t, 1)$. Again we use the trick to convert a sum of products into a product of sums

$$\begin{aligned} \Xi(A, B) &= \sum_{n_1, n_2, n_3, \dots} \prod_j \frac{\exp\{(Aj + B)n_j\}}{n_j!} = \prod_j \sum_n \frac{\exp\{(Aj + B)n\}}{n!} \\ &= \prod_j \exp\{e^{(Aj+B)}\} = \exp\left\{\sum_j e^{Aj} e^B\right\} = \exp\left\{\frac{e^{A+B}}{1 - e^A}\right\}. \end{aligned}$$

where we need $e^A < 1$ to sum the geometric series—this however, is precisely our condition that $\|\kappa E_{11}\| < 1$.

4. Conclusions. We have established a Markov limit in the sense of [18] which we may write as

$$\vec{T}\left\{\exp -i \int_0^t E_{\alpha\beta} \otimes (\Phi_s^{(-)})^\alpha (\Phi_s^{(+)})^\beta ds\right\} \rightsquigarrow \vec{T}\left\{\exp -i \int_0^t E_{\alpha\beta} \otimes (\mathbf{a}_s^\dagger)^\alpha (\mathbf{a}_s)^\beta ds\right\}.$$

On the left hand side we have a unitary which can be expanded as a normal ordered expression of the quantum fields in terms of Goldstone diagrams. The right hand side can be developed as an expansion over time-consecutive can be understood as Hudson–Parthasarathy unitary quantum stochastic process. We have shown the non-time-consecutive terms on the left hand side make a negligible contribution in the Markovian limit. Interpreting Weyl order as Stratonovich form and Wick order as Itô form, the above result can be considered as a non-commutative version of the Wong–Zakai limit theorem for classical processes.

The same holds for Fermi fields, however, the proof is complicated because we have to take the limit in matrix elements of appropriately scaled number states [19]. The same basic estimates suffice once more and in the limit we end up with the same process except with the $\Lambda^{\alpha\beta}$ now being Fermionic noises. As one might suspect, we have to bother ourselves collecting factors of -1 , and one would expect to obtain the same result if we dealt with q -commutation relations [20].

We remark that the time-ordered exponentials developed in [21] differ from the notions presented here, as we are time-ordering quantum white noises and not Itô differentials, though they do arise in models for Markov limits of discrete time systems [22].

Finally, we mention that we also have the convergence of the Heisenberg dynamics

$$U_t(\lambda)^\dagger (X \otimes 1) U_t(\lambda) \quad \text{to} \quad U_t^\dagger (X \otimes 1) U_t$$

[7]. This requires a slightly deeper analysis, however, the basic estimates above are again at the heart of things. We invite the reader to try and imagine the Goldstone diagram expansion of $U_t(\lambda)^\dagger (X \otimes 1) U_t(\lambda)$ to get an idea of what is involved.

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